2.2a: Distance Functions

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for Ohio University MATH 3110/5110 College Geometry

Topics:

- Definition of Distance Function
- Examples of Distance Functions
  - The absolute value distance function $d_{\mathbb{R}}$ on $\mathbb{R}$
  - The Euclidean distance function d_E on $\mathbb{R}^2$
  - The taxicab distance function d_T on $\mathbb{R}^2$
  - The max (or supremum) distance function d_S on $\mathbb{R}^2$
  - The Poincaré distance function d_H on $\mathbb{H}$
- Circles

Reading: pages 27 – 29 of Section 2.2 Metric Geometry in the book *Geometry: A Metric Approach with Models, Second Edition* by Millman & Parker (Springer, 1991, ISBN 3-540-97412-1)

Homework: Section 2.2 # 1, 2, 7ii, 18i, 19

Recall important definitions from Section 2.1

Definition of Abstract Geometry

An ***abstract geometry*** $\mathcal{A}$ is an ordered pair $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ where $\mathcal{P}$ denotes a set whose elements are called **points** and $\mathcal{L}$ denotes a non-empty set whose elements are called **lines**, which are **sets of points** satisfying the following two requirements, called ***axioms***:

(i) For every two distinct points $A, B \in \mathcal{P}$, there exists **at least one** line $l \in \mathcal{L}$ such that $A \in l$ and $B \in l$.

(ii) For every line $l \in L$ there exist at least two distinct points that are elements of the line.

Definition of Incidence Geometry

An ***incidence geometry*** $\mathcal{A}$ is an *abstract geometry* $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ that satisfies the following two additional requirements, called ***axioms***:

(i) For every two distinct points $A, B \in \mathcal{P}$, there exists **exactly one** line $l \in \mathcal{L}$ such that $A \in l$ and $B \in l$.

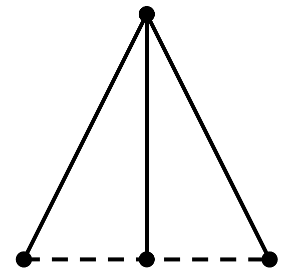
(ii) There exist (at least) three *non-collinear* points.

We saw various examples of *Abstract Geometries*

Finite Geometries, such as

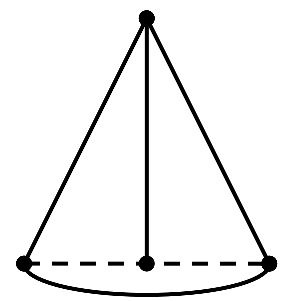
the pair $(\mathcal{P}, \mathcal{L})$ with

- points $\mathcal{P} = \{A, B, C, D\}$
- lines $\mathcal{L} = \big\{\{A, B\}, \{A, C\}, \{A, D\}, \{B, C, D\}\big\}$



the pair $(\mathcal{P}, \mathcal{L})$ with

- points $\mathcal{P} = \{A, B, C, D\}$
- lines $\mathcal{L} = \big\{\{A, B\}, \{A, C\}, \{A, D\}, \{B, C, D\}, \{B, D\}\big\}$



The Cartesian Plane, $\mathcal{C} = (\mathbb{R}^2, \mathcal{L}_E)$

The Poincaré Plane, $\mathcal{H} = (\mathbb{H}, \mathcal{L}_H)$

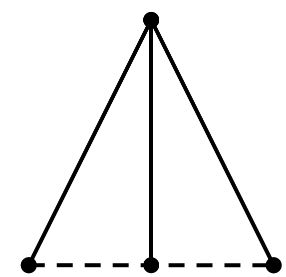
The Riemann Sphere, $R = (S^2, \mathcal{L}_R)$

We saw that some, but not all, *Abstract Geometries* were qualified to be called *Incidence Geometries*.

Some Finite Geometries, such as

the pair $(\mathcal{P}, \mathcal{L})$ with

- points $\mathcal{P} = \{A, B, C, D\}$
- lines $\mathcal{L} = \{\{A, B\}, \{A, C\}, \{A, D\}, \{B, C, D\}\}$



The Cartesian Plane, $\mathcal{C} = (\mathbb{R}^2, \mathcal{L}_E)$

The Poincaré Plane, $\mathcal{H} = (\mathbb{H}, \mathcal{L}_H)$

(The Riemann Sphere, $R = (S^2, \mathcal{L}_R)$ is not qualified to be called an *incidence geometry*.)

We see that there is nothing in the *abstract geometry* or *incidence geometry* axioms that requires that every line have an infinite set of points that lie on it. In Section 2.2, we will see the introduction of *metric geometry*. We will see that definition of metric geometry will ensure that, in addition to other behavior, every line will have an infinite set of points that lie on it.

In the current video, Video 2.2a, we will learn about *distance functions on a set*, and see examples of distance functions for some of the sets that we have learned about in previous videos.

In the next video, Video 2.1b, we will learn about *rulers* and *metric geometry*.

Recall the Procedure for Finding the Poincaré Line Through Two Distinct Points in $\mathbb{H}$

In Section 2.1 (and in Video 2.1a), we discussed this procedure, which ws extracted from the book's proof of **Proposition 2.1.2**.

Procedure for Finding the *Poincaré Line* Passing Through Two Distinct Points in $\mathbb{H}$

Suppose $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are any two distinct points of $\mathbb{H}$.

If $x_1 = x_2$ then let $a = x_1 = x_2$. In this case, ${}_aL \in \mathcal{L}_H$ and $P, Q \in {}_aL$.

If $x_1 \neq x_2$ then define constants c, r by the following formulas:

$$c = \frac{x_2^2 - x_1^2 + y_2^2 - y_1^2}{2(x_2 - x_1)}$$

$$r = \sqrt{(x_1 - c)^2 + y_1^2}$$

In this case, ${}_cL_r \in \mathcal{L}_H$ and $P, Q \in {}_cL_r$.

Distance Functions

The definition of *distance function* is very general, and applies to many mathematical situations that one might not at first consider very "geometrical".

Definition of Distance Function

words: d is a distance function on set S

meaning: d is a function $d: S \times S \to \mathbb{R}$ that satisfies these requirements

(i) $\forall P, Q \in S(d(P, Q) \geq 0)$

(ii) $d(P, Q) = 0$ if and only if $P = Q$

(iii) $d(P, Q) = d(Q, P)$

Examples of Distance Functions

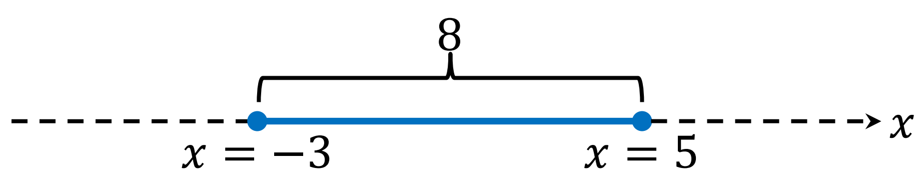
The Absolute Value Distance Function on $\mathbb{R}$

> **Definition of the Absolute Value Distance Function on $\mathbb{R}$**
>
> **symbol:** $d_{\mathbb{R}}$
>
> **meaning:** the function $d_{\mathbb{R}}: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $d_{\mathbb{R}}(x, y) = |x - y|$

[Example 1]

(a) Find $d_{\mathbb{R}}(5, -3)$ and illustrate using a number line.

Solution: $d_{\mathbb{R}}(5, -3) = |5 - (-3)| = 8$



(b) Find $d_{\mathbb{R}}(-3,5)$

Solution: $d_{\mathbb{R}}(-3,5) = |(-3) - 5| = |-8| = 8$

(c) Find $d_{\mathbb{R}}(5,5)$ and illustrate using a number line.

Solution: $d_{\mathbb{R}}(5,5) = |5 - 5| = 0$. The point $x = 5$ on the number line does not define a segment.

End of [Example 1]

Three Distance Functions on $\mathbb{R}^2$

Definition of the Euclidean Distance Function on $\mathbb{R}^2$

symbol: d_E

meaning: the function $d_E\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$d_E\big((x_1, y_1), (x_2, y_2)\big) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Definition of the Taxicab Distance Function on $\mathbb{R}^2$

symbol: d_T

meaning: the function $d_T\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$d_T\big((x_1, y_1), (x_2, y_2)\big) = |x_1 - x_2| + |y_1 - y_2|$$

Definition of the Max (or Supremum) Distance Function on $\mathbb{R}^2$

symbol: d_S

meaning: the function $d_S\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$d_S\big((x_1, y_1), (x_2, y_2)\big) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$$

[Example 2] Let $P = (1,1) \in \mathbb{R}^2$ and $Q = (4,5) \in \mathbb{R}^2$.

(a) Find $d_E(P,Q)$.

Solution: $d_E(P,Q) = \sqrt{(1-4)^2 + (1-5)^2} = \sqrt{(-3)^2 + (-4)^2} = \sqrt{9+16} = 5$

(b) Find $d_T(P,Q)$.

Solution: $d_T(P,Q) = |1-4| + |1-5| = |-3| + |-4| = 3 + 4 = 7$

(c) Find $d_S(P,Q)$.

Solution: $d_S(P,Q) = \max\{|1-4|, |1-5|\} = \max\{|-3|, |-4|\} = \max\{3,4\} = 4$

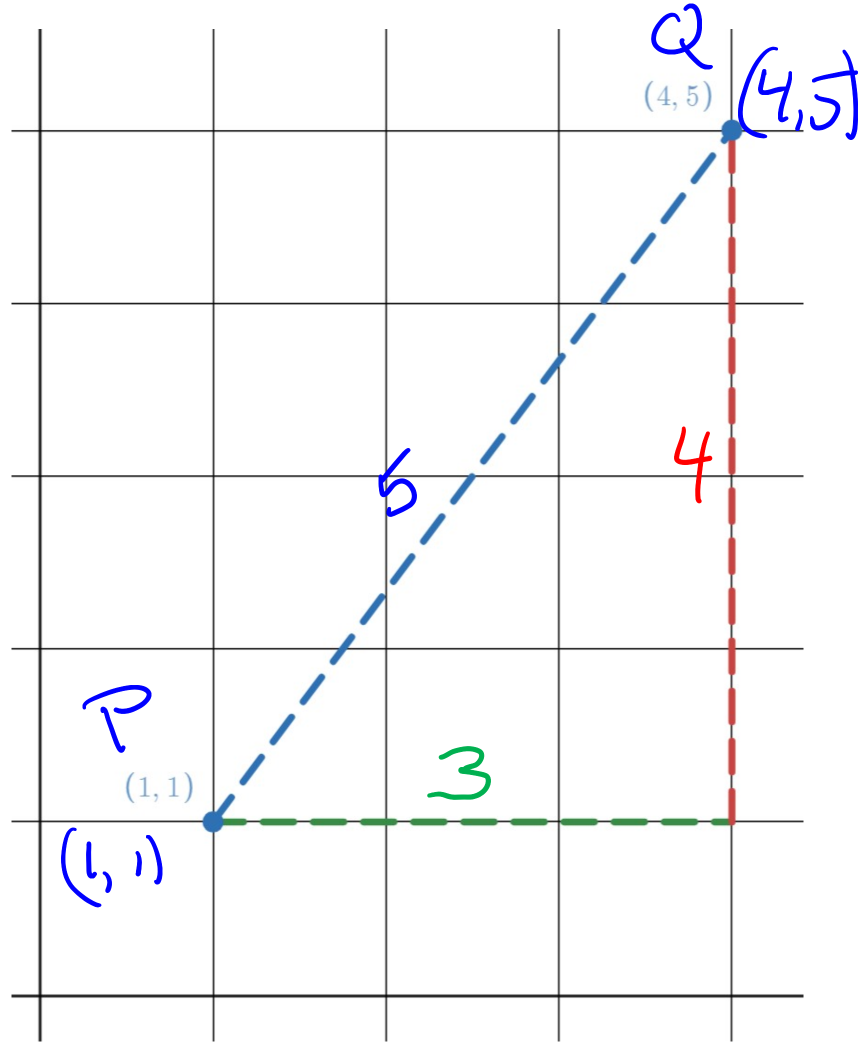
(d) Illustrate your results from **(a)**, **(b)**, **(c)**.

Solution: Observe that points P and Q are the acute vertices of a 3,4,5 right triangle whose legs are horizontal and vertical segments.

The *Euclidean distance* $d_E(P,Q) = 5$ measures the length of the hypotenuse of this triangle.

The *taxicab distance* $d_T(P,Q) = 7$ measures the sum of the lengths of the legs of this triangle.

The *max distance* $d_S(P,Q) = 4$ measures ~~the sum~~ of the length of the longer leg of this triangle.



It is important to note that in *all three geometries*, the *line* that points P and Q lie on is the blue dotted *Cartesian line* $L_{\frac{4}{3},-\frac{1}{3}}$. The green and red segments are involved in the calculation of *taxicab distance* and *max distance*, but they are not part of the line.

End of [Example 2]

The Poincaré Distance Function on $\mathbb{H}$

Definition of the Poincaré Distance Function on $\mathbb{H}$

symbol: d_H

meaning: the function $d_H: \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ defined by in the following way

Suppose $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are any two points of $\mathbb{H}$.

If $x_1 = x_2$ then compute the distance between them using the formula

$$d_H(P,Q) = \left|\ln\left(\frac{y_2}{y_1}\right)\right|$$

If $x_1 \neq x_2$ then compute the distance between them using the formula

$$d_H(P,Q) = \left|\ln\left(\frac{\frac{x_1 - c + r}{y_1}}{\frac{x_2 - c + r}{y_2}}\right)\right|$$

where c, r are the constants describing the *type II* line that passes through P and Q.

[Example 3] Consider the following four points

$A = (x_A, y_A) = (1,3), B = (x_B, y_B) = (1,4), C = (x_C, y_C) = (1,5), D = (x_D, y_D) = (8,3).$

(a) Find $d_H(A,B), d_H(B,C), d_H(A,C), d_H(B,D)$.

Give *exact, simplified answers* and a *decimal approximation*, rounded to 3 decimal places.

Solution:

Since $x_A = x_B$, we use the simple formula for computing the distance between A and B.

$$d_H(A,B) = \left|\ln\left(\frac{y_B}{y_A}\right)\right| = \left|\ln\left(\frac{4}{3}\right)\right| = \ln\left(\frac{4}{3}\right) \approx 0.288$$

exact approximation

Since $x_B = x_C$, we use the simple formula for computing the distance between B and C.

$$d_H(B,C) = \left|\ln\left(\frac{y_C}{y_B}\right)\right| = \left|\ln\left(\frac{5}{4}\right)\right| = \ln\left(\frac{5}{4}\right) \approx 0.223$$

exact approximation

Since $x_A = x_C$, we use the simple formula for computing the distance between A and C.

$$d_H(A,C) = \left|\ln\left(\frac{y_C}{y_A}\right)\right| = \left|\ln\left(\frac{5}{3}\right)\right| = \ln\left(\frac{5}{3}\right) \approx 0.511$$

Since $x_B \neq x_D$, we must use the harder formula for computing the distance between B and D. That entails first finding c and r for the *Poincaré line* that passes through B and D.

$$c = \frac{x_D^2 - x_B^2 + y_D^2 - y_B^2}{2(x_D - x_B)} = \frac{8^2 - 1^2 + 3^2 - 4^2}{2(8-1)} = 4$$

$$r = \sqrt{(x_B - c)^2 + y_B^2} = \sqrt{(1-4)^2 + 4^2} = 5$$

Using these numbers in the *Poincaré distance* formula, we obtain

$$d_H(B,D) = \left| \ln\left(\frac{\frac{x_B - c + r}{y_B}}{\frac{x_D - c + r}{y_D}} \right) \right| = \left| \ln\left(\frac{\frac{1-4+5}{4}}{\frac{8-4+5}{3}} \right) \right| = \left| \ln\left(\frac{1}{6}\right) \right| = |-\ln(6)| = \ln(6) \approx 1.792$$

exact simplified

(b) Find the corresponding *Euclidean distances*, $d_E(A,B), d_E(B,C), d_E(A,C), d_E(B,D)$.

Solution: These are easy

$$d_E(A,B) = 1$$

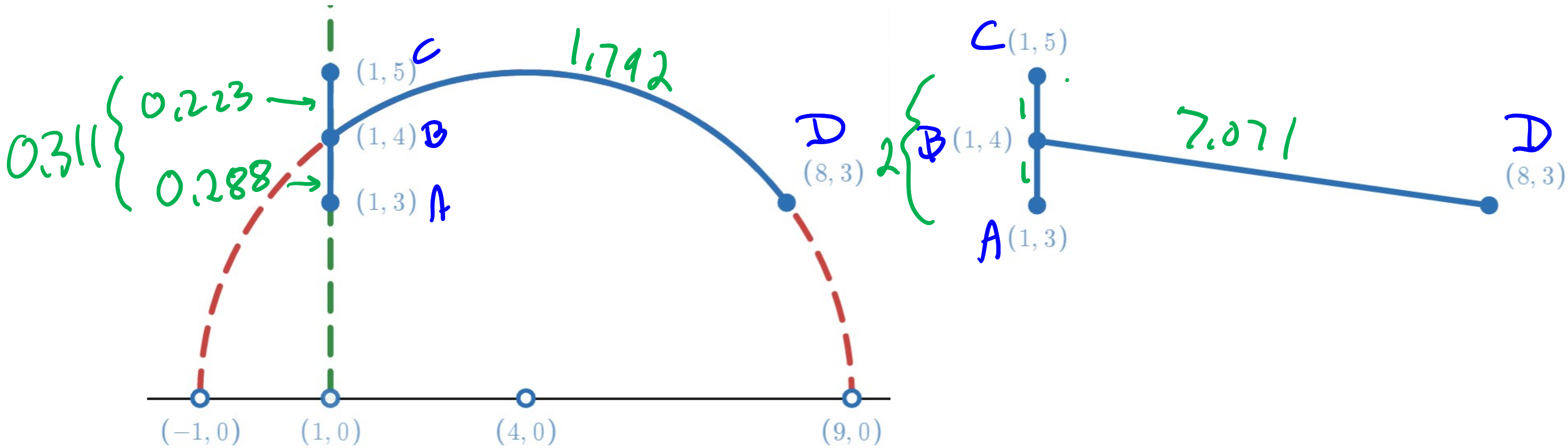
$$d_E(B,C) = 1$$

$$d_E(A,C) = 2$$

$$d_E(B,D) = \sqrt{(1-8)^2 + (4-3)^2} = \sqrt{50} \approx 7.071$$

exact approx

(c) Illustrate your results from **(a)** and **(b)**.



Solution: The distances computed in parts **(a)** and **(b)** are the lengths of the segments shown in the left and right drawings, respectively.

Remarks:

Notice that the *Poincaré distances* do not equal the corrsponding *Euclidean distances*.

And notice that the $d_H(A, B)$ and $d_H(B, C)$ do not equal each other, either.

However, notice that even though $d_H(A, B) \neq d_H(B, C)$, their values do add up to $d_H(A, C)$

$$d_H(A, B) + d_H(B, C) = \ln\left(\frac{4}{3}\right) + \ln\left(\frac{5}{4}\right) = \ln\left(\frac{4}{3} \cdot \frac{5}{4}\right) = \ln\left(\frac{5}{3}\right) = d_H(A, C)$$

This is important and will be discussed further in the next ~~video~~ section, when we discuss rulers.

End of [Example 3]

Remarks on the qualifications of the distance functions that I have presented

I have presented five "distance functions" $d_{\mathbb{R}}, d_E, d_T, d_S, d_H$ without actually proving that they are qualified to be called *distance functions*. That is, I have not proved that they satisfy the requirements set forth in the *definition of distance function* presented earlier in this video.

The real number distance function $d_{\mathbb{R}}$ clearly satisfies the requirements.

But what about the other four "distance functions" d_E, d_T, d_S, d_H?

The book does provide a proof that one of the four d_E, d_T, d_S, d_H really does qualify to be called a *distance function*.

Proposition 2.2.1

The *taxicab distance function* d_T satisfies the requirements to be called a *distance function on* $\mathbb{R}^2$.

In your homework exercise 2.2#1, you will prove that the *Euclidean distance function* d_E satisfies the requirements to be called a *distance function on* $\mathbb{R}^2$. And in your homework exercise 2.2#2, you will prove that the *Poincaré distance function* d_H satisfies the requirements to be called a *distance function on* $\mathbb{H}^2$. You should model your proof on the book's proof of Proposition 2.2.1.

We will assume without proof that the *max distance function* d_S does meet the requirements to be called a distance function on $\mathbb{R}^2$.

Circles

Definition of Circle

symbol: $\text{circle}(P, r)$

words: the circle with center P and radius r

usage: There is some set S with distance function d in the discussion, and $P \in S$, and $r > 0$.

meaning: the set

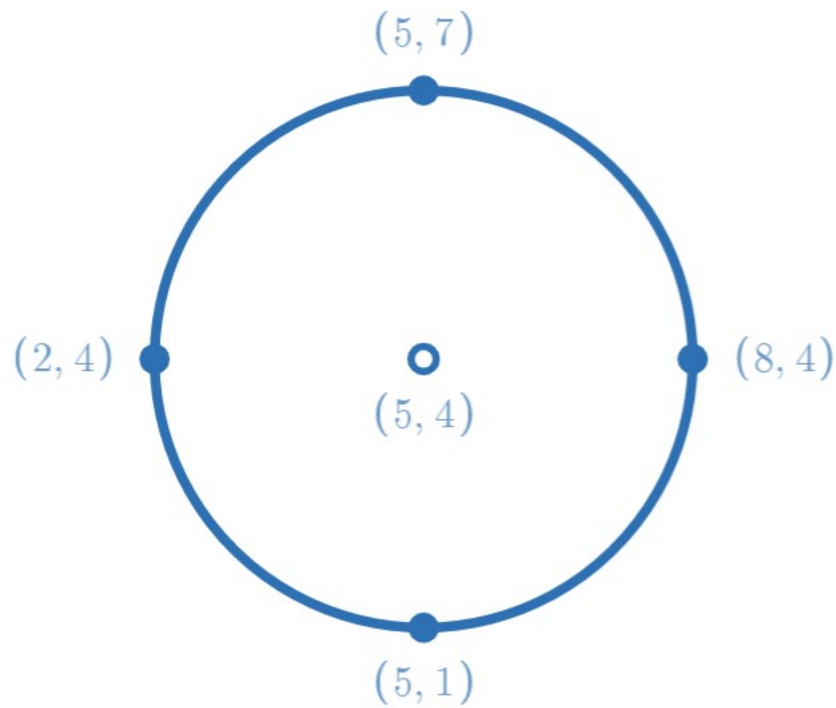
$$\text{circle}(P, r) = \{Q \in S | d(P, Q) = r\}$$

[Example 4] Circles in $\mathbb{R}^2$ using different distance functions.

(a) Draw the circle in $\mathbb{R}^2$ centered at $(5,4)$ with radius 3 using *Euclidean distance*.

Solution:

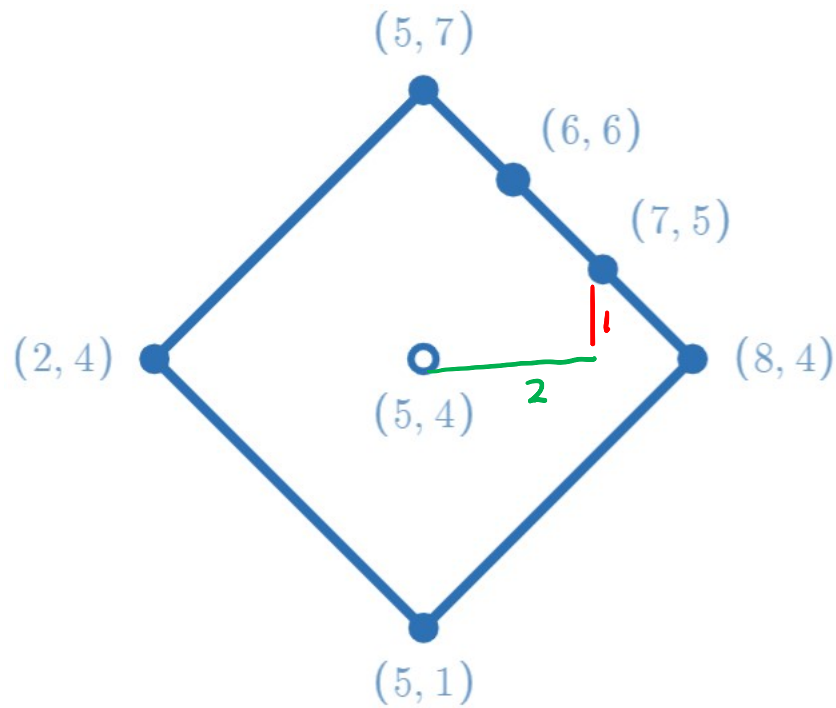
The circle is shown below. It needs no explanation.



(b) Draw the circle in $\mathbb{R}^2$ centered at (5,4) with radius 3 using *taxicab distance*.

Solution:

The circle is shown at right.



It has a surprising shape. It is instructive to consider why some of the points are on it.

$$d_T\big((5,4),(8,4)\big) = |5-8| + |4-4| = 3 + 0 = 3$$

$$d_T\big((5,4),(7,5)\big) = |5-7| + |4-5| = 2 + 1 = 3$$

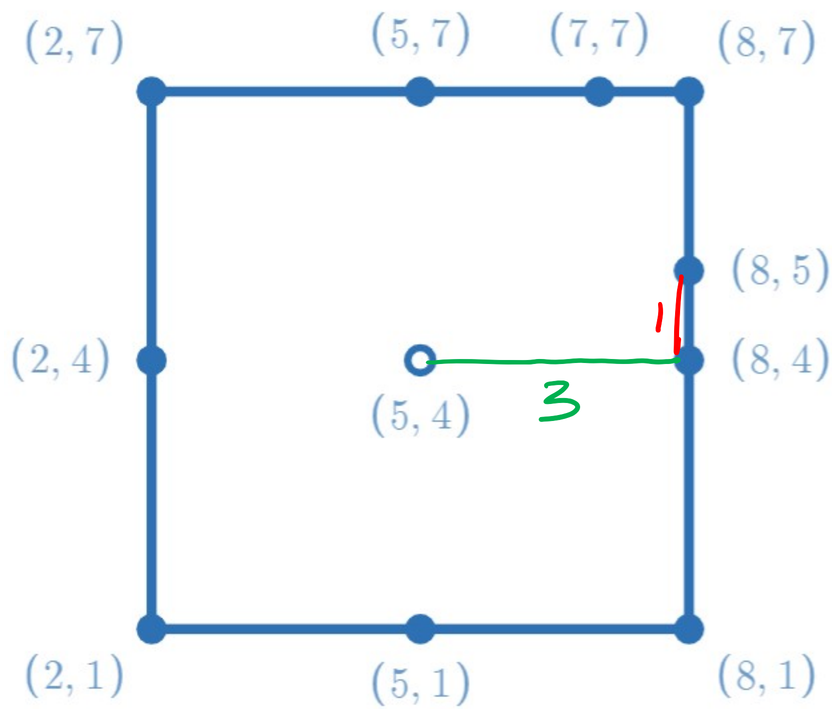
$$d_T\big((5,4),(6,6)\big) = |5-6| + |4-6| = 1 + 2 = 3$$

$$d_T\big((5,4),(5,7)\big) = |5-5| + |4-7| = 0 + 3 = 3$$

(c) Draw the circle in $\mathbb{R}^2$ centered at (5,4) with radius 3 using *max distance*.

Solution:

The circle is shown at right.



It also has a surprising shape. It is instructive to consider why some of the points are on it.

$$d_T\big((5,4),(8,4)\big) = \max\{|5-8|,|4-4|\} = \max\{|-3|,|0|\} = \max\{3,0\} = 3$$

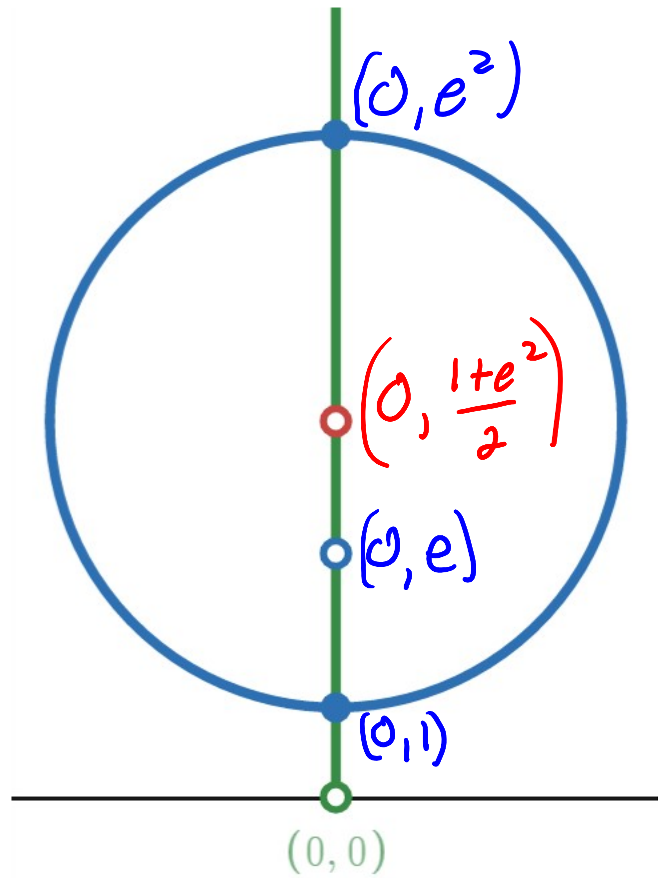
$$d_T\big((5,4),(8,5)\big) = \max\{|5-8|,|4-5|\} = \max\{|-3|,|-1|\} = \max\{3,1\} = 3$$

$$d_T\big((5,4),(8,7)\big) = \max\{|5-8|,|4-7|\} = \max\{|-3|,|-3|\} = \max\{3,3\} = 3$$

$$d_T\big((5,4),(7,7)\big) = \max\{|5-7|,|4-7|\} = \max\{|-2|,|-3|\} = \max\{2,3\} = 3$$

(d) Draw the circle in $\mathbb{H}$ centered at $(0, e)$ with radius 1 using *Poincaré distance*.

Solution: It is a remarkable fact that circles in $\mathbb{H}$ using *Poincaré distance* are sets of (x, y) pairs that also qualify as circles in $\mathbb{R}^2$ using *Euclidean distance*. The proof of that fact is beyond our level at this point in the course. But it is instructive to look at an example and study what we can say about it. The circle centered at $(0, e)$ with *Poincaré radius* 1 is shown at right. All of the points on the blue circle are the same *Poincaré distance* from $(0, e)$. What is fascinating is that the *Poincaré center* $(0, e)$ is the open *blue* dot, *not* the open *red* dot! To see why, consider two easy points that we know must be on the circle.



- The point $(0,1)$ is on the circle because $d_H\big((0,1),(0,e)\big) = \left|\ln\left(\frac{e}{1}\right)\right| = |\ln(e)| = |1| = 1$
- The point $(0, e^2)$ is on the circle because $d_H\big((0,e),(0,e^2)\big) = \left|\ln\left(\frac{e^2}{e}\right)\right| = |\ln(e)| = |1| = 1$

The points $(0,1)$ and $(0, e^2)$ will be at the bottom and top of the circle. They are the solid blue dots in the picture.

Observe that $e \approx 2.718$ while $e^2 \approx 7.389$. This tells us that the three points $(0,1), (0,e), (0,e^2)$ will not appear equally spaced along the y axis. The *Poincaré center* $(0,e)$ will appear closer to $(0,1)$ than to $(0,e^2)$. That is why the open blue dot looks like it is off center, closer to the bottom..

An obvious question is, what is the *red* open dot that *looks like* it is at the center of the blue circle?

It will be the pair on the y axis that is the same *Euclidean distance* from $(0,1)$ and $(0,e^2)$. That will be the pair $(0,y)$ where y is the average of 1 and e^2. That is,

$$y = \frac{1+e^2}{2} \approx 4.195$$

Because $\frac{1+e^2}{2} > e$, we see that the ordered pair $\left(0, \frac{1+e^2}{2}\right)$ will appear above the pair $(0,e)$ on the y axis. It is shown as the open *red* dot on the drawing above. It is the *Euclidean center* of the blue circle. That is, all of the points on the blue circle are the same *Euclidean distance* from the point $\left(0, \frac{1+e^2}{2}\right)$.

End of [Example 4]

End of Video