1.2b: Relations on a Set

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for Ohio University MATH 3110/5110 College Geometry

Subject: Relations on a Set

- The Cartesian Product of Sets
- Relations on a Set
- Properties of that Relations on a Set may or may not have
- Equivalence Relations

Textbook: Millman & Parker, *Geometry: A Metric Approach with Models, Second Edition*
(Springer, 1991, ISBN 3-540-97412-1)

Reading: Section 1.2 Sets and Equivalence Relations, pages 4 - 7

Homework: Section 1.2 # 6,8,9,13,14,15,18,19

The Cartesian Products of Sets

Definition of *Ordered Pair*

Let A and B are sets. An **ordered pair** is a symbol (a, b) where $a \in A$ and $b \in B$.

Two ordered pairs (a, b) and (c, d) are **equal** if $a = c$ and $b = d$.

Definition of *Cartesian Product*

Symbol: $A \times B$ (pronounced $A\ cross\ B$)

Usage: A and B are sets.

Spoken: The **Cartesian product** of A and B

Meaning: the set $A \times B = \{(a, b) | a \in A\ and\ b \in B\}$

Power Notation: The cartesian product of a set with itself is sometimes denoted using exponent notation. That is, $A \times A$ is sometime denoted A^2.

[Example 2] The symbol $\mathbb{R}^2$ denotes the Cartesian product $\mathbb{R} \times \mathbb{R}$. In this cartesian product, the symbols $(2,3)$ and $(3,2)$ denote different ordered pairs. Order matters in ordered pairs. Compare this to set notation, where order does not matter. For example, $\{2,3\} = \{3,2\}$.

Relations on a Set

Definition of *Binary Relation on a Set*

Words: R is a **binary relation** on S

Meaning: S is a set, and $R \subset S \times S$. That is, R is a set containing ordered pairs from $S \times S$.

Additional Terminology and Notation

words: s *is related to* t

symbols: ${}_sR_t$ or $s \sim t$, but many other symbols can also be used.

meaning: $(s, t) \in R$

words: s *is not related to* t

symbols: ${}_s\not{R}_t$ or $s \nsim t$

meaning: $(s, t) \notin R$

It is often helpful to illustrate a binary relation on a set. In general, there can be more than one way to do this. For binary relations on the set $\mathbb{R}$, the illustration can be in the form of a graph in the $\mathbb{R}^2$ plane.

[Example 3] The Less Than Relation

Define $R = \{(x, y) \in \mathbb{R}^2 | x < y\}$

We could say that the symbol ${}_xR_y$ means $x < y$, or say that $x \sim y$ means $x < y$.

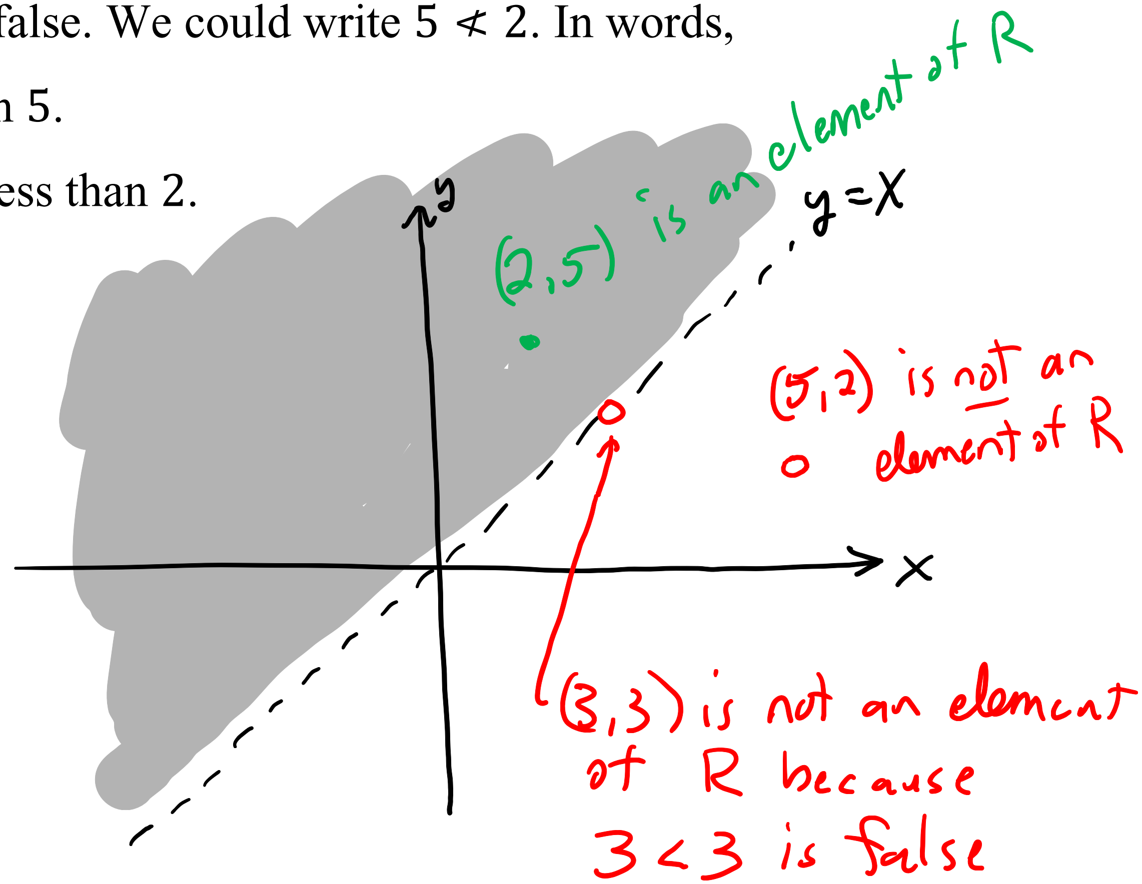
But it is simpler to just use the symbol $<$ to denote this relation.

Notice that statements involving relation symbols can be true or false. For example, the statement $2 < 5$ is true while the statement $5 < 2$ is false. We could write $5 \not< 2$. In words,

2 is related to 5. That is, 2 is less than 5.

5 is not related to 2. That is, 5 is not less than 2.

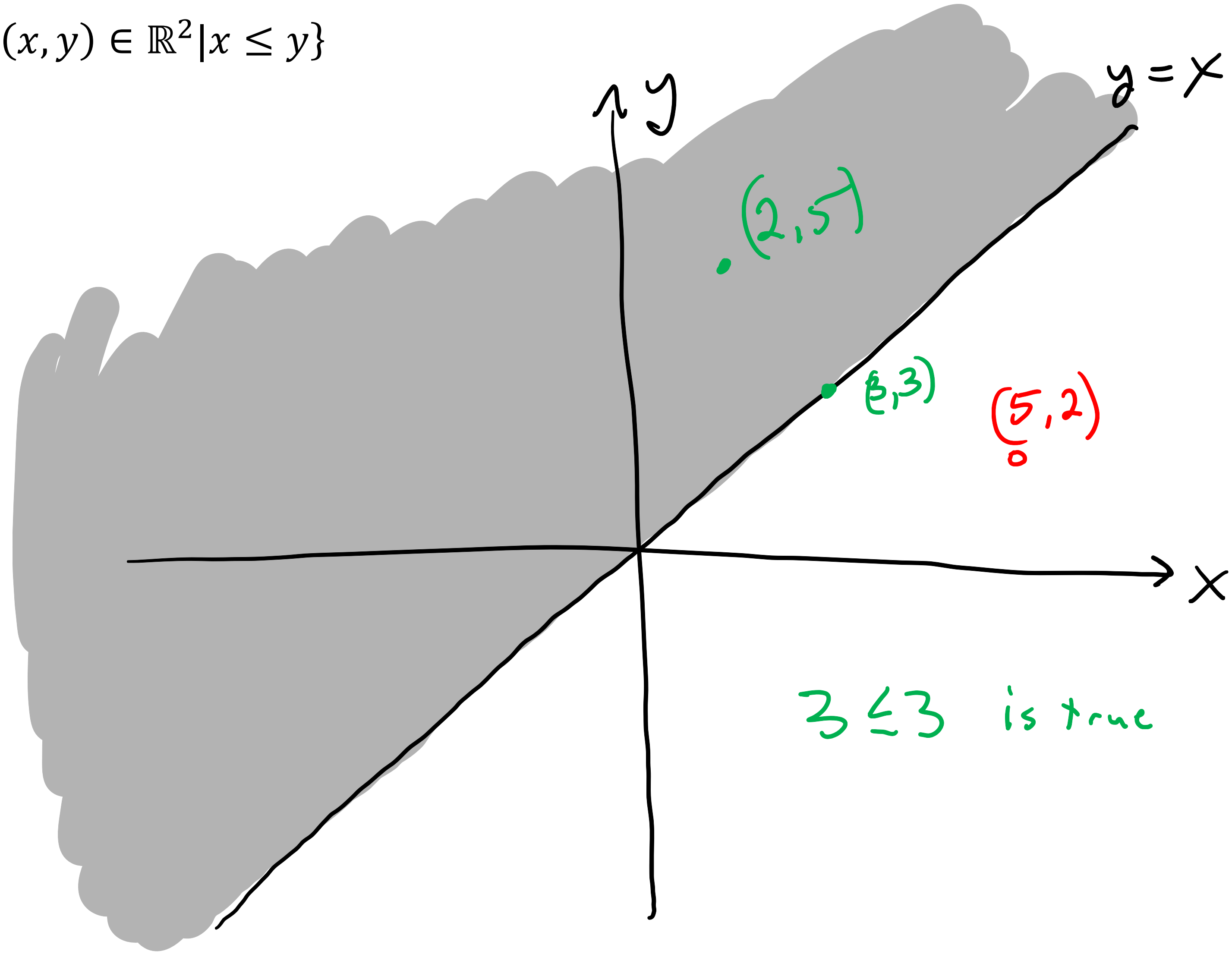
Here is the graph of the less than relation.



End of [Example 3]

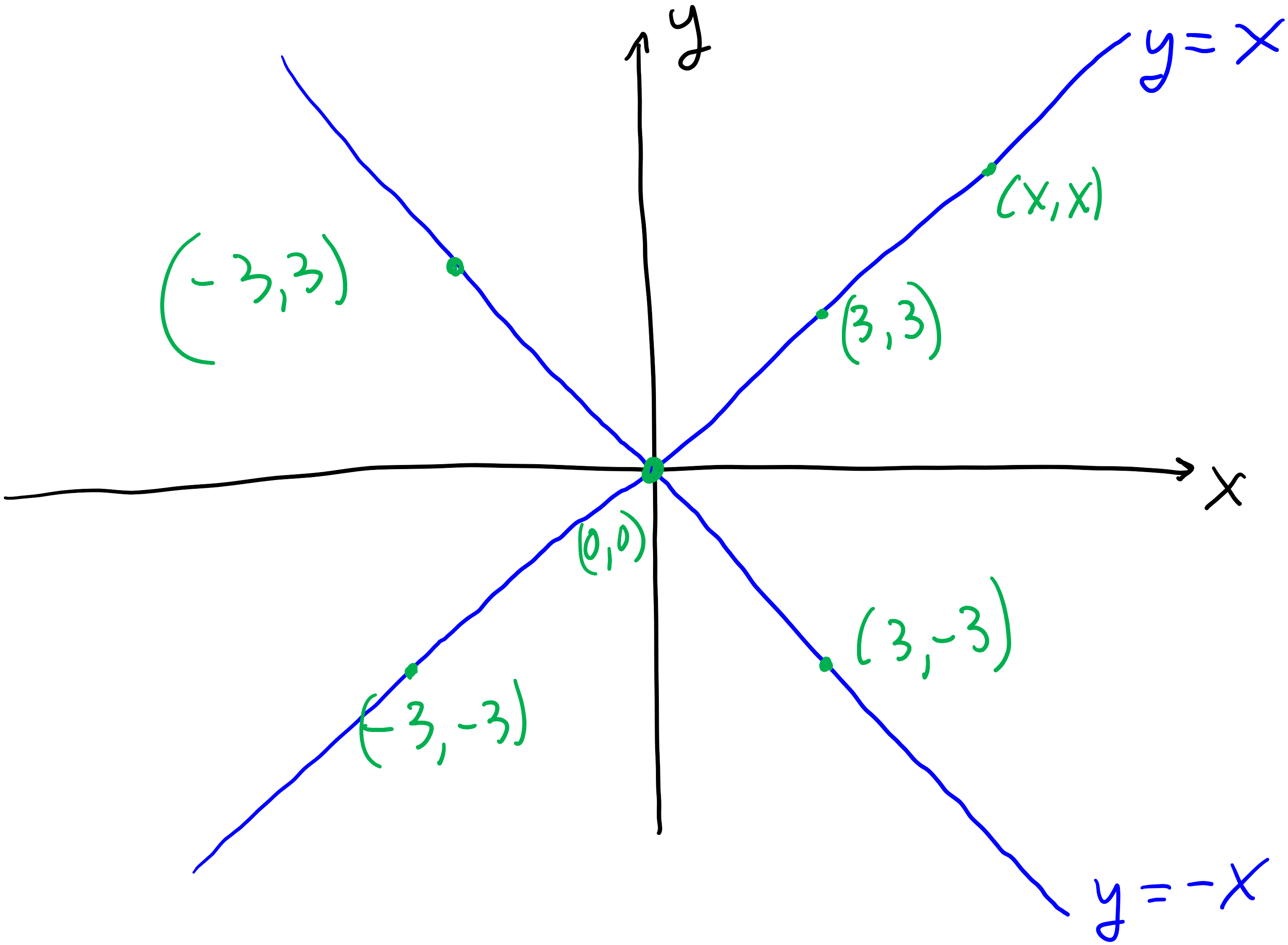
[Example 4] The Less Than Or Equal To Relation

Define $R = \{(x, y) \in \mathbb{R}^2 | x \leq y\}$



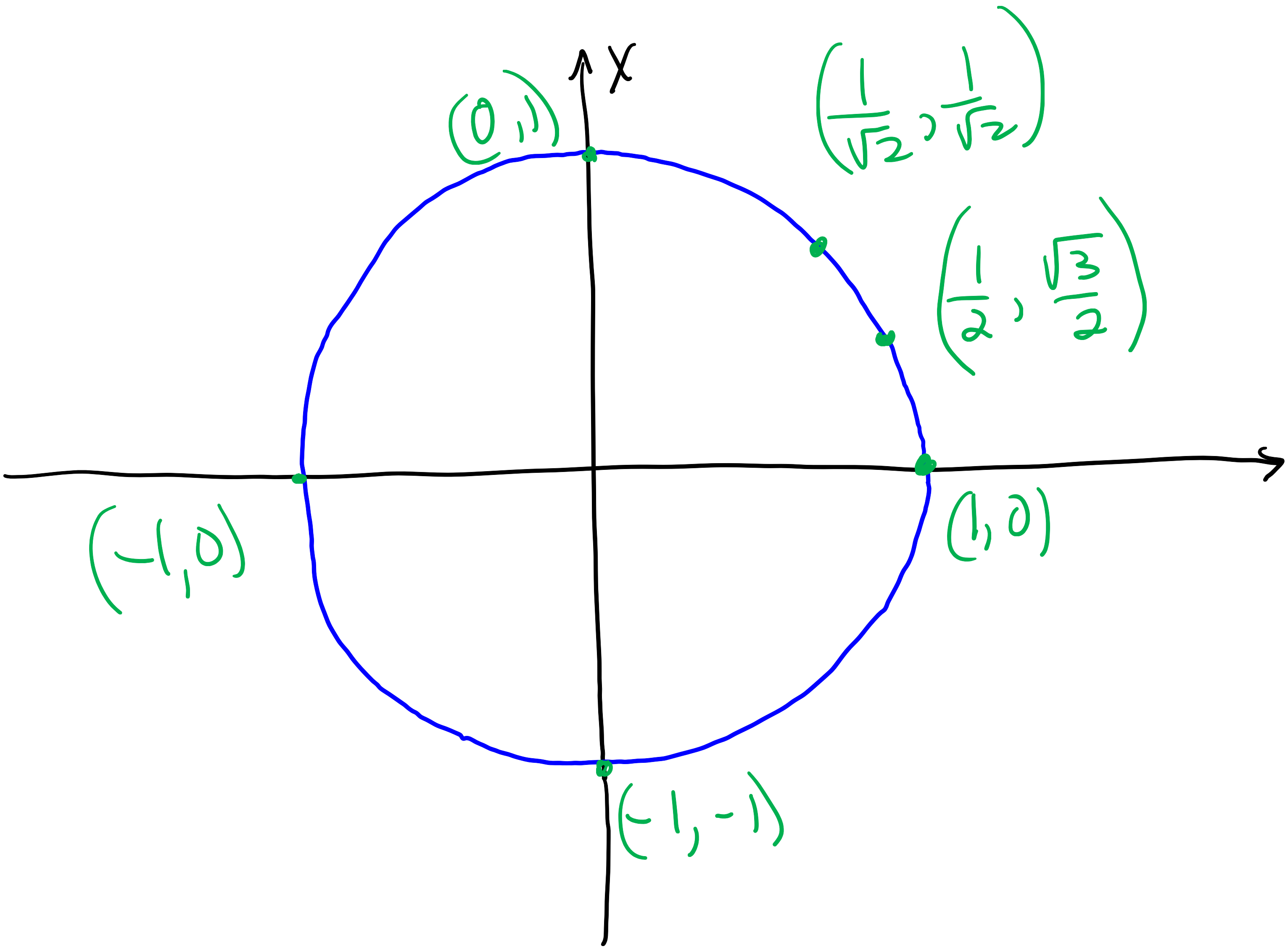
End of [Example 4]

[Example 5] Define $R = \{(x, y) \in \mathbb{R}^2 | x^2 = y^2\}$



End of [Example 5]

[Example 6] Define $R = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$



End of [Example 6]

Three Properties That Relations on a Set May or May Not Have

Definition of *Refexive Relation*

Words: R is a **reflexive relation** on S.

Meaning: Every element in S is related to itself

Meaning written formally: $\forall a \in S(a \sim a)$

Definition of *Symmetric Relationt*

Words: R is a **symmetric relation** on S.

Meaning: For every a, b in S, if a is related to b, then b is related to a.

Meaning written formally: $\forall a, b \in S(\text{If } a \sim b \text{ then } b \sim a)$

Definition of *Transitive Relation*

Words: R is a **transitive relation** on S.

Meaning: For every a, b, c in S, if a is related to b and b is related to c, then a is related to c.

Meaning written formally: $\forall a, b, c \in S\big(\text{If } \big((a \sim b) \text{ and } (b \sim c)\big) \text{ then } a \sim c\big)$

We are often interested in determining whether or not a particular relation has any of the three properties above.

Each of the three properties of relations is a logical statement. Each may be true or false. If the statement of one of the properties is *true* for a certain relation, then we say that the given relation *has that property*. If the statement of one of the properties is *false* for a certain relation, then we say that the given relation *does not have that property*. If the statement of one of the properties is *false*, then the *negation* of that statement will be *true*. Therefore, it is important to understand how to find the negations of each of the three statements above.

Finding the negation of quantified statements may have been discussed in your MATH 3050 or CS 3000 course. But we will need to review the concept here, just in case it was not.

Negation of the statement of Reflexivity

Words: R is a **reflexive relation** on S.

Meaning written formally: $\forall a \in S(a \sim a)$

To find the negation of this statement, we proceed in stages:

$\text{NOT}(\forall a \in S\ (a \sim a))$ ← easiest way to build the negation

simplifying by moving the negation to the right

$\exists a \in S\ (\text{NOT}(a \sim a))$

existential quantifier

$\exists a \in S\ (a \nsim a)$ ← clearer statement of the negation

There exists an element a in S such that a is not related to itself.

what it means for a relation to not be Reflexive

Negation of the statement of Symmetry

Words: R is a **symmetric relation** on S.

Meaning written formally: $\forall a, b \in S$(If $a \sim b$ then $b \sim a$)

When finding the negation , we will need this fact about the negation of conditional statements:

The negation of a *conditional* statement is an *and* statement:

$$NOT(\text{If } P \text{ then } Q) \equiv P \text{ and } NOT(Q)$$

$NOT(\forall a,b \in S \,(\text{If } a \sim b \text{ then } b \sim a))$

$\exists a,b \in S\,(NOT(\text{If } a \sim b \text{ then } b \sim a))$

$\exists a,b \in S\,(a \sim b \text{ and } NOT(b \sim a))$

$\exists a,b \in S\,(a \sim b \text{ and } b \not\sim a)$

What it means for a relation to <u>not</u> be symmetric

There exist elements a, b in S such that a is related to b and b is <u>not</u> related to a.

Negation of the statement of Transitivity

Words: R is a **transitive relation** on S.

Meaning written formally: $\forall a, b, c \in S\big(\text{If } \big((a \sim b) \text{ and } (b \sim c)\big) \text{ then } a \sim c\big)$

$\text{NOT}\big(\forall a,b,c \in S\,(\text{If } ((a\sim b) \text{ and } (b\sim c)) \text{ then } (a\sim c))\big)$

$\exists a,b,c \in S\,\big(\text{NOT}(\text{If } ((a\sim b) \text{ and } (b\sim c)) \text{ then } (a\sim c))\big)$

$\exists a,b,c \in S\,\big(((a\sim b) \text{ and } (b\sim c)) \text{ and } \text{NOT}(a\sim c)\big)$

$\exists a,b,c \in S\,((a\sim b) \text{ and } (b\sim c) \text{ and } (a\not\sim c))$

There exist elements $a,b,c \in S$ such that (a is related to b) and (b is related to c) and (a is not related to c)

What it means for a relation to not be transitive.

[Example 3, Revisited] Recall the *Less Than Relation*.

$$R = \{(x, y) \in \mathbb{R}^2 | x < y\}$$

Which properties does the *Less Than Relation* have? Justify your answers.

The relation is not reflexive. For example, 3 is not related to itself.

$3 < 3$ is false

The relation is not symmetric

For example $2 < 5$ and $5 \not< 2$

The relation is transitive

$\forall a, b, c \in \mathbb{R} \left(\text{If } (a < b \text{ and } b < c) \text{ then } (a < c) \right)$

is just the transitive property of real number inequality.

[Example 4, Revisited] Recall the *Less Than or Equal To Relation*.

$$R = \{(x, y) \in \mathbb{R}^2 | x \leq y\}$$

Which properties does the *Less Than OR Equal To Relation* have? Justify your answers.

This relation is reflexive

Proof: Let x be any real number (generic element of $\mathbb{R}$)

Then $x \leq x$ is true. So x is related to itself.

This relation is not symmetric

For example, let $x=2, y=5$

Observe that $2 \leq 5$ is true while $5 \leq 2$ is false.

The relation is transitive

$\forall a, b, c \in \mathbb{R}$ (If ($a \leq b$ and $b \leq c$) then ($a \leq c$))

is just the transitive property of real number inequality.

[Example 5, Revisited] Recall the relation from **[Example 5]**.

$$R = \{(x, y) \in \mathbb{R}^2 | x^2 = y^2\}$$

Which properties does the relation have? Justify your answers.

The Relation is reflexive.

Proof

(1) Let x be any real number (generic particular element of $\mathbb{R}$)

(2) Then $x^2 = x^2$ is true. (reflexive property of real number equality)

(3) So, x is related to itself.

Relation Is Symmetric

Proof

(1) Suppose x, y are real numbers such that $x \sim y$

(2) Then $x^2 = y^2$ (by (1) and definition of the relation

(3) Then $y^2 = x^2$ (by symmetric property of real number equality)

(4) Therefore $y \sim x$ (by (**) and definition of the relation

End of proof.

Relation Is Transitive

(Direct proof structure)

Proof

(1) Suppose $x, y, z \in \mathbb{R}$ such that $(x \sim y)$ and $(y \sim z)$ generic particular elements

(2) Then $x^2 = y^2$ (by (1) and definition of relation)

(3) and $y^2 = z^2$ (by (1) and definition of relation)

(4) Then $x^2 = z^2$ (by (2), (3), and transitive property of real number equality)

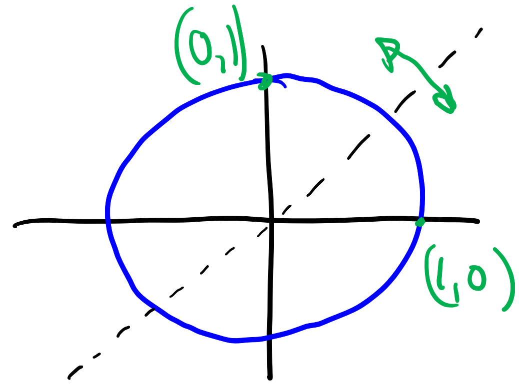
(5) Therefore $x \sim z$ (by (**) and definition of the Relation)

End of proof

[Example 6, Revisited] Recall the relation from **[Example 6]**.

$$R = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$$

Which properties does the relation have? Justify your answers.



The relation is <u>not</u> reflexive.

For example. Let $x = 1$

Then $x^2 + x^2 = 1^2 + 1^2 = 2 \neq 1$

So x is not related to itself.

The relation is <u>symmetric</u>

<u>Proof</u> (Direct Proof)

(1) Suppose <u>x, y are real numbers</u> such that $x \sim y$

(2) So $x^2 + y^2 = 1$ (by (1) and definition of relation)

(3) Then $y^2 + x^2 = 1$ (by (2) and commutative property of addition)

(4) Therefore $y \sim x$ (by (3) and definition of the relation)

End of proof

The relation is <u>not</u> transitive

For example: Let $x = 1$, $y = 0$, $z = 1$

Then observe $x^2 + y^2 = 1$ and $y^2 + z^2 = 1$, but $x^2 + z^2 \neq 1$.

[Example 7] Define a relation on the set $\mathbb{Z}$ of integers as follows.

$$R = \{(m, n) \in \mathbb{Z}^2 | m - n \text{ is a multiple of } 5\}$$

Which properties does the relation have? Justify your answers.

This relation is reflexive

Proof (1) let $m \in \mathbb{Z}$ be any integer

(2) observe that $m - m = 0 = 5 \cdot 0$, and that 0 is an integer

Let $k = 0$. Then k is an integer

(3) Observe that $m - m = 5 \cdot k$ where k is an integer (by (2))

(4) $m - m$ is a multiple of 5 (by (3) and definition of multiple)

(5) Therefore m is related to itself (by (4) and definition of the relation)

End of proof

Proof that the relation is symmetric. (Direct Proof)

(1) Suppose $a, b \in \mathbb{Z}$ and that $a \sim b$ (generic elements)

(2) $a - b$ is a multiple of 5 (by (1) and definition of the relation)

(3) There is some integer k such that $a - b = 5k$ (by (2) and definition of multiple)

(4) Then $b - a = -(a-b) = -5k$ (by (3))

$$b - a = 5(-k)$$

(5) Let $j = -k$. Observe that j is an integer and $b - a = 5j$

(6) There exists an integer j such that $b - a = 5j$ (by (5))

(7) Then $b - a$ is a multiple of 5 (by (6) and definition of multiple)

(8) Then $b \sim a$ (by (7) and definition of the relation)

End of Proof.

The Relation is also transitive   Direct Proof

Proof

(1) Let $a, b, c \in \mathbb{Z}$ such that $a \sim b$ and $b \sim c$

(2) $a-b$ is a multiple of 5   (by (1) and definition of relation)

(3) There exists integer k such that $a-b = 5k$ (by 2) and definition of multiple)

(4) $b-c$ is a multiple of 5 (by (1) and definition of relation)

(5) There exists an integer j such that $b-c = 5j$ (by (4) and definition of relation

(6) Then $a-c = a-b+b-c$ (trick)

$= 5k + 5j$ (by (3),(4))

$= 5(k+j)$

(7) Let $h = k+j$ observe that h is an integer and $a-c = 5h$

(8) There exists an integer h such that $a-c = 5h$ (by (7))

(9) $a-c$ is a multiple of 5 (by (8) and definition of multiple)

(10) Therefore $a \sim c$ (by (9) and definition of relation)

End of Proof

Equivalence Relations

Definition of *Equivalence Relation*

Words: R is an **equivalence relation** on S.

Meaning: R is reflexive and symmetric and transitive.

[Example 8] Which of the **[Examples 4,5,6,7]** are Equivalence Relations? Justify your answers.

Relation	Reflexive?	Symmetric?	Transitive?	Equivalence
3	no	no	yes	no
4	yes	no	yes	no
5	yes	yes	yes	yes
6	no	yes	no	no
7	yes	yes	yes	yes

Equivalence Classes

Definition of Equivalence Class

Symbol: $[s]$

Usage: There is an equivalence relation R on a set S in the discussion, and $s \in S$.

Spoken: *The equivalence class of* s.

Meaning: the subset of S consisting of all the elements of S that are related to s.

Meaning written formally: $[s] = \{x \in S | x \sim s\}$

Additional Terminology: The element $s \in S$ is called a representative of the equivalence class $[s]$.

[Example 5, Revisited] Recall that the relation from **[Example 5]** is an equivalence relation.

$$R = \{(x, y) \in \mathbb{R}^2 | x^2 = y^2\}$$

(a) What is [2]? $\{-2, 2\}$

(b) What is [5]? $\{-5, 5\}$

(c) What is [−2]? $\{-2, 2\}$

(d) What would be a general description of all equivalence classes?

Sets of the form $\{x, -x\}$ where $x \in \mathbb{R}$

(e) What is [0]? Does this agree with the answer from **(d)**?

$\{0\}$

yes!

$\cancel{\{0, -0\}} = \{0, 0\} = \{0\}$

[Example 7, Revisited] Recall that the relation from **[Example 7]** is an equivalence relation.

$$R = \{(m,n) \in \mathbb{Z}^2 \mid m - n \text{ is a multiple of } 5\}$$

(a) What is [2]? $\{\ldots, -8, -3, 2, 7, 12, \ldots\}$

(b) What is [4]? $\{\ldots, -6, -1, 4, 9, 14, \ldots\}$

(c) What is [−3]? $\{\ldots, -8, -3, 2, 7, 12, \ldots\}$

(d) What would be a general description of all equivalence classes?

Sets of the form

$[0] = \{0 + 5k \mid k \in \mathbb{Z}\}$

$[1] = \{1 + 5k \mid k \in \mathbb{Z}\}$

$[2] = \{2 + 5k \mid k \in \mathbb{Z}\}$

$[3] = \{3 + 5k \mid k \in \mathbb{Z}\}$

$[4] = \{4 + 5k \mid k \in \mathbb{Z}\}$

Theorem about Equivalence Classes

Our discussion of **[Examples 5,7]** should make the following theorem believable:

> **Theorem 1.2.7 about Equivalence Classes**
>
> If $\sim$ is an equivalence relation on a set S and $s, t \in S$, then either $[s] \cap [t] = \phi$ or $[s] = [t]$.

This tells us that either two equivalence classes are actually the exact same set, or they are completely disjoint. There is no partial overlap of equivalence classes.

It also tells us that it doesn't matter which element of an equivalence class we use as a representative of an equivalence class when we write the symbol for the equivalence class. For example, in **[Example 7]**, there are lots of ways to denote the equivalence class of 2.

$$[2] = \{\dots, -8, -3, 2, 7, 12, \dots\} = [-3] = [12]$$

Put another way, there can be many *representatives* of a particular equivalence class.

End of Video