2.1b: Incidence Geometry

produced by Mark Barsamian, 2021.02.04

for Ohio University MATH 3110/5110 College Geometry

Topics:

- Incidence Geometry
  - Definition
  - Models
    - Finite Geometries
    - Cartesian Plane
    - Poincaré Plane
    - (the Riemann Sphere is *not* a model of Incidence Geometry
- Theorem about intersecting lines in Incidence Geometry

Reading: pages 22 – 24 of Section 2.1 Definition and Models of Incidence Geometry in the book

Millman & Parker, *Geometry: A Metric Approach with Models, Second Edition*

(Springer, 1991, ISBN 3-540-97412-1)

Homework: Section 2.1 # 13, 16, 18, 19, 24

Abstract Geometry

Definition of Abstract Geometry

An ***abstract geometry*** $\mathcal{A}$ is an ordered pair $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ where $\mathcal{P}$ denotes a set whose elements are called **points** and $\mathcal{L}$ denotes a non-empty set whose elements are called **lines**, which are **sets of points** satisfying the following two requirements, called ***axioms***:

(i) For every two distinct points $A, B \in \mathcal{P}$, there exists at least one line $l \in \mathcal{L}$ such that $A \in l$ and $B \in l$.

(ii) For every line $l \in L$ there exist at least two distinct points that are elements of the line.

Additional Terminology

Words: P *lies on* l or l *passes through* P.

Usage: $P \in \mathcal{P}$ and $l \in \mathcal{L}$

Meaning: $P \in l$

Requirements (i),(ii) are called the ***abstract geometry axioms***. They are simply the requriements that sets $\mathcal{P}, \mathcal{L}$ must satisfy (in addition to $\mathcal{L}$ being non-empty) in order for the pair $(\mathcal{P}, \mathcal{L})$ to be qualified to be called an *abstract geometry* .

Incidence Geometry

Definition of Incidence Geometry

An ***incidence geometry*** $\mathcal{A}$ is an *abstract geometry* $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ that satisfies the following two additional requirements, called ***axioms***:

(i) For every two distinct points $A, B \in \mathcal{P}$, there exists **exactly one** line $l \in \mathcal{L}$ such that $A \in l$ and $B \in l$.

(ii) There exist (at least) three *non-collinear* points.

Requirements (i),(ii) are called the ***incidence geometry axioms***. Keep in mind that in order for a pair $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ to be qualified to be called an *incidence geometry*, the pair $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ must satisfy the two *abstract geometry axioms* and the two *incidence geometry axioms*.

Models of Incidence Geometries

Finite Geometries

[Example 1](Revisiting [Example 1] from Video 2.1a)

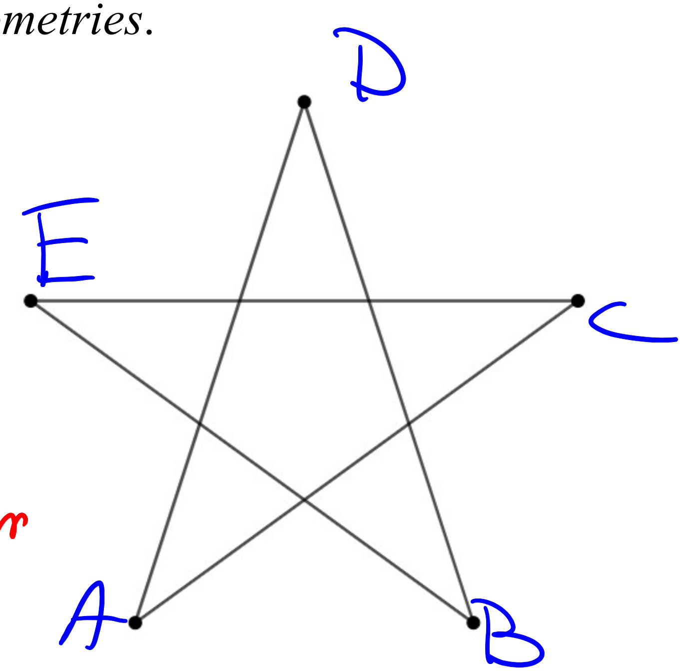
Finite sets that may or may not qualify to be called *incidence geometries*.

(a) Consider the pair $(\mathcal{P}, \mathcal{L})$ with

- points $\mathcal{P} = \{A, B, C, D, E\}$
- lines $\mathcal{L} = \{\{A, C\}, \{A, D\}, \cancel{\{A, E\}}, \{B, D\}, \{B, E\}, \{C, E\},\}$

not an abstract geom.

So not an incidence geom

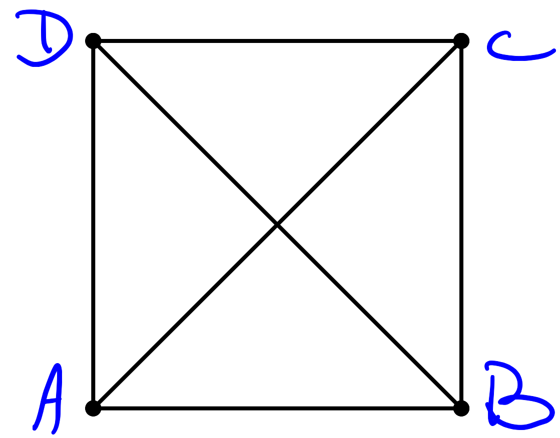


Is $(\mathcal{P}, \mathcal{L})$ qualified to be called an *incidence geometry*? Explain why or why not.

No! Fails Abstract geometry axiom(i), so it is not an abstract geometry. So it cannot be an incidence geometry

(b) Consider the pair $(\mathcal{P}, \mathcal{L})$ with

- points $\mathcal{P} = \{A, B, C, D, E\}$
- lines $\mathcal{L} = \{\{A,B\},\{A,C\},\{B,C\},\{B,D\},\{C,D\},\{D,A\}\}$



abstract geometry ✓

incidence geometry

Is $(\mathcal{P}, \mathcal{L})$ qualified to be called an *incidence geometry*? Explain why or why not.

This is an abstract geometry ✓

Satisfies incidence axiom (i)

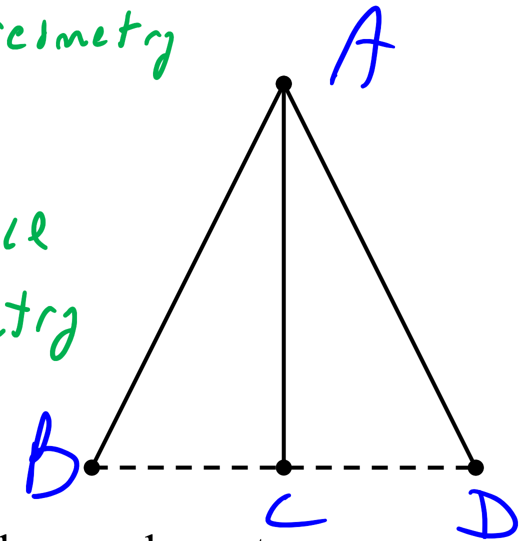
Passes axiom (ii) For example, A, B, C are non-collinear

(c) Consider the pair $(\mathcal{P}, \mathcal{L})$ with

- points $\mathcal{P} = \{A, B, C, D\}$
- lines $\mathcal{L} = \{\{A, B\}, \{A, C\}, \{A, D\}, \{B, C, D\},\}$

abstract geometry

incidence geometry



Is $(\mathcal{P}, \mathcal{L})$ qualified to be called an *incidence geometry*? Explain why or why not.

It is an abstract geometry

Satisfies abstract axiom(i) ✓

Satisfies abstract axiom(ii) ✓

Also satisfies incidence axioms

Incidence axiom (i) ✓

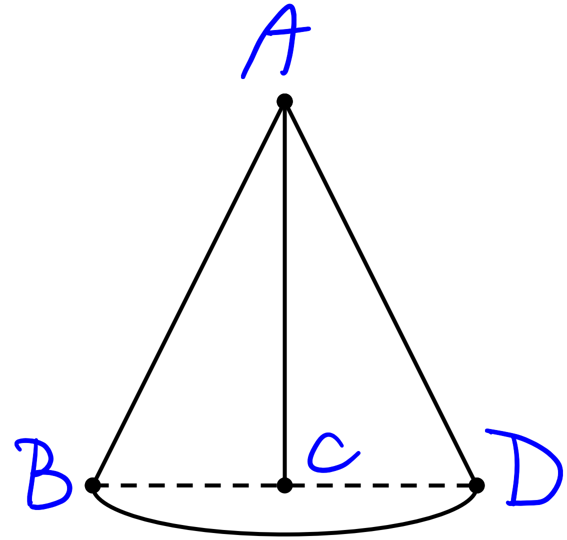
Incidence axiom (ii) ✓ (Points A, B, C are non-collinear)

(d) Consider the pair $(\mathcal{P}, \mathcal{L})$ with

- points $\mathcal{P} = \{A, B, C, D\}$
- lines $\mathcal{L} = \{\{A, B\}, \{A, C\}, \{A, D\}, \{B, C, D\}, \{B, D\}\}$

abstract geometry

not an incidence geom



Is $(\mathcal{P}, \mathcal{L})$ qualified to be called an *incidence geometry*? Explain why or why not.

abstract axiom (i) ✓

abstract axiom (ii) ✓

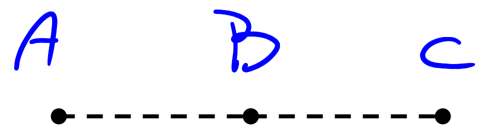
incidence axiom (i) Fails B, D both lie on $\{B, D\}$ and $\{B, C, D\}$

incidence axiom (ii) ✓

(e) Consider the pair $(\mathcal{P}, \mathcal{L})$ with

- points $\mathcal{P} = \cancel{\{B, C\}}$ $\{A, B, C\}$
- lines $\mathcal{L} = \{\{A, B, C\}\}$

abstract geometry



~~not~~ an incidence geometry

Is $(\mathcal{P}, \mathcal{L})$ qualified to be called an *incidence geometry*? Explain why or why not.

abstract axiom (i) ✓

abstract axiom (ii) ✓

incidence axiom (i) ✓

incidence axiom (ii) ✗ there are not three non-collinear points

End of [Example 1]

The *Cartesian Plane* is an Example (a *Model*) of *Incidence Geometry*.

In the previous video, we saw that the *Cartesian Plane* is qualified to be called an *abstract geometry* . That fact is proven in Proposition 2.1.1 of the book. In Proposition 2.1.4 of the book, it is proven that the *Cartesian Plane* is also qualified to be called an *incidence geometry*.

The bulk of the book's proof is taken up in proving that *incidence geometry* axiom (i) is satisfied.

> **(i)** For every two distinct points $A, B \in \mathcal{P}$, there exists **exactly one** line $l \in \mathcal{L}$ such that $A \in l$ and $B \in l$.

That portion of the proof is clear enough, and there is no need for me to discuss it here.

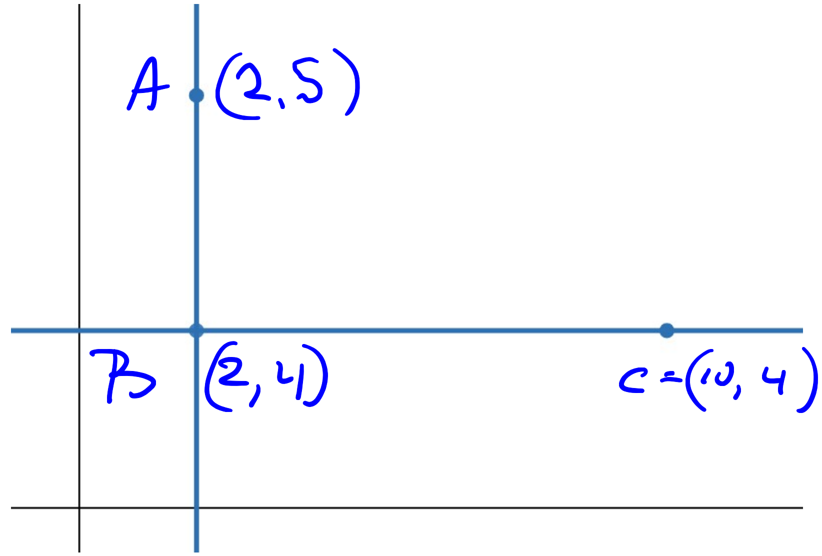
But the authors leave it to the reader to prove that *incidence geometry a*xiom (ii) is satisfied

> **(ii)** There exist (at least) three *non-collinear* points.

To prove that this axiom is satisfied, one must produce an *example* of three *non-collinear* points. In presenting an example, one must not just present the points, but also explain clearly *why* they are *non-collinear*. An example is most useful if it is very simple, involving no computations. It is worth presenting such an example here.

[Example 2] Three *non-collinear* points in the *Cartesian Plane*

Consider $A = (x_a, y_a) = (2,5), B = (x_b, y_b) = (2,4), C = (x_c, y_c) = (10,4)$.



- Observe that $x_a = x_b$, so that the *Cartesian line* that passes through A and B must be a *vertical line*.
- Observe that $x_b \neq x_c$, so that the *Cartesian line* that passes through B and C must be a *non vertical* line.
- Therefore the *Cartesian line* that passes through A and B cannot be the same as the *Cartesian line* that passes through B and C.

Conclude that A, B, C are *non-collinear* in the *Cartesian Plane.*

End of [Example 2]

The Poincaré Plane is an Example (a *Model*) of *Incidence Geometry*.

In the previous video, we saw that the *Poincaré Plane* is qualified to be called an *abstract geometry*. That fact is proven in Proposition 2.1.2 of the book. In Proposition 2.1.5 of the book, it is proven that the *Poincaré Plane* is also qualified to be called an *incidence geometry*.

As with its proof of Proposition 2.1.4, the bulk of the book's proof of Proposition 2.1.5 is taken up in proving that *incidence geometry* axiom (i) is satisfied.

> **(i)** For every two distinct points $A, B \in \mathcal{P}$, there exists **exactly one** line $l \in \mathcal{L}$ such that $A \in l$ and $B \in l$.

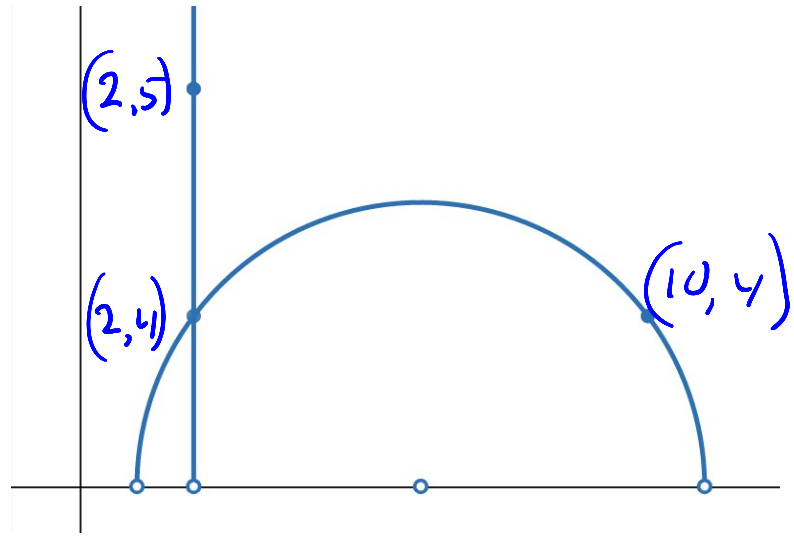
That portion of the proof is clear enough, and there is no need for me to discuss it here.
But as in the book's proof of Proposition 2.1.4, in their proof of Proposition 2.1.5, the the authors leave it to the reader to prove that *incidence geometry* axiom (ii) is satisfied.

> **(ii)** There exist (at least) three *non-collinear* points.

To prove that this axiom is satisfied, one must produce an *example* of three *non-collinear* points. Because computations involving *Poincaré lines* can be so messy, it is particularly useful to have a simple example involving no computations. Here is such an example.

[Example 3] Three *non-collinear* points in the *Poincaré Plane*

Consider $A = (x_a, y_a) = (2,5), B = (x_b, y_b) = (2,4), C = (x_c, y_c) = (10,4)$.

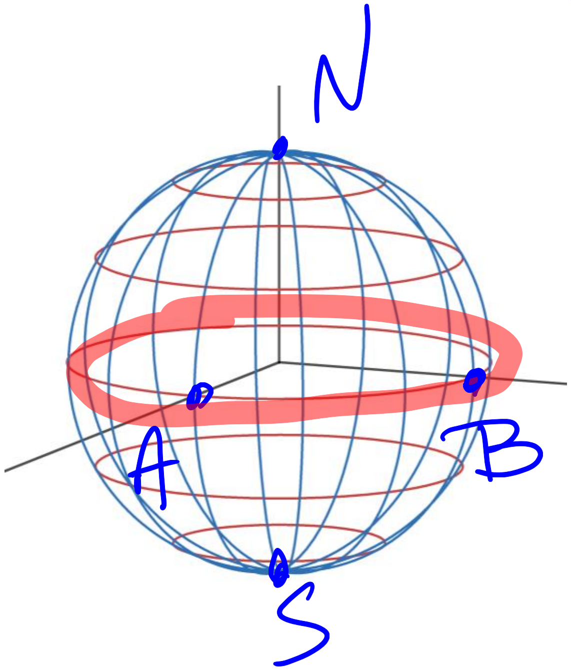


- Observe that $x_a = x_b$, so that the *Poincaré line* that passes through A and B must be a *type I* line.
- Observe that $x_b \neq x_c$, so that the *Poincaré line* that passes through B and C must be a *type II* line.
- Therefore the *Poincaré line that* passes through A and B cannot be the same as the *Poincaré line* that passes through B and C.

Conclude that A, B, C are *non-collinear* in the *Poincaré Plane*.

End of [Example 3]

The *Riemann Sphere* is *not* a model of *Incidence Geometry*



It is an abstract geometry

abstract axiom(i) ✓

abstract axiom(ii) ✓

Not an incidence geometry

incidence axiom(i) ✗ fails

For example, north & south pole lie on an infinite collection of lines

incidence axiom (ii) ✓

A, B, N are non-collinear

Theorem about Intersecting Lines in *Incidence Geometry*

Theorem 2.1.6 Given two lines l_1 and l_2 in an *incidence geometry*,

If $l_1 \cap l_2$ has two or more distinct points,

then l_1 and l_2 are the same line. That is, $l_1 = l_2$.

The *contrapositive* of the statement of Theorem 2.1.6 can be stated as a *corollary*.

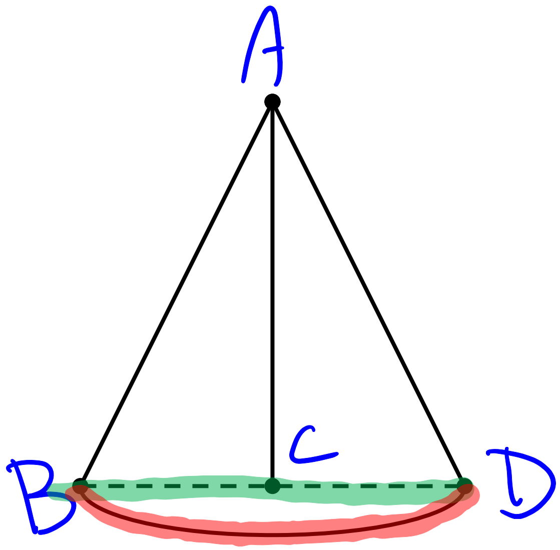
Corollary 2.1.7 (contrapositive of Theorem 2.1.6)

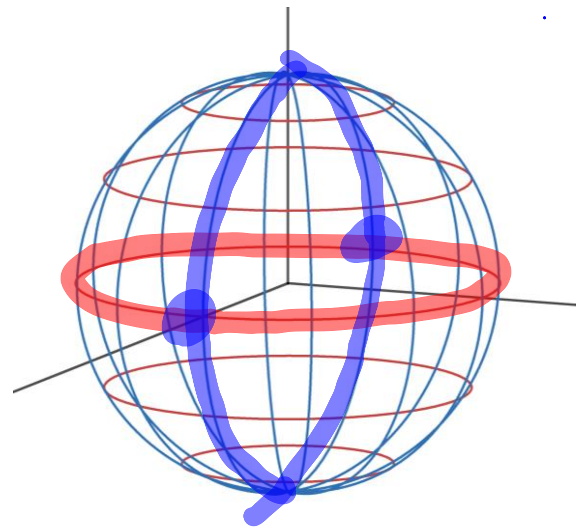
Given two lines l_1 and l_2 in an *incidence geometry*,

If lines l_1 and l_2 are known to be distinct lines (that is, $l_1 \neq l_2$),

then either lines l_1 and l_2 do not intersect or they intersect in exactly one point.

Observe that Theorem 2.1.5 and its Corollary 2.1.8 are *not* theorems of *abstract geometry* . Indeed, ~~it~~ we have seen examples of *abstract geometry* in which two distinct lines intersect in *more than one* point.





{B, C, D} and {B, D} are distinct lines
and they intersect in more than one point

Of course, neither of these examples qualifies as an *incidence geometry*. That’s the point.

End of Video