1.3a: Introduction to Functions

produced by Mark Barsamian, 2021.01.28

for Ohio University MATH 3110/5110 College Geometry

Subject: Functions

- Relations from One Set to Another Set
  - Definition
  - Illustrating a Relation from One Set to Another Set
  - The Inverse Relation
- Functions
  - Definition of Function, Domain, Range, Image of an Element.
  - Image of a Set, Image of a Function
  - Preimage
- Properties of Functions: surjective, injective, bijective

Textbook: Millman & Parker, *Geometry: A Metric Approach with Models, Second Edition*
(Springer, 1991, ISBN 3-540-97412-1)

Reading: Section 1.3 Functions, pages 9 – 12

Homework: Section 1.3 # 1, 2ab, 3, 4, 5, 12

Relations from One Set to Another Set

(This concept is not discussed in Millman & Parker Section 1.3, but it is a useful concept that will help clarify the concepts that *are* discussed there.)

Recall the definition of *Binary Relation on a Set* from Section 1.2:

Definition of *Binary Relation on a Set*

Words: R is a **binary relation** on S

Meaning: S is a set, and $R \subset S \times S$. That is, R is a set containing ordered pairs from $S \times S$.

This idea can be generalized as follows.

Definition of *Relation from One Set to Another Set*

(s, t)

Words: R is a **relation from *S* to *T***

Meaning: S, T are sets, and $R \subset S \times T$. That is, R is a set containing ordered pairs from $S \times T$.

Additional Terminology: Set S is called the **domain** of the relation R; Set T is called the **range**.

Realize that the earlier concept of a *Binary Relation on a Set* is just a special case of a *Relation from One Set to Another Set*. We can use the general term *Relation* to mean *Relation from One Set to Another Set*, with the understanding that the term includes *Relations on a Set*.

[Example 1]

Let $X = \{A, B, D, E\}$ denoting people *Ann*, *Bob*, *Dave*, *Edith*.
And let $Y = \{c, p, s, i\}$, denoting *cake*, *pie*, *sorbet*, *ice cream*.

Define relation R from X to Y by saying that $x \sim y$ means that person x ate dessert item y at the picnic, with set R defined explicitly by

$$R = \{(A, i), (B, c), (B, i), (D, i), (D, p)\}$$

Note two features of this relation:

- There is more than one pair of the form (B, y), indicating that Bob had more than one item. (Dave also had more than one item.)
- There is no pair of the form (E, y), indicating that Edith did not have any dessert.
- There is no pair of the form (x, s), indicating that no one had sorbet.

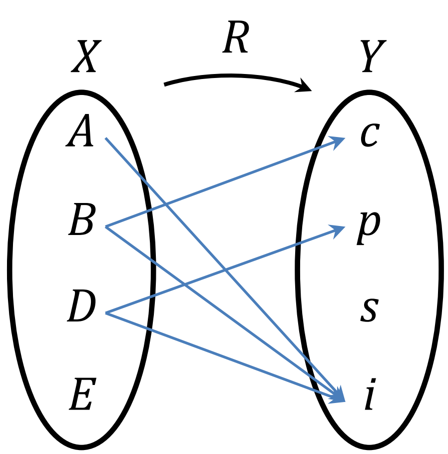
End of [Example 1]

Illustrating a Relation from One Set to Another Set

Relations can take many forms, and so there is a variety of ways to illustrate them. It is good to practice illustrating relations in different ways, because not all types of illustrations are possible for a given relation and because one style of illustration may be more useful than another in certain situations.

When the domain and range are both finite sets, two common types of illustrations are the *arrow diagram* and the *table*. The convention for arrow diagrams is to put the domain set on the left. The convention for tables is to put the elements of the domain set as the headings for the rows.

[Example 1, Revisited] Illustrating the Relation



R		Y			
		c	p	s	i
X	A				•
	B	•			•
	D		•		•
	E				

End of [Example 1, Revisited]

The Inverse Relation

Every relation has an inverse relation.

Definition of Inverse Relation

Symbol: R^{-1}

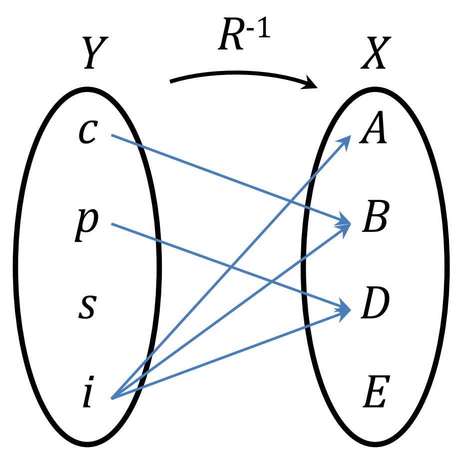
Spoken: R *inverse,* or *the inverse relation for* R.

Usage: There is a relation R from a set S to a set T in the discussion.

Meaning: R^{-1} is the relation from set T to set S (that is, R^{-1} is a subset of $T \times S$) defined by $R^{-1} = \{(t, s) | (s, t) \in R\}$. That is, the ordered pairs for the relation R^{-1} are obtained by reversing the order of the elements in the ordered pairs for R.

[Example 1, Revisited] For the relation R from **[Example 1]**, the inverse relation R^{-1} is

$$R^{-1} = \{(i, A), (c, B), (i, B), (i, D), (p, D)\}$$



R^{-1}		X: A	B	D	E
Y	c		•		
	p			•	
	s				
	i	•	•	•	

End of [Example 1, Revisited]

Functions

Definition of *Function*

Symbol: $f: X \to Y$

Words: $\boldsymbol{f}$ is a function from $\boldsymbol{X}$ to $\boldsymbol{Y}$

Meaning: X, Y are sets and f is a relation from X to Y (that is, $f \subset X \times Y$) that has this extra property: For each $x \in X$, there is exactly one $y \in Y$ such that $(x, y) \in f$. This unique element $y \in Y$ is denoted by the symbol $f(x)$. That is, $y = f(x)$, so $\big(x, f(x)\big) \in f$

Meaning Expressed Formally: $\forall x \in X \left(\exists!\, y \in Y\big(y = f(x)\big)\right)$

Additional Terminology:

- Set X is called the **domain** of the function f; set Y is called the **range** of f. In our book, the range is sometimes denoted as $\text{Range}(f)$.
- The element $y = f(x) \in T$ is called
    - ***f of x.***
    - **the value of $\boldsymbol{f}$ for the input $\boldsymbol{x}$**
    - **the output of $\boldsymbol{f}$ at $\boldsymbol{x}$**
    - **the image of $\boldsymbol{x}$**, or **the image of $\boldsymbol{x}$ under the function $\boldsymbol{f}$**

[Example 1, Revisited]

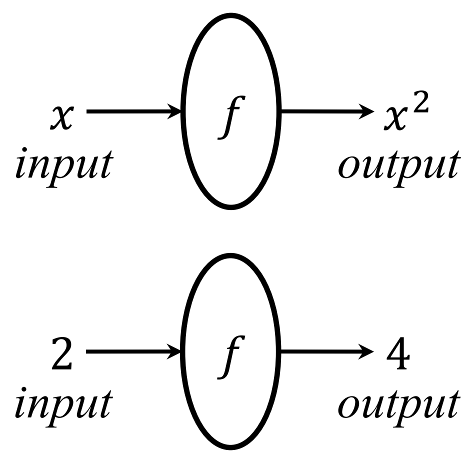
Observe that the relation R from **[Example 1]** is *not* qualified to be called a *function* because there is no output for the input E. (Edith did not have any dessert.)

Also notice that the inverse relation R^{-1} from **[Example 1]** is not qualified to be called a function either, because there is no output for the input s. (Nobody ate sorbet for dessert.)

End of [Example 1, Revisited]

[Example 2] Consider the function $f: \mathbb{R} \to \mathbb{R}$ defined by the formula $f(x) = x^2$

For this function, *the image of 2* is the number 4. That is, $f(2) = 4$. This can be illustrated by a machine diagram.



End of [Example 2]

[Example 3] Consider the equation $y = \dfrac{x-2}{x-3}$

For most real numbers x, the equation can be used to obtain a value of y. But for $x = 3$, there is no corresponding value of y.

Now, consider this proposed definition of a function: not a valid definition

Define a function $f: \mathbb{R} \to \mathbb{R}$ by the formula $f(x) = \dfrac{x-2}{x-3}$

This would not qualify as a function, because $x = 3$ is in the domain $\mathbb{R}$ and yet $f(3)$ does not exist. Mathematicians would say that the proposed function f is ***not well defined***. This means that f fails some requirement that all functions must satisfy.

Finally, consider this alternate proposed definition of a function: Valid definition

Define a function $f: \mathbb{R} - \{3\} \to \mathbb{R}$ by the formula $f(x) = \dfrac{x-2}{x-3}$

This *does* qualify as a function because it satisfies all of the requirements that a function must satisfy. Mathematicians would say that this function ***is well-defined***.

End of [Example 3]

[Example 4] A Particularly Simple Function: The *Identity Function*

The Identity Function on a Set

Symbol: id_S

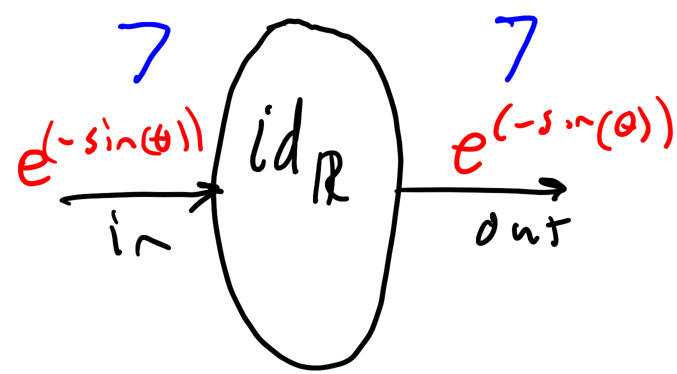
Spoken: the *identity function* on S

Usage: S is a set

Meaning : the function $id_S\colon S \to S$ defined by $id_S(x) = x$ for every $x \in S$

$(\boldsymbol{a})$ $id_{\mathbb{R}}(7) =$ 7

$(\boldsymbol{b})$ $id_{\mathbb{R}}\left(e^{(-\sin(\theta))}\right) =$ $e^{(-\sin(\theta))}$



[End of Example 4]

Graphs of *functions*.

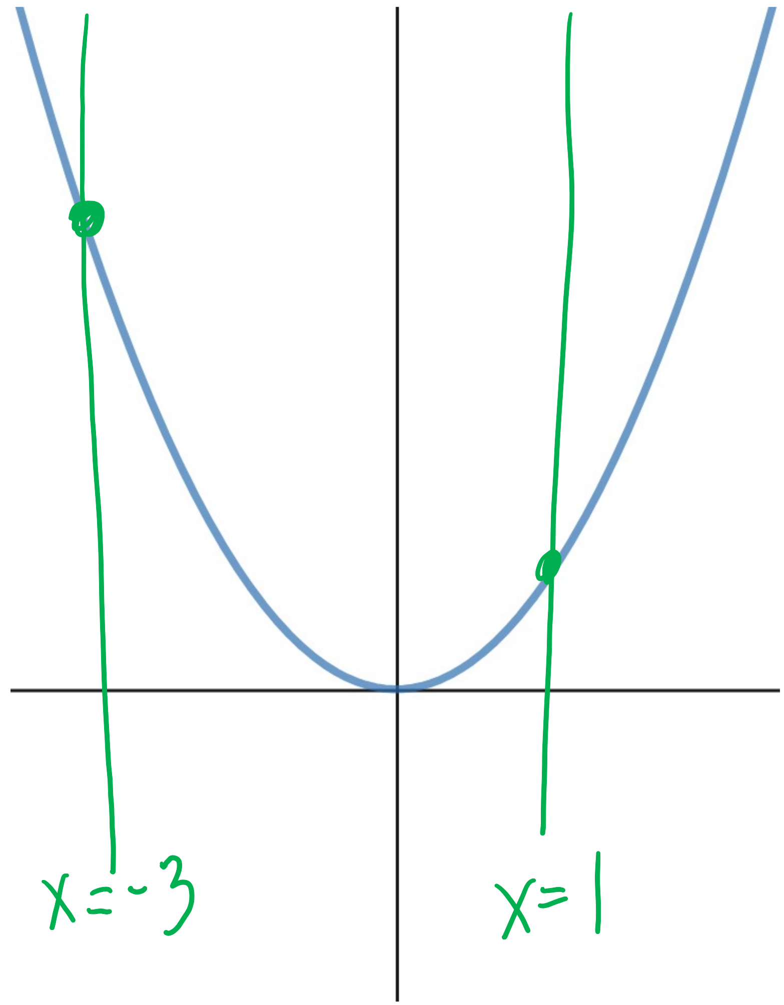
A function whose domain and codomain are subsets of the real numbers $\mathbb{R}$ can be illustrated using a graph. But there are graphs that do not correspond to functions. The *vertical line test* articulates which graphs qualify to be graphs of functions

The *Vertical Line Test for a Graph to be the Graph of a Function*

- If, for every $a \in A$, the vertical line $x = a$ intersects the graph exactly once, then the graph is the graph of a function with domain A. (The graph *passes* the *vertical line test*.)
- If there exists an $a \in A$ such that the vertical line $x = a$ does not intersect the the graph, or intersects the graph more than once, then the graph is not the graph of a function with domain A. (The graph *fails* the *vertical line test*.)

[Example 2, Revisited] Recall the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$.

The graph of this function is a parabola, facing up. Every vertical line touches this graph exactly once. That is consistent with the fact that the domain of the function is the set of all real numbers.

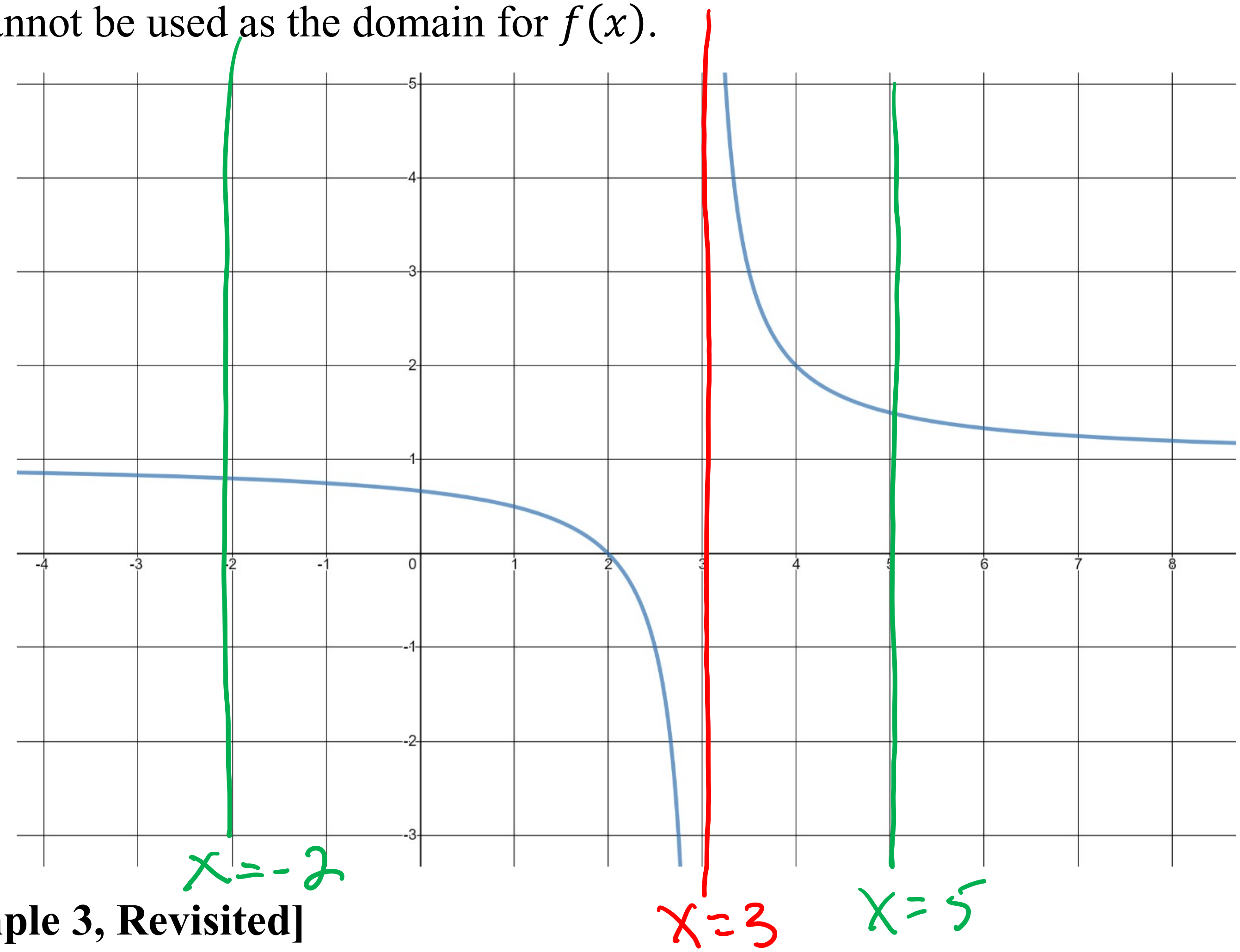


End of [Example 2, Revisited]

[Example 3, revisited] Recall the function $f: \mathbb{R} - \{3\} \to \mathbb{R}$ defined by $f(x) = \dfrac{x-2}{x-3}$

Observe that every vertical line $x = a$ with $a \neq 3$ touches the graph exactly once. That is consistent with the fact that the domain of the function is the set : $\mathbb{R} - \{3\}$.

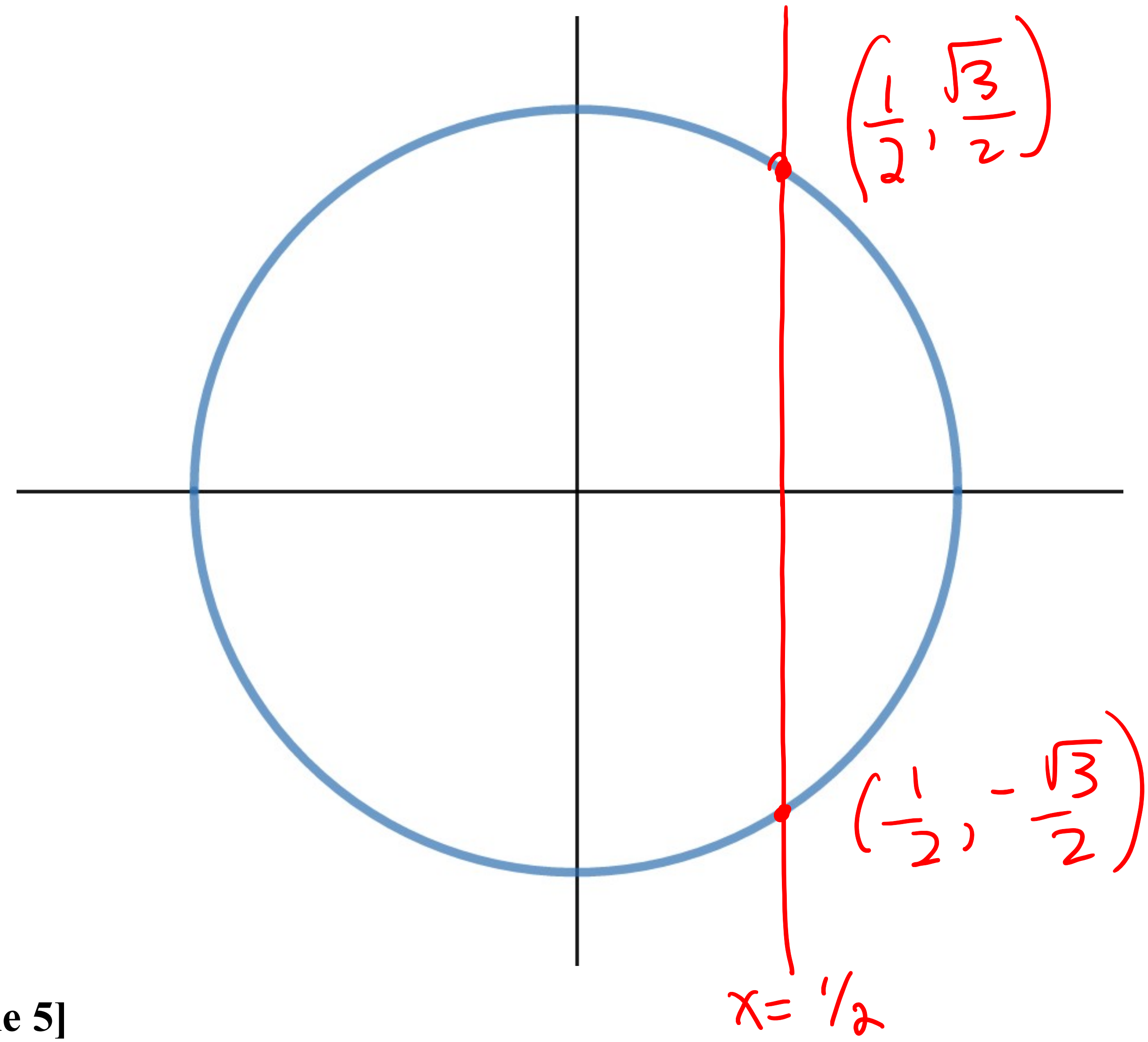
Also observe that the vertical line $x = 3$ does not touch the graph at all. That illustrates why the the whole set $\mathbb{R}$ cannot be used as the domain for $f(x)$.



End of [Example 3, Revisited]

[Example 5] Consider the equation $x^2 + y^2 = 1$.

The graph of this function is circle. Some vertical lines touch this graph more than once. This indicates that the equation cannot be used to define y as a function of x.



End of [Example 5]

The Image of a Set and the Image of a Function

In our definition of function, we considered the output of a function corresponding to a *single* input. That output was called the *image of the input*. It is often helpful to consider the *set of outputs* that correspond to a *set of inputs*. That is the notion of the *image of a set*.

Definition of the Image of a Set, Image of a Function

Symbol: $f(A)$

Spoken: ***f of A***, or ***the image of A***, or ***the image of A under the function f***,

Usage: There is a function $f: X \to Y$ from a set X to a set Y in the discussion and $A \subset X$.

Meaning: $f(A)$ is the set of all outputs that result when elements of set A are used as inputs. That is, $f(A) = \{f(a) | a \in A\}$

Remark: Observe that $f(A) \subset Y$.

Special Case and Additional Terminology: Observe that the entire domain X can be considered as a subset $X \subset X$. So the notion of the *image of a set* A includes the special case of the *image of the whole domain* X. In that special case, there is additional terminology.

Words: the image of f, or the **Symbol:** $\text{Im}(f)$

Meaning: The set set $f(X)$. That is, the set of all outputs that result when all of the elements of the domain X are used as inputs. That is, $\text{Im}(f) = f(X) = \{f(x) | x \in X\}$

[Example 2, Revisited] Recall the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$.

For this function,

$$f(\{-2,-1,0,1,2,3\}) = \{0,1,4,9\}$$

$$f\big((-2,3)\big) = [0,9)$$

interval

interval

$$f\big((-\infty,\infty)\big) = [0,\infty)$$

interval interval

$$f(\mathbb{R}) = \mathbb{R}^{nonneg}$$

$$\text{Im}(f) = \mathbb{R}^{nonneg}$$

Observe that $\text{Range}(f) = \mathbb{R}$, so $\text{Im}(f) \subsetneq \text{Range}(f)$

End of [Example 2, Revisited]

Remarks about the Image and Range of a Function

Remark 1: Why is $\text{Range}(\boldsymbol{f})$ useful? Isn't $\text{Im}(\boldsymbol{f})$ the more useful concept?

In **[Example 2]**, the function $f: \mathbb{R} \to \mathbb{R}$ was defined by the formula $f(x) = x^2$. For this function, we observed that $\text{Im}(f) = \mathbb{R}^{nonneg}$, while $\text{Range}(f) = \mathbb{R}$, so $\text{Im}(f) \subsetneq \text{Range}(f)$. That is, the set of real number y values that are actually possible as outputs of f is not the whole set $\mathbb{R}$.

It seems like we would always be most interested in the actual y values that will be the outputs of a function, that is, the *image* of the function. Why would we want to have a *range* that is larger than the *image*?

Put another way, we could have defined the squaring function differently: We could have defined $f: \mathbb{R} \to \mathbb{R}^{nonneg}$ by the formula $f(x) = x^2$. For this function, $\text{Im}(f) = \mathbb{R}^{nonneg} = \text{Range}(f)$.

Why wouldn't we always want this?

The answer to this is a little subtle.

Sometimes, one does want to know *exactly* what outputs will come out of a function.

But sometimes, one only needs to know *what kinds* of outputs will come out.

Sometimes both questions are easy to answer. But sometimes the first question is harder to answer than the second.

[Example 3, revisited] Recall the function $f: \mathbb{R} - \{3\} \rightarrow \mathbb{R}$ defined by $f(x) = \dfrac{x-2}{x-3}$

It is easy to see that the output of this function will be real numbers. But it is harder to see exactly what real numbers can actually be achieved as outputs. To find out, we can start with the equation

$$y = \frac{x-2}{x-3}$$

and solve the equation for x in terms of y. The result is the new equation

$$x = \frac{3y-2}{y-1}$$

This equation has no solutions when $y = 1$. That is, there are no x values that correspond to $y = 1$. This means that for the original function,

$$y = f(x) = \frac{x-2}{x-3}$$

there are no x values that will give a y value of 1. (This sort of makes sense now: Notice that the numerator and denominator of that fraction are not the same, so their ratio can never equal 1.) We have determined that the image of f is $\mathrm{Im}(f) = \mathbb{R} - \{1\}$. Observe that it was more difficult to determine the image than it was to determine the range.

End of [Example 3, revisited]

Remark 2 about the Image and Range of a Function: Different Use of the Terminology

Different books use the terms *image* and *range* to mean different things. I will compare our geometry book to Susanna Epp's book *Discrete Mathematics*.

	The Set of Inputs	TheSset of Outputs that Result from those Inputs	The Set where the Outputs Live
Our Geometry Book	Domain	Image of f	Range of f
Epp's Discrete Math Book	Domain	Range of f	Codomain

Preimage

Having learned about the following three concepts

- image of an element
- image of a set
- the inverse relation

we are equipped to understand the concept of *preimage*.

Definition of Preimage

Symbols: $f^{-1}(y)$

Words: the preimage of *y*

Usage: f is a function, $f: X \to Y$

Meaning: $f^{-1}(y) = \{x \in X | f(x) = y\}$

Variations:

- If $f^{-1}(y)$ is a single element set, then $f^{-1}(y)$ is often written without brackets.
- The preimage of a set $B \subset Y$ is defined in the obvious way: $f^{-1}(B) = \{x \in X | f(x) \in B\}$

[Example 2, Revisited] For the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$, some preimages are

- The preimage of the single element $y = 9$ is a set: $f^{-1}(9) = \{-3,3\}$
- The preimage of the single element $y = 0$ is a set containing one element: $f^{-1}(0) = \{0\}$. Because the preimage is a set containing only one element, it is common to omit the curly bracket in the result and simply write $f^{-1}(0) = 0$.
- The preimage of the single element $y = -5$ is the empty set: $f^{-1}(-5) = \phi$
- The preimage of an interval can be an interval: $f^{-1}([-5,4)) = (-2,2)$
- The preimage of an interval can be a single number: $f^{-1}((-5,0]) = 0$
- The preimage of an interval can be the empty set: $f^{-1}\big((-5,0)\big) = \phi$

↑ interval

End of [Example 2, Revisited]

Observe that although $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is a *function*, the symbol f^{-1} *does not* represent a function.

- For some single inputs, like $y = 9$, the output is a *set* containing two elements, not a *number*.
- For some single inputs, like $y = -5$, the output is the *empty set*, not a *number*.

But the symbol f^{-1} does represent a *relation from* $\mathbb{R}$ *to* $\mathbb{R}$.

In general, if $f: X \to Y$ is a function, then the symbol f^{-1} might not represent a *function*, but it will always represent a *relation from* Y *to* X.

For certain functions, however, the symbol f^{-1} *does* represent a *function*. In the next video, we will discuss the criteria that will indicate when that is true.

Properties that Functions May or May Not Have: Surjective, Injective, Bijective

Surjective Functions

Definition of Surjective Function

Words: f *is* ***surjective***, or f *is a* ***surjection***

Alternate Words: f *is* ***onto***

Usage: f is a function, $f: X \to Y$

Meaning: For every element in the range, there exists an element of the domain (***at least one***) that can be used as input to cause that element of the range to be output.

Meaning Written Formally: $\forall y \in Y\big(\exists x \in X(f(x) = y)\big)$

Other Wording: For every element in the range, the number n of elements of the domain that can be used as input to cause that element of the range to be output is $n \geq 1$.

Remark: The formal expression might be clearer if the words ***at least one*** were included.

$$\forall y \in Y\big(\exists \text{ at least one } x \in X(f(x) = y)\big)$$

But this is redundant. The symbol $\exists \boldsymbol{x} \in \boldsymbol{X}$, which is spoken ***there exits an x in X* such that**, is understood to mean ***there exits at least one x in X* such that**.

Injective Functions

original If P then Q

contrapositive If NOT(Q) then NOT(P).

Definition of Injective Function

Words: f *is* ***injective***, or f *is an* ***injection***, or f is ***one-to-one***

Usage: f is a function, $f: X \to Y$

Meaning: If two inputs cause outputs that are equal, then the inputs must be equal.

Meaning Written Formally: $\forall x_1, x_2 \in X(If\ f(x_1) = f(x_2)\ then\ x_1 = x_2)$

Contrapositive Meaning: If two inputs are not equal, then the outputs will not be equal.

Contrapositive Written Formally: $\forall x_1, x_2 \in X\big(If\ x_1 \neq x_2\ \ then\ f(x_1) \neq f(x_2)\big)$

Alternate Wording: For every element in the range, there exists ***at most one*** element of the domain that can be used as input to cause that element of the range to be output.

Remark: The alternate wording does not imply that there are any elements of the domain that will work. It just says that there is ***not more than one***.

Written Formally: $\forall y \in Y\big(\exists \text{ at most one } x \in X(f(x) = y)\big)$

Other Wording: For every element in the range, the number n of elements of the domain that can be used as input to cause that element of the range to be output is $n \leq 1$.

Remark: this means that $n = 0$ or $n = 1$.

Negations of the Statements of Surjective and Injective

We are often interested in determining whether or not a particular function has either of the two properties above. Therefore, it is important to know the *negations* of the statements of the properties.

Negating the statement of Surjective

Original Statement: f *is* ***surjective***

Meaning: For every element in the range, there exists an element of the domain (***at least one***) that can be used as input to cause that element of the range to be output.

Meaning Written Formally: f is surjective $\equiv \forall y \in Y\big(\exists x \in X(f(x) = y)\big)$

Negation: f *is* ***not surjective***

Negation Written Formally:

$$
\begin{aligned}
f \text{ is not surjective} &\equiv NOT\Big(\forall y \in Y\big(\exists x \in X(f(x) = y)\big)\Big) \\
&\equiv \exists y \in Y\Big(NOT\big(\exists x \in X(f(x) = y)\big)\Big) \\
&\equiv \exists y \in Y\Big(\forall x \in X\big(NOT(f(x) = y)\big)\Big) \\
&\equiv \exists y \in Y\big(\forall x \in X(f(x) \neq y)\big)
\end{aligned}
$$

Negation in Words:

- There exists an element in the range such that ***no*** element of the domain can be used as input to cause that element of the range to be output.
- There exists an element in the range such that when any element of the domain is used as input, that element of the range is ***not*** output.

Negating the statement of Injective

There are multiple versions of the statement that a function f is injective. It is instructive to consider the negations of the different versions.

Original Statement: f *is* ***injective***

Meaning : If two inputs cause outputs that are equal, then the inputs must be equal.

Meaning Written Formally: $\forall x_1, x_2 \in X(If\ f(x_1) = f(x_2)\ then\ x_1 = x_2)$

Negation: f *is* ***not injective***

Negation Written Formally:

$$\begin{aligned} f \text{ is not injective} &\equiv NOT\big(\forall x_1, x_2 \in X(If\ f(x_1) = f(x_2)\ then\ x_1 = x_2)\big) \\ &\equiv \exists x_1, x_2 \in X\big(NOT(If\ f(x_1) = f(x_2)\ then\ x_1 = x_2)\big) \\ &\equiv \exists x_1, x_2 \in X\big(f(x_1) = f(x_2)\ and\ NOT(x_1 = x_2)\big) \\ &\equiv \exists x_1, x_2 \in X(f(x_1) = f(x_2)\ and\ x_1 \neq x_2) \end{aligned}$$

Negation in Words: There exist two inputs that are not equal but that cause the same output.

Original Statement: f *is* ***injective***

Contrapositive Meaning : If two inputs are not equal, then the outputs will not be equal.

Contrapositive Written Formally: $\forall x_1, x_2 \in X\big(If\ x_1 \neq x_2\ \ then\ f(x_1) \neq f(x_2)\big)$

Negation: f *is* ***not injective***

Negation Written Formally:

$$
\begin{aligned}
f \text{ is not injective} &\equiv NOT\Big(\forall x_1, x_2 \in X\big(If\ x_1 \neq x_2\ \ then\ f(x_1) \neq f(x_2)\big)\Big) \\
&\equiv \exists x_1, x_2 \in X\Big(NOT\big(If\ x_1 \neq x_2\ \ then\ f(x_1) \neq f(x_2)\big)\Big) \\
&\equiv \exists x_1, x_2 \in X\Big(x_1 \neq x_2\ \ and\ NOT\big(f(x_1) \neq f(x_2)\big)\Big) \\
&\equiv \exists x_1, x_2 \in X\big(x_1 \neq x_2\ and\ f(x_1) = f(x_2)\big)
\end{aligned}
$$

Negation in Words: There exist two inputs that are not equal but that cause the same output.

Notice that the negation of the contrapositive statement is exactly the same as the negation of the original statement. This makes sense, because the contrapositive is logically equivalent to the original statement.

Original Statement: f *is* ***injective***

Other Wording: For every element in the range, the number n of elements of the domain that can be used as input to cause that element of the range to be output is $n \leq 1$.

Negation: f *is* ***not injective***

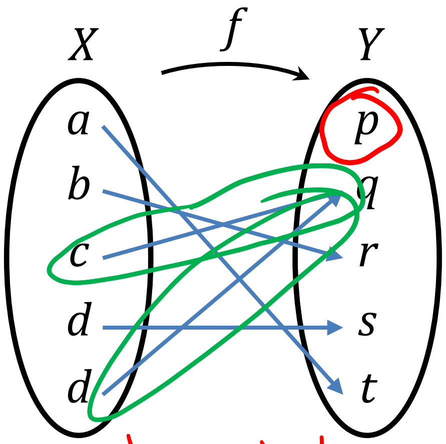
In Words: There exists an element in the range such that the number n of elements of the domain that can be used as input to cause that element of the range to be output **does not satisfy** $n \leq 1$.

Clearer Wording: There exists an element in the range such that the number n of elements of the domain that can be used as input to cause that element of the range to be output has the property that $n > 1$.

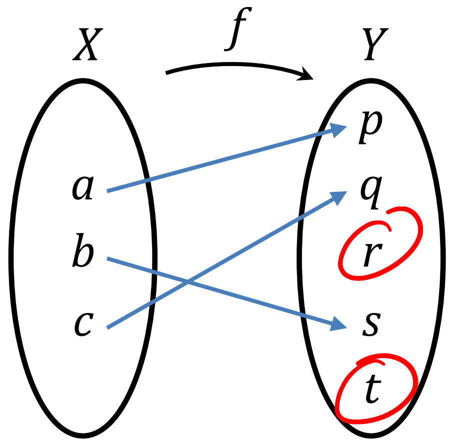
Even Clearer Wording: There exists an element in the range such that more than one elements of the domain can be used as input to cause that element of the range to be output.

We have articulated two different ways of wording the statement that a function f is *not injective*. We will see both can be useful in different situations.

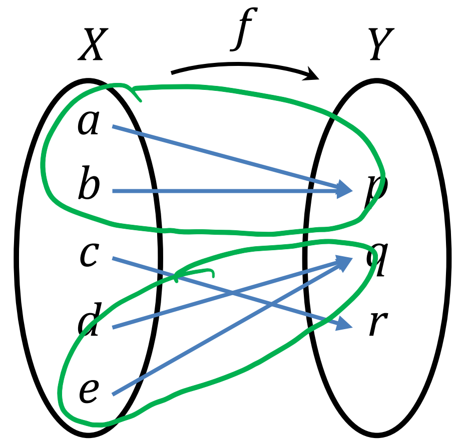
[Example 6] Which of these functions is surjective? Which is injective?



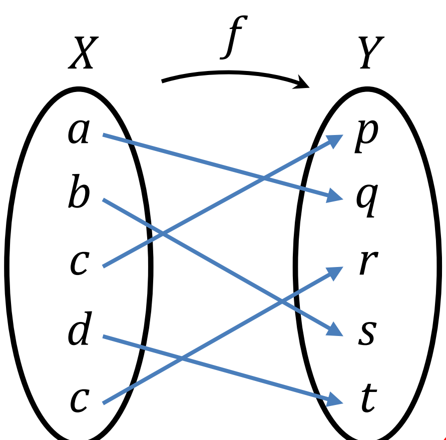
not surjective
not injective



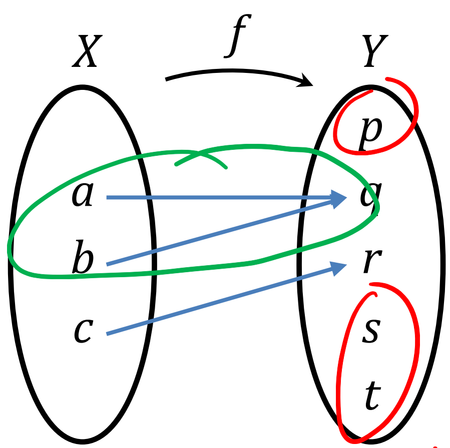
not surjective
injective ✓



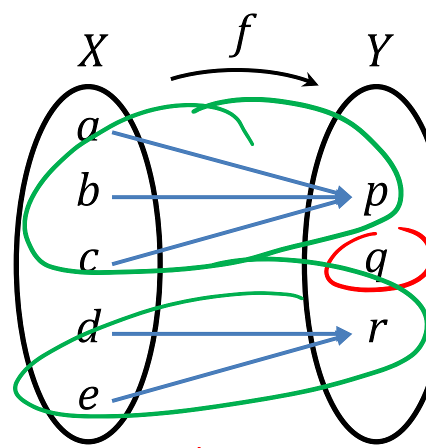
Surjective ✓
not injective



Surjective ✓
injective ✓



not surjective
not injective



not surjective
not injective

Graphs of *surjective* and *injective* functions.

For a function whose domain and codomain are subsets of the real numbers $\mathbb{R}$, the surjective and injective properties of the function have corresponding behaviors in the graph of the function.

The *Horizontal Line Tests for Surjectivity and Injectivity*

Suppose $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ and $f: A \to B$

surjectivity of f	$\leftrightarrow$	behavior of the graph of f
f is *surjective*	$\leftrightarrow$	For every $b \in B$,the horizontal line $y = b$ intersects the graph of f at least once. (f *passes* the *horizontal line test*.)
f is *not surjective*	$\leftrightarrow$	There exists a $b \in B$ such that the horizontal line $y = b$ does not intersect the graph of f. (f *fails* the *test*.)
injectivity of f	$\leftrightarrow$	behavior of the graph of f
f is *injective*	$\leftrightarrow$	For every $b \in B$,the horizontal line $y = b$ intersects the graph of f at most once. (f *passes* the *horizontal line test*.)
f is *not injective*	$\leftrightarrow$	There exists a $b \in B$ such that the horizontal line $y = b$ intersects the graph of f more than once. (f *fails* the *test*.)

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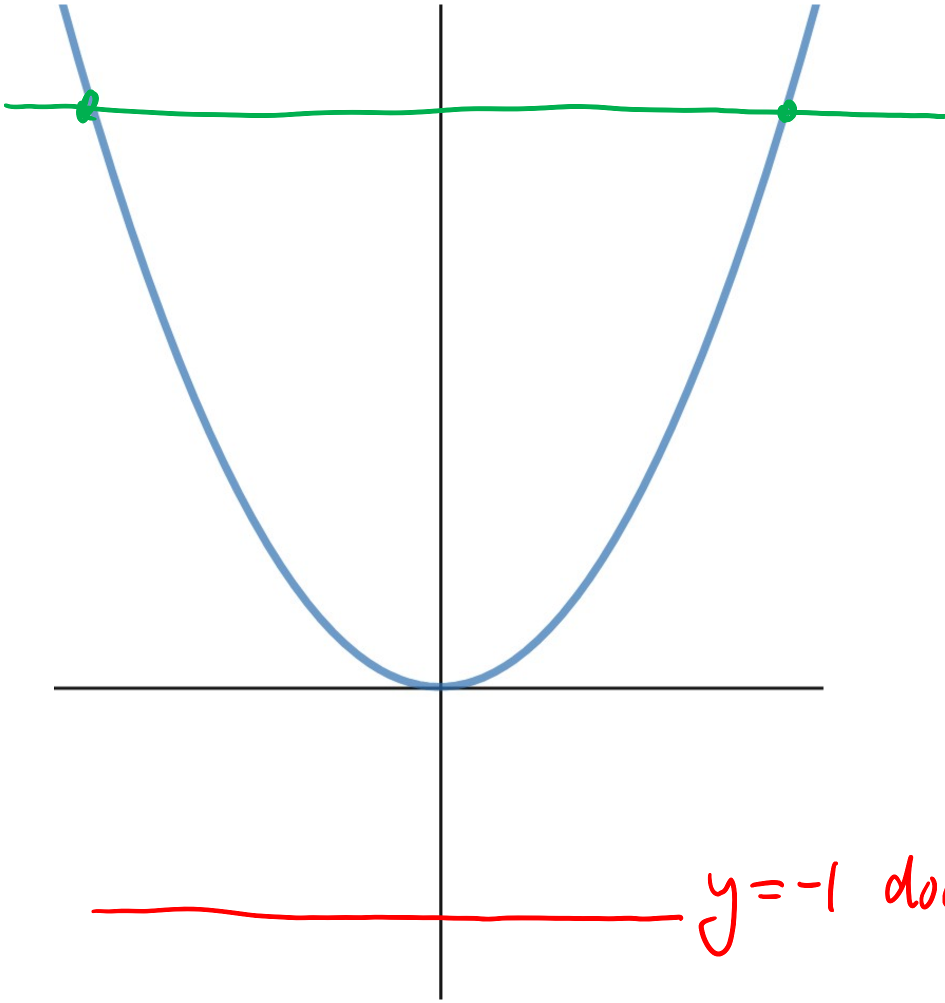
Determining whether a function given by a formula is surjective or injective.

[Example 2, revisited] Recall the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$

(a) Using the given graph of f, explain what *surjectivity* & *injectivity* properties you think f has.



$y=2$ intersects the graph twice, so f is not injective.

$y=-1$ does not intersect the graph so f is not surjective

(b) Write a formal proof of the claims that you made in **(a)**.

f is not surjective

Proof: let $y = -1$. There is no x such that $f(x) = -1$.

End of proof

f is not injective

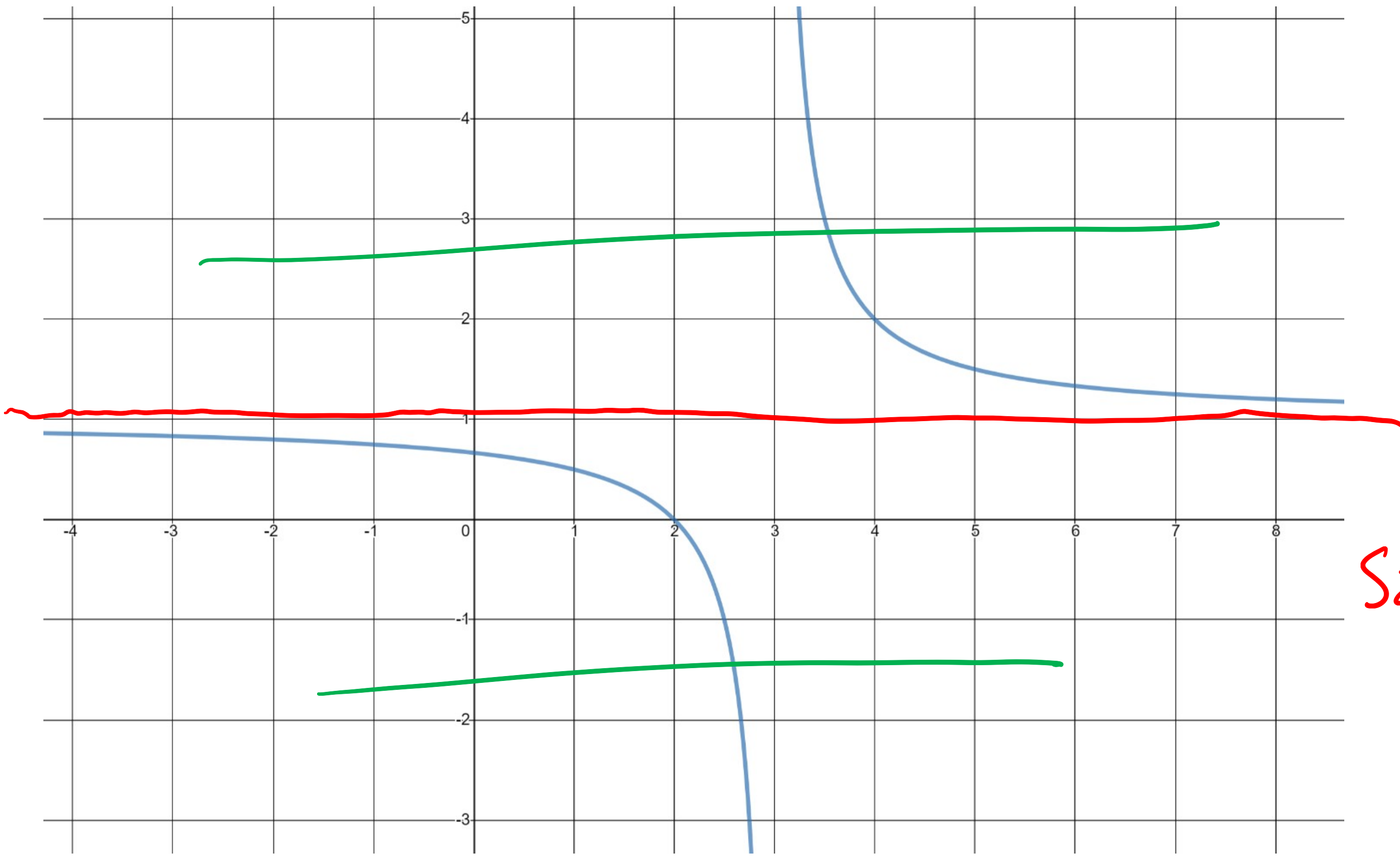
Proof: Observe $-2 \neq 2$ but $f(-2) = 4 = f(2)$

End of proof.

End of [Example 2, revisited]

[Example 3, revisited] Recall the function $f : \mathbb{R} - \{3\} \to \mathbb{R}$ defined by $f(x) = \dfrac{x-2}{x-3}$

(a) Using the given graph of f, explain what *surjectivity* & *injectivity* properties you think f has.



$y=1$ does not touch the graph
So f is not surjective

There are no horizontal lines that touch the graph more than once.
f is injective.

(b) Write a formal proof of the claims that you made in **(a)**.

Proof that f is not surjective

Let $y = 1$

We saw earlier that $y = 1$ is not in the image of f.

So f is not surjective.

Proof that f is injective

(1) Suppose $x_1, x_2 \in \mathbb{R} - \{3\}$ and $f(x_1) = f(x_2)$

(2) Recall that the equation $y = \frac{x-2}{x-3}$ can be solved for x in terms of y

$$x = \frac{3y-2}{y-1}$$

(3) $x_1 = \frac{3f(x_1)-2}{f(x_1)-1} = \frac{3f(x_2)-2}{f(x_2)-1} = x_2$

because $f(x_1) = f(x_2)$

(*) Therefore $x_1 = x_2$

End

[Example 7] Define the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 - 3x^2 + 3x + 1$.

(a) Using the given graph of f, explain what *surjectivity* & *injectivity* properties you think f has.

every horizontal line touches the graph exactly once.

So f is both surjective and injective.

(b) Write a formal proof of the claims that you made in **(a)**.

f can be rewritten

$$f(x) = x^3 - 3x^2 + 3x + 1 =$$

$$= x^3 - 3x^2 + 3x - 1 + 2$$

trick

$$= (x-1)^3 + 2$$

$$y = (x-1)^3 + 2$$ corresponding equation.

Solve for x

$$y - 2 = (x-1)^3$$

$$(y-2)^{1/3} = x - 1$$

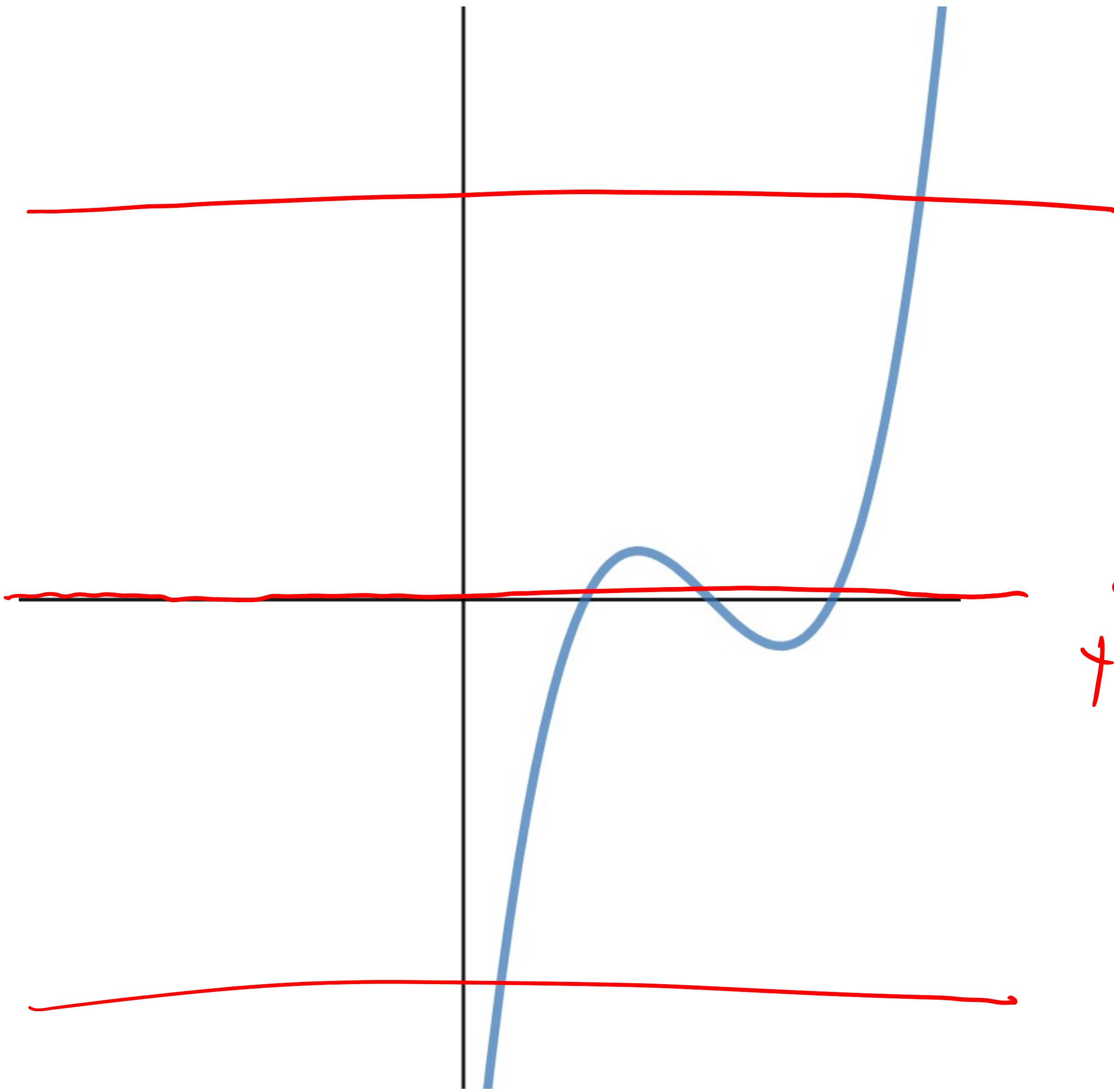
$$x = (y-2)^{1/3} + 1$$

The fact that we can solve the equation for x in terms of y tells us that the function is both surjective and injective.

End of [Example 7]

[Example 8] Define the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^3 - 6x^2 + 11x - 6$.

(a) Using the given graph of f, explain what *surjectivity* & *injectivity* properties you think f has.



Every horizontal line touches the graph at least once. So f is surjective.

Some horizontal lines touch the graph more than once, so f is not injective.

(b) Write a formal proof of the claims that you made in **(a)**.

$f(x) = x^3 - 6x^2 + 11x - 6$ can be factored

$f(x) = (x-1)(x-2)(x-3)$

observe x intercepts at $x = 1, 2, 3$

Notice that $1 \neq 2$ but $f(1) = 0 = f(2)$

not injective.

Skip proof of surjective

End of [Example 8]

Bijective Functions

Definition of Bijective Function

Words: f *is* ***bijective***, or f *is a* ***bijection*** (or f *is a* ***one-to-one correspondence***)

Usage: f is a function, $f: X \to Y$

Meaning: f is both *injective* and *surjective*. (f is both *one-to-one* and *onto*.)

Other Wording: For every element in the range, there exists ***exactly one*** element of the domain that can be used as input to cause that element of the range to be output.

Meaning Written Formally: $\forall y \in Y\big(\exists!\, x \in X(f(x) = y)\big)$

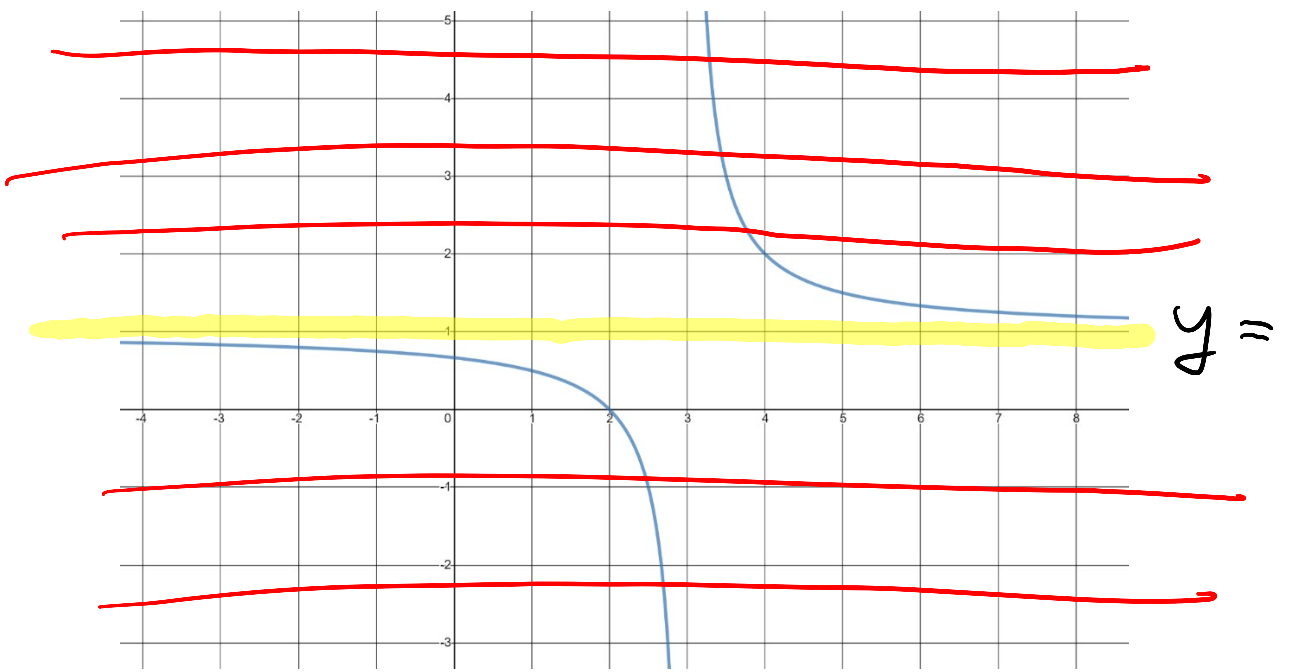
[Examples 2,3,6,7,8 Revisited] For each function in these examples, indicate whether the function is bijective.

[Example 3, revisited] Recall that the function $f: \mathbb{R} - \{3\} \to \mathbb{R}$ defined by $f(x) = \dfrac{x-2}{x-3}$ is injective but not surjective: There is no value of x in the domain $\mathbb{R} - \{3\}$ such that $f(x) = 1$.

Realize that the same formula can be used to define a bijective function if we use a different range:

Define a new function $f: \mathbb{R} - \{3\} \to \mathbb{R} - \{1\}$ by $f(x) = \dfrac{x-2}{x-3}$

Observe that every horizontal line $y = b$ where $b \neq 1$ touches the graph of f exactly once.



In other words, this new function is both surjective and injective. So it is bijective.

End of [Example 3, Revisited]

End of Video