<u>2.2b: Rulers and Metric Geometry</u>

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for Ohio University MATH 3110/5110 College Geometry

Topics:

- Rulers
    - Definition of Ruler
    - Diagram illustrating the Ruler Equation
    - Examples of Rulers
        - Rulers for the $d_{\mathbb{R}}, d_E, d_T, d_S$ distance functions on $\mathbb{R}^2$
        - Rulers for the Poincaré distance function d_H on $\mathbb{H}$
- Metric Geometry
    - Definition of Metric Geometry
    - ~~Corollary about the set of points that lie on a line in a Metric Geometry~~
    - Examples of Metric Geometry

Reading: pages 30 – 35 of Section 2.2 Metric Geometry in the book *Geometry: A Metric Approach with Models, Second Edition* by Millman & Parker (Springer, 1991, ISBN 3-540-97412-1)

Homework: Section 2.2 # 4, 5, 6, 9, 10, 11, 12, 17, 20

Recall important definitions from Section 2.1

Definition of Abstract Geometry

An ***abstract geometry*** $\mathcal{A}$ is an ordered pair $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ where $\mathcal{P}$ denotes a set whose elements are called **points** and $\mathcal{L}$ denotes a non-empty set whose elements are called **lines**, which are **sets of points** satisfying the following two requirements, called ***axioms***:

(i) For every two distinct points $A, B \in \mathcal{P}$, there exists **at least one** line $l \in \mathcal{L}$ such that $A \in l$ and $B \in l$.

(ii) For every line $l \in L$ there exist at least two distinct points that are elements of the line.

Definition of Incidence Geometry

An ***incidence geometry*** $\mathcal{A}$ is an *abstract geometry* $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ that satisfies the following two additional requirements, called ***axioms***:

(i) For every two distinct points $A, B \in \mathcal{P}$, there exists **exactly one** line $l \in \mathcal{L}$ such that $A \in l$ and $B \in l$.

(ii) There exist (at least) three *non-collinear* points.

Procedure for Finding the *Cartesian* Line Passing through Two Distinct Points in $\mathbb{R}^2$

Suppose $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are any two distinct points of $\mathbb{R}^2$.

If $x_1 = x_2$ then let $a = x_1 = x_2$. In this case, $L_a \in \mathcal{L}_H$ and $P, Q \in L_a$.

If $x_1 \neq x_2$ then define constants m, b by the following formulas:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$b = y_2 - mx_2$$

Then $P, \in L_{m,b}$ and $Q \in L_{m,b}$.

Procedure for Finding the *Poincaré Line* Passing Through Two Distinct Points in $\mathbb{H}$

Suppose $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are any two distinct points of $\mathbb{H}$.

If $x_1 = x_2$ then let $a = x_1 = x_2$. In this case, ${}_aL \in \mathcal{L}_H$ and $P, Q \in {}_aL$.

If $x_1 \neq x_2$ then define constants c, r by the following formulas:

$$c = \frac{x_2^2 - x_1^2 + y_2^2 - y_1^2}{2(x_2 - x_1)}$$

$$r = \sqrt{(x_1 - c)^2 + y_1^2}$$

In this case, ${}_cL_r \in \mathcal{L}_H$ and $P, Q \in {}_cL_r$.

And important definitions from Section 2.2 that were discussed in the previous video

Definition of Distance Function

words: d is a distance function on set S

meaning: d is a function $d: S \times S \to \mathbb{R}$ that satisfies these requirements

(i) $\forall P, Q \in S(d(P,Q) \geq 0)$

(ii) $d(P,Q) = 0$ if and only if $P = Q$

(iii) $d(P,Q) = d(Q,P)$

Definition of the Absolute Value Distance Function on $\mathbb{R}$

symbol: $d_{\mathbb{R}}$

meaning: the function $d_{\mathbb{R}}: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $d_{\mathbb{R}}(x,y) = |x - y|$

It should be clear to the reader that the *Absolute Value Distance Function on* $\mathbb{R}$ really does satisfy the requirements to qualify to be called a *distance function on* $\mathbb{R}$.

Three Distance Functions on $\mathbb{R}^2$

Definition of the Euclidean Distance Function on $\mathbb{R}^2$

symbol: d_E

meaning: the function $d_E\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$d_E\big((x_1, y_1), (x_2, y_2)\big) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Definition of the Taxicab Distance Function on $\mathbb{R}^2$

symbol: d_T

meaning: the function $d_T\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$d_T\big((x_1, y_1), (x_2, y_2)\big) = |x_1 - x_2| + |y_1 - y_2|$$

Definition of the Max (or Supremum) Distance Function on $\mathbb{R}^2$

symbol: d_S

meaning: the function $d_S\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$d_S\big((x_1, y_1), (x_2, y_2)\big) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$$

The Poincaré Distance Function on $\mathbb{H}$

Definition of the Poincaré Distance Function on $\mathbb{H}$

symbol: d_H

meaning: the function $d_H\colon \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ defined by in the following way

Suppose $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are any two points of $\mathbb{H}$.

If $x_1 = x_2$ then compute the distance between them using the formula

$$d_H(P, Q) = \left|\ln\left(\frac{y_2}{y_1}\right)\right|$$

If $x_1 \neq x_2$ then compute the distance between them using the formula

$$d_H(P, Q) = \left|\ln\left(\frac{\dfrac{x_1 - c + r}{y_1}}{\dfrac{x_2 - c + r}{y_2}}\right)\right|$$

where c, r are the constants describing the *type II* line that passes through P and Q.

Rulers as a Way of Measuring Distance in Drawings

We have seen introduced three distance functions for the *Cartesian Plane* $(\mathbb{R}^2, \mathcal{L}_E)$ and a distance function for the Poincaré Plane $(\mathbb{H}, \mathcal{L}_H)$. Those distance functions enable one to find the distance $d(P, Q)$ between points P and Q by using the (x, y) coordinates of the two points. If our goal was just to be able to measure distance, then we are done: we have a great way to do that.

But at the beginning of the current section, 2.2, it was observed that there is nothing in the axioms for *incidence geometry* that requires that lines contain an *infinite* set of points. Indeed, we have seen a bunch of examples of finite geometries that are qualified to be called *incidence geometries*.

One might at first think that we could simply add an axiom that requires that all lines ~~have~~ contain an infinite set of points. But it is not that simple. We want the behavior of lines that is prescribed by the axioms to mimic the behavior of lines that we observe in our informal straight-line drawings. One aspect of that behavior has to do with the way that we measure distance in drawings.

In our drawings, we do not measure distance between two points P and Q by using the Euclidean distance formula. We find the distance between P and Q by putting a ruler alongside the line that contains P and Q. We use the ruler to get numbers that correspond to the points P and Q. These numbers are called the *coordinates* of P and Q. We then subtract the *coordinates* of P and Q to find the *distance between P and Q*.

We will define something analogous to this process in our incidence geometries. The thing that we will introduce is fittingly called a *ruler*. Just as a ruler in a drawing is related to the distance in a drawing, the ruler in an *incidence geometry* will be related to the *distance function* (if there is one). In other words, the definition of a *ruler* will only apply to *incidence geometries* that have *distance functions*.

Rulers in Axiomatic Geometry

Definition of a Ruler for a Line

words: f *is a ruler for line* l

alternate words: f *is a coordinate system for line* l

alternate words: f *is a coordinate function for line* l

usage: There is an incidence geometry $(\mathcal{P}, \mathcal{L})$ in the discussion, and there is a distance function d on the set of points $\mathcal{P}$ in the discussion, and $l \in \mathcal{L}$.

meaning: f is a function $f: l \to \mathbb{R}$ that satisfies these requirements

(i) f is a bijection.

(ii) f "agrees with" the distance function d in the following way:

For each pair of points P and Q (not necessarily distinct) on line l, this equation is true:

$$|f(P) - f(Q)| = d(P, Q)$$

Additional Terminology:

The equation above is called the **Ruler Equation**.

The number $f(P)$ is called the **coordinate of P with respect to f**.

Illustrating the Relationship Between a Ruler and the Distance Function

Observe that given points P and Q on a line l, there are two different processes that can be used to produce a real number.

Process #1:

Feed the pair of points (P, Q) into the *Distance Function on the Set of Points*, the function d, to get a real number, denoted $d(P, Q)$ called the *distance between P and Q*. This process could be illustrated with an arrow diagram:

$$\begin{array}{ccc} (P,Q) & \longmapsto & d(P,Q) \\ \mathcal{P} \times \mathcal{P} & \xrightarrow{d} & \mathbb{R} \end{array}$$

The bottom half of the diagram, we have seen before. It is the arrow diagram that tells us that the symbol d represents a function with domain $\mathcal{P} \times \mathcal{P}$ and range $\mathbb{R}$. The top part of the diagram has been added. It shows what happens to an actual pair of points.

$$d : \mathcal{P} \times \mathcal{P} \to \mathbb{R}$$

Process #2: (This is a two-step process.)

First Step: Let l be the line passing through points P and Q and let f be a *ruler* for line l. Feed point P into f to get a real number $f(P)$ (the *coordinate of* P) and feed point Q into f to get a real number $f(Q)$ (the *coordinate of* Q). This gives us a pair of real numbers, $\big(f(P), f(Q)\big)$.

Second Step: Feed the pair of real numbers $\big(f(P), f(Q)\big)$ into the *Distance Function on the Set of Real Numbers*, the function $d_{\mathbb{R}}$, to get a real number, denoted $d_{\mathbb{R}}\big(f(P), f(Q)\big)$. We know exactly how the Distance Function on the Set of Real Numbers works. The real number $d_{\mathbb{R}}\big(f(P), f(Q)\big)$ is just $|f(P) - f(Q)|$.

The two-step process can be illustrated with a two-step arrow diagram:

$$\begin{array}{ccccc} (P, Q) & \longmapsto & \big(f(P), f(Q)\big) & \longmapsto & |f(P) - f(Q)| \\ \mathcal{P} \times \mathcal{P} & \xrightarrow{\quad f \times f \quad} & \mathbb{R} \times \mathbb{R} & \xrightarrow{\quad d_{\mathbb{R}} \quad} & \mathbb{R} \end{array}$$

Having identified two different processes that can be used to turn a pair of points on a line l into a single real number, an obvious question is this:

Obvious Question: *Do the two processes give the same result*? That is, for any points P and Q on line l, and a ruler f for line l, does $d(P,Q)$ equal $|f(P)-f(Q)|$?

Answer: The fact that f is a ruler guarantees that the two results will always match.

$$d(P,Q) = |f(P)-f(Q)|$$

$$result\ of\ process\ \#1 = result\ of\ process\ \#2$$

The fact that these two processes always yield the same result can be *illustrated* by combining the two arrow diagrams into a single, larger diagram. In order to improve readability, we will bend the diagram for process #2. The resulting diagram is

$$\begin{tikzcd}
\mathcal{P} \times \mathcal{P} \arrow[r, "f \times f"] \arrow[dr, swap, "d"] & \mathbb{R} \times \mathbb{R} \arrow[d, "d_{\mathbb{R}}"] \arrow[dl, phantom, "\circleddash"] \\
& \mathbb{R}
\end{tikzcd}$$

In the diagram, we see that there are two different routes to get from a pair of points (that is, an element of $\mathcal{P} \times \mathcal{P}$) to the set of real numbers, $\mathbb{R}$. The slanting arrow is Process #1. The two-step path that goes straight across and then straight down is Process #2. The circled equal sign in the middle of the diagram indicates that these two paths always yield the same result. In diagram jargon, we say that the diagram *commutes*.

We can superimpose on the diagram some additional symbols that show what happens to an actual pair of points.

$$\begin{tikzcd}
(P,Q) \arrow[rrr, mapsto] \arrow[dddrr, mapsto] & & & \big(f(P),f(Q)\big) \arrow[ddd, mapsto] \\
& \mathcal{P}\times\mathcal{P} \arrow[r, "f\times f"] \arrow[dr, "d"'] & \mathbb{R}\times\mathbb{R} \arrow[d, "d_{\mathbb{R}}"] \arrow[dl, phantom, "="] & \\
& & \mathbb{R} & \\
& & d(P,Q) & |f(P)-f(Q)|
\end{tikzcd}$$

The two diagrams above may seem rather strange to you, but these sorts of diagrams are very common in higher-level math. Remember that the two diagrams are merely illustrations of what it means when we say that a function f is a ruler. They illustrate the relationship between a ruler f and the distance function d.

Metric Geometry

Definition of Metric Geometry

A ***metric geometry*** $\mathcal{M}$ is an ordered triple $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$ that satisfies the following:

- $(\mathcal{P}, \mathcal{L})$ is an *incidence geometry*.
- d is a *distance function* on the set of points $\mathcal{P}$
- Every line $l \in \mathcal{L}$ has a *ruler*. This is requirement is called the **Ruler Postulate**.

Four Examples of Metric Geometries

Proposition 2.2.4: The triple $(\mathbb{R}^2, \mathcal{L}_E, d_E)$ is consisting of the *Cartesian Plane* $\mathcal{C} = (\mathbb{R}^2, \mathcal{L}_E)$ *incidence geometry* along with the *Euclidean distance function* d_E satisfies the *Ruler Postulate*, so the triple $(\mathbb{R}^2, \mathcal{L}_E, d_E)$ is qualified to be called a *metric geometry*.

Definition: The **Euclidean Plane** $\mathcal{E}$ is defined to be the *metric geometry* $\mathcal{E} = (\mathbb{R}^2, \mathcal{L}_E, d_E)$.

Proposition 2.2.7: The triple $(\mathbb{R}^2, \mathcal{L}_E, d_T)$ is consisting of the *Cartesian Plane* $\mathcal{C} = (\mathbb{R}^2, \mathcal{L}_E)$ *incidence geometry* along with the *taxicab distance function* d_T satisfies the *Ruler Postulate*, so the triple $(\mathbb{R}^2, \mathcal{L}_E, d_T)$ is qualified to be called a *metric geometry*.

Definition: The **Taxicab Plane** $\mathcal{T}$ is defined to be the *metric geometry* $\mathcal{T} = (\mathbb{R}^2, \mathcal{L}_E, d_T)$.

Fact: The triple $(\mathbb{R}^2, \mathcal{L}_E, d_S)$ is consisting of the *Cartesian Plane* $\mathcal{C} = (\mathbb{R}^2, \mathcal{L}_E)$ *incidence geometry* along with the *max distance function* d_S satisfies the *Ruler Postulate*, so the triple $(\mathbb{R}^2, \mathcal{L}_E, d_S)$ is qualified to be called a *metric geometry*.

Definition: The **Max Plane** $\mathcal{M}$ is defined to be the *metric geometry* $\mathcal{M} = (\mathbb{R}^2, \mathcal{L}_E, d_S)$.

Proposition 2.2.6 The triple $(\mathbb{H}, \mathcal{L}_H, d_H)$ is consisting of the *Poincaré Plane* $\mathcal{H} = (\mathbb{H}, \mathcal{L}_H)$ *incidence geometry* along with the *Poincaré distance function* d_H satisfies the *Ruler Postulate*, so the triple $(\mathbb{H}, \mathcal{L}_H, d_H)$ is qualified to be called a *metric geometry*.

Definition: The **Poincaré Plane** $\mathcal{H}$ is defined to be the *metric geometry* $(\mathbb{H}, \mathcal{L}_H, d_H)$.

I will not provide proofs that any of these four "metric geometries" that I have presented are actually qualified to be called *metric geometries*.

- The book provides a detailed proof of Proposition 2.2.4 about the *Euclidean plane*.
- The book provides a partial proof of Proposition 2.2.7 about the *Taxicab plane*. You will be asked to finish that proof in your homework exercise 2.2#12.
- We will accept without proof that the Max plane $\mathcal{M}$ is qualified to be called a *metric geometry*.
- The book provides a detailed proof of Proposition 2.2.6 about the *Poincaré plane*.

Though I won't discuss the proof here, I will point out that part of the proofs involve producing ruler functions that work. The table on the next page presents rulers that do work..

Rulers for Some of our Incidence Geometries

The table below presents Rulers for the Cartesian plane with three different distance functions (*Euclidean distance*, *taxicab distance*, *max distance*), as well as for the *Poincaré plane* with the *Poincaré distance*.

$x\sqrt{1+m^2}$

Incidence Geometry	**Distance Function**	**Type of Line**	**Standard Ruler**
$\mathcal{C} = (\mathbb{R}^2, \mathcal{L}_E)$	d_E	$L_a = \{(a, y) \in \mathbb{R}^2\}$ $L_{m,b} = \{(x, y) \in \mathbb{R}^2 \mid y = mx + b\}$	$f(a, y) = y$ $f(x, y) = \cancel{x\sqrt{a + m^2}}$
$\mathcal{C} = (\mathbb{R}^2, \mathcal{L}_E)$	d_T	$L_a = \{(a, y) \in \mathbb{R}^2\}$ $L_{m,b} = \{(x, y) \in \mathbb{R}^2 \mid y = mx + b\}$	$f(a, y) = y$ $f(x, y) = x(1 + \lvert m \rvert)$
$\mathcal{C} = (\mathbb{R}^2, \mathcal{L}_E)$	d_S	$L_a = \{(a, y) \in \mathbb{R}^2\}$ $L_{m,b} = \{(x, y) \in \mathbb{R}^2 \mid y = mx + b\}$	We won't discuss
$\mathcal{H} = (\mathbb{H}, \mathcal{L}_H)$	d_H	$L_a = \{(a, y) \in \mathbb{H}\}$ ${}_cL_r = \{(x, y) \in \mathbb{H} \mid (x - c)^2 + y^2 = \cancel{r\ 2}\}$	$f(a, y) = \ln(y)$ $f(x, y) = \ln\left(\frac{x - c + r}{y}\right)$

r^2

Two Examples Involving Computing Coordinates of Given Points

[Example 1] Points on a *type I* line in the Poincaré plane

Let $A = (x_A, y_A) = (1,3)$ and $C = (x_C, y_C) = (1,5)$.

(a) Find the coordinates of A, C on the *Poincaré line* l that passes through them, using the standard ruler for line l. notice $X_A = X_C = 1$

So the poincaré line that passes through A, C is the type I line $_1L$

So we use the ruler $f(1,y) = \ln(y)$

$f(A) = f(1,3) = \ln(3) \approx 1.099$ $\quad f(C) = f(1,5) = \ln(5) \approx 1.609$

exact approx

(b) Use the coordinates to find ~~the distance between A and C.~~

$|f(A) - f(C)|$

$$|f(A) - f(C)| = |\ln(3) - \ln(5)| = \left|\ln\left(\frac{3}{5}\right)\right| = \left|-\ln\left(\frac{5}{3}\right)\right| = \ln\left(\frac{5}{3}\right) \approx 0.511$$

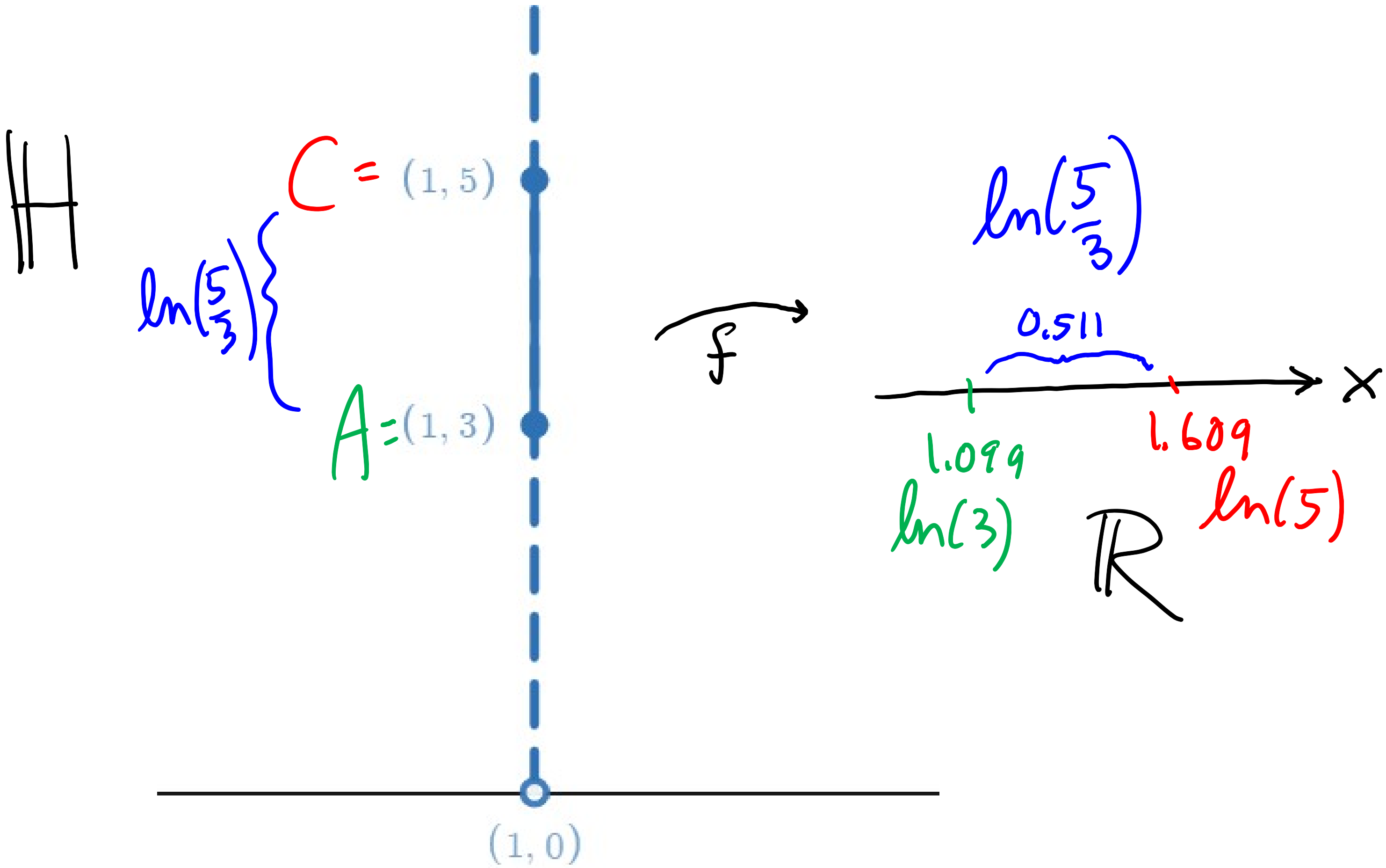
(c) Use the *distance function* d_H to find the distance between A and C.

$$d_H(A, C) = \left|\ln\left(\frac{X_C}{X_A}\right)\right| = \left|\ln\left(\frac{5}{3}\right)\right| = \ln\left(\frac{5}{3}\right) \approx 0.511$$

(d) Compare the results from **(b)** and **(c)**. Is the *ruler equation* satisfied?

yes! $|f(A) - f(C)| = \ln\left(\frac{5}{3}\right) = d_H(A, C)$

(e) Illustrate your results from **(a)**, **(c)**.



End of [Example 1]

[Example 2] Points With Different x Coordinates in the *Euclidean plane*

Let $B = (x_B, y_B) = (1,4)$ and $D = (x_D, y_D) = (8,3)$.

(a) Find the coordinates of B, D on the *line* l that passes through them, using the *standard ruler* f for that line in the *Euclidean plane* $\mathcal{E}$

Observe that $x_B \neq x_D$ so line l is a non-vertical line.

Slope $m = \frac{\Delta y}{\Delta x} = \frac{3-4}{8-1} = -\frac{1}{7}$

So the ruler is $f(x,y) = x\sqrt{1+m^2} = x\sqrt{1+\left(-\frac{1}{7}\right)^2} = x\sqrt{1+\frac{1}{49}} = x\sqrt{\frac{50}{49}} = \frac{x\sqrt{50}}{7}$

$f(B) = f(1,4) = \frac{1\sqrt{50}}{7} = \frac{\sqrt{50}}{7} \approx 1.010$

$f(D) = f(8,3) = \frac{8\sqrt{50}}{7} \approx 8.081$

(b) Compute $|f(B) - f(D)|$

$|f(B) - f(D)| = \left|\frac{\sqrt{50}}{7} - \frac{8\sqrt{50}}{\sqrt{7}}\right| = \left|\frac{-7\sqrt{50}}{7}\right| = \sqrt{50} \approx 7.07$

exact

approx

(c) Use the *Euclidean distance function* d_E to find the *Euclidean distance* between B and D.

Solution

$$d_E(B,D) = \sqrt{\Delta x^2 + \Delta y^2} = \sqrt{(1-8)^2 + (4-3)^2} = \sqrt{(-7)^2 + 1^2}$$

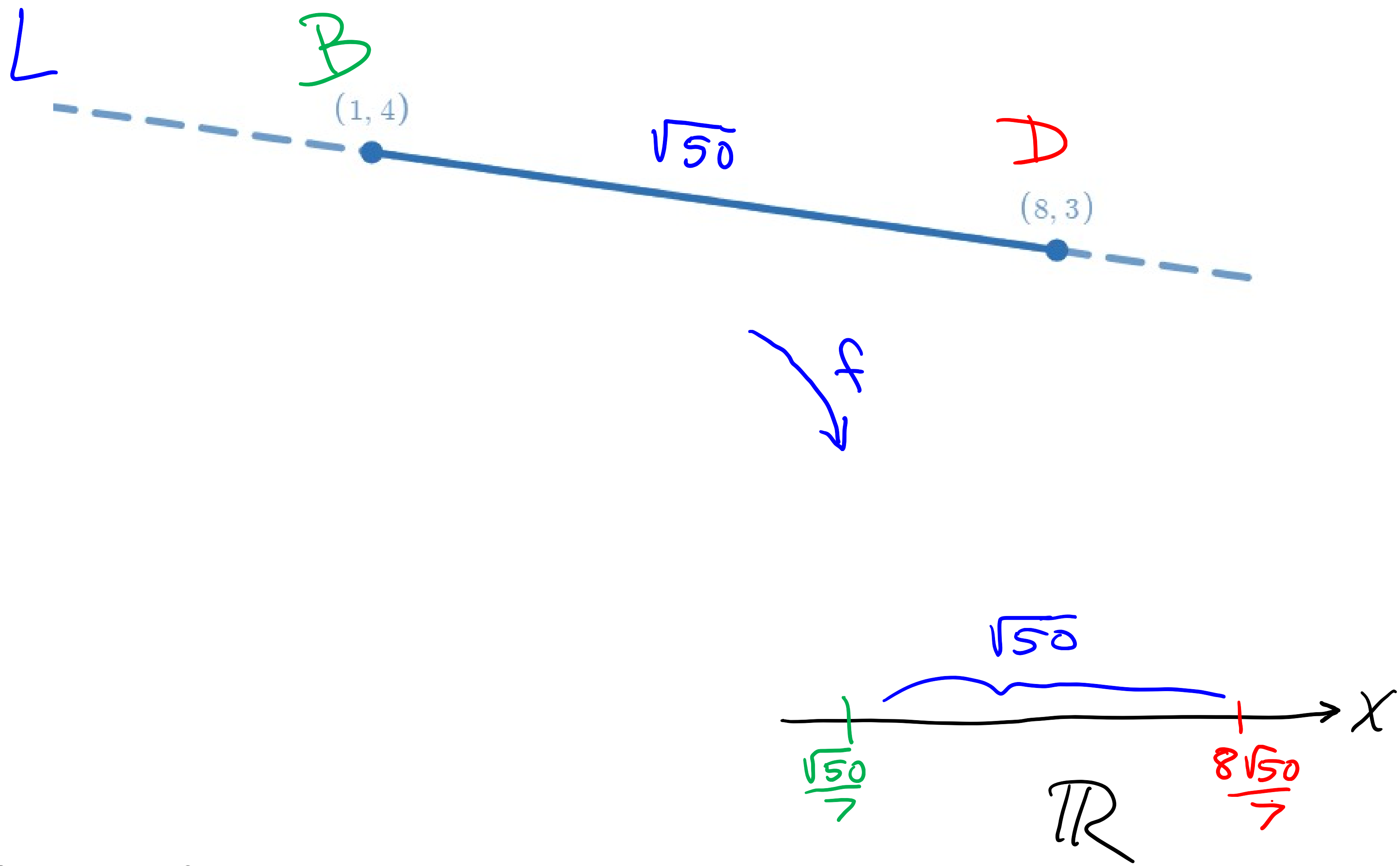
$$= \sqrt{50} \approx 7.07$$

(d) Compare the results from **(b)** and **(c)**. Is the *ruler equation* satisfied?

yes

$$|f(B) - f(D)| = \sqrt{50} = d_E(B,D)$$

(e) Illustrate your results from **(a)**, **(c)**.



End of [Example 2]

[Example 3] Points With Different x Coordinates in the *Taxicab Plane*

Let $B = (x_B, y_B) = (1,4)$ and $D = (x_D, y_D) = (8,3)$.

(a) Find the coordinates of B, D on the *line l* that passes through them, using the *standard ruler f* for that line in the *taxicab plane $\mathcal{T}$*

Line l is non-vertical, with slope $m = -\frac{1}{7}$

So the standard ruler has formula

$$f(x,y) = x(1+|m|) = x\left(1+\left|-\frac{1}{7}\right|\right) = x\left(1+\frac{1}{7}\right) = \frac{x \cdot 8}{7}$$

$$f(B) = f(1,4) = 1\left(\frac{8}{7}\right) = \frac{8}{7} \approx 1.143$$

$$f(D) = f(8,3) = 8\left(\frac{8}{7}\right) = \frac{64}{7} \approx 9.143$$

(b) Compute $|f(B) - f(D)|$

$$|f(B) - f(D)| = \left|\frac{8}{7} - \frac{64}{7}\right| = \left|\frac{-56}{7}\right| = |-8| = 8$$

(c) Use the *Taxicab distance function* d_T to find the *Taxicab distance* between B and D.

$$d_T(B,D) = |x_1 - x_2| + |y_1 - y_2| = |1-8| + |4-3| =$$

$$= |-7| + |-1| = 7+1$$

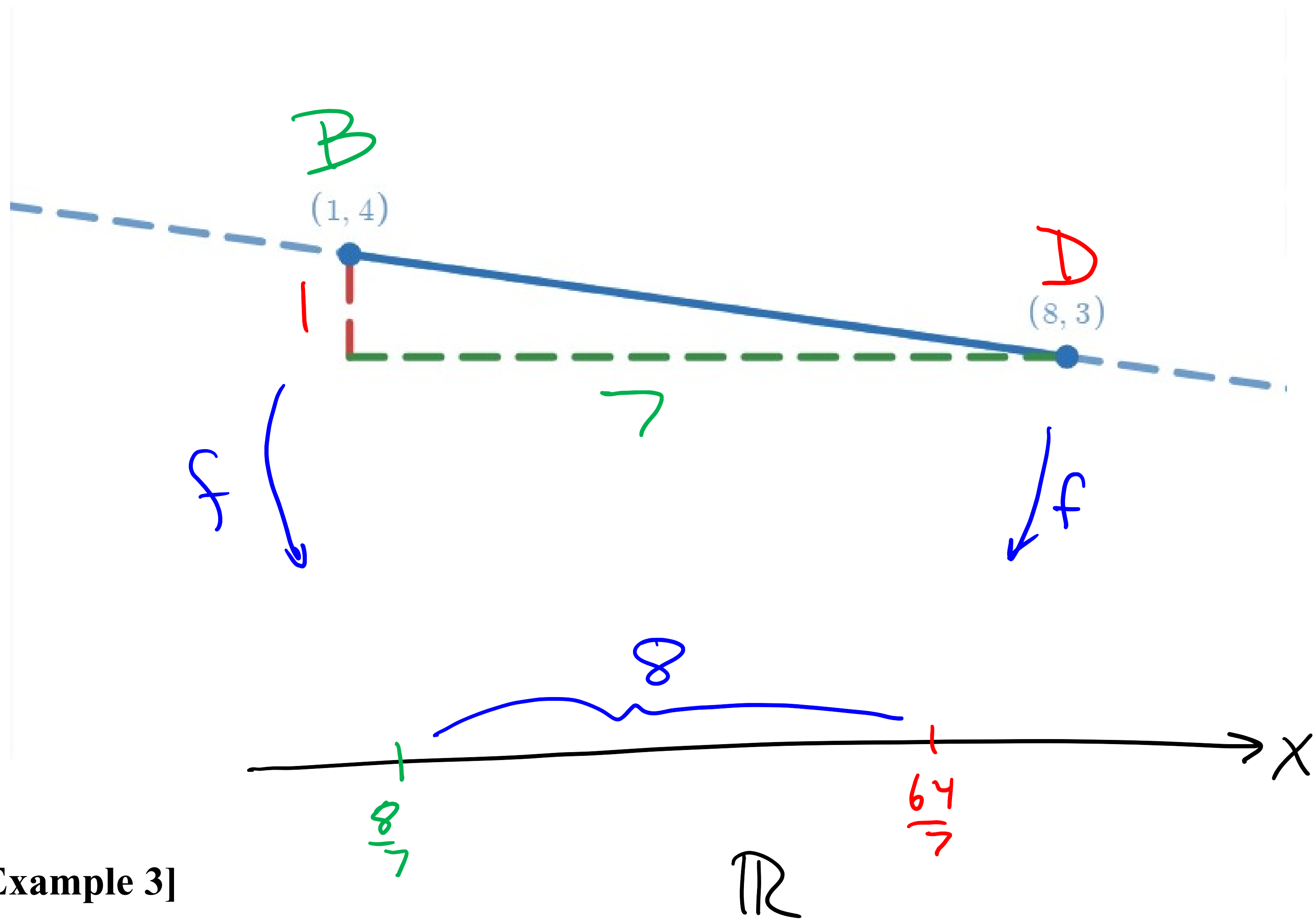
$$= 8$$

(d) Compare the results from **(b)** and **(c)**. Is the *ruler equation* satisfied?

yes

$$|f(B) - f(D)| = 8 = d_T(B,D)$$

(e) Illustrate your results from **(a)**, **(c)**.



End of [Example 3]

[Example 4] Points With Different x Coordinates in the *Poincaré plane*

Let $B = (x_B, y_B) = (1,4)$ and $D = (x_D, y_D) = (8,3)$.

(a) Find the coordinates of B, D on the *line* l that passes through them, using the *standard ruler* f for that line in the *Poincaré plane* $\mathcal{H}$.

The line is a type II line because $X_B \neq X_D$

must find its center

$$c = \frac{x_2^2 - x_1^2 + y_2^2 - y_1^2}{2(x_2 - x_1)} = \frac{8^2 - 1^2 + 3^2 - 4^2}{2(8-1)} = \cdots = 4$$

$$r = \sqrt{(x_1 - c)^2 + y_1^2} = \sqrt{(1-4)^2 + 4^2} = \sqrt{3^2 + 4^2} = 5$$

So the ruler function is

$$f(x,y) = \ln\left(\frac{x - c + r}{y}\right) = \ln\left(\frac{x - 4 + 5}{y}\right) = \ln\left(\frac{x+1}{y}\right)$$

$$f(B) = f(1,4) = \ln\left(\frac{1+1}{4}\right) = \ln\left(\frac{2}{4}\right) = \ln\left(\frac{1}{2}\right) = \underline{-\ln(2)} \approx \underline{-0.693}$$

$$f(D) = f(8,3) = \ln\left(\frac{8+1}{3}\right) = \underline{\ln(3)} \approx \underline{1.099}$$

(b) Compute $|f(B) - f(D)|$

$$|f(B) - f(D)| = |-\ln(2) - \ln(3)| = \left|\ln\left(\tfrac{1}{2}\right) + \ln\left(\tfrac{1}{3}\right)\right|$$

$$= \left|\ln\left(\tfrac{1}{2} \cdot \tfrac{1}{3}\right)\right| = \left|\ln\left(\tfrac{1}{6}\right)\right| = |-\ln(6)|$$

$$= \underline{\ln(6)} \approx \underline{1,792}$$

(c) Use the *Poincaré distance function* d_H to find the *Poincaré distance* between B and D.

$$d_H(B,D) = \left|\ln\left(\frac{\frac{x_1-c+r}{y_1}}{\frac{x_2-c+r}{y_2}}\right)\right| = \left|\ln\left(\frac{\frac{1-4+5}{4}}{\frac{8-4+5}{3}}\right)\right| =$$

$$= \left|\ln\left(\frac{\frac{2}{4}}{\frac{9}{3}}\right)\right| = \left|\ln\left(\frac{\frac{1}{2}}{3}\right)\right| = \left|\ln\left(\frac{1}{6}\right)\right|$$

$$= \left|-\ln(6)\right|$$

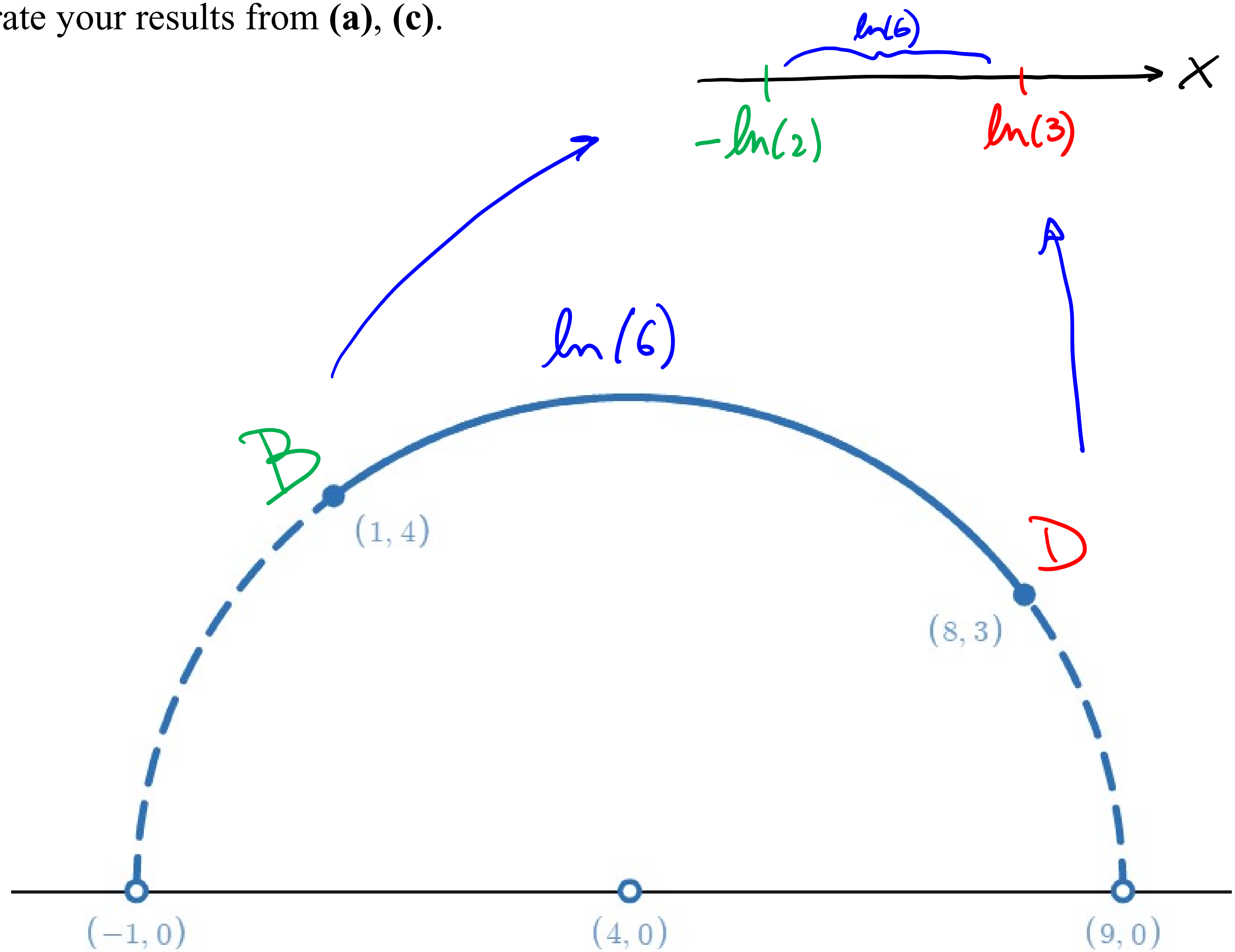
$$= \ln(6) \approx 1.792$$

(d) Compare the results from **(b)** and **(c)**. Is the *ruler equation* satisfied?

yes

$$|f(B) - f(D)| = \ln(6) = d_{+1}(B, D)$$

(e) Illustrate your results from **(a)**, **(c)**.



End of [Example 4]

Finding the Point That Has a Given Coordinate on a Given Line With a Given Ruler

Suppose that a line L in a metric geometry $(\mathcal{P}, \mathcal{L}, d)$ has a ruler f.

Then f is a *function*, $f: L \rightarrow \mathbb{R}$.

Therefore, the *inverse relation* f^{-1} will be a *relation* from $\mathbb{R}$ to L. Furthermore, since f is *bijective*, the *inverse relation* f^{-1} will be qualified to be called an *inverse function*.

It is possible to find the *general formula* for the inverse function f^{-1} for each of the *standard rulers*. This will involve solving equations. Some of the equations are terribly messy, though. For that reason, I won't discuss finding the general formula for the inverse function f^{-1} for in these notes. (The textbook also does not discuss it.) But I will present some examples involving finding the point that has a given coordinate on a given line with a given ruler. You have some homework exercises of this type.

In solving these problems, it is helpful to use the symbol λ to denote the value of a coordinate.

[Example 5] Finding point with specified coordinate on lines in the *Euclidean plane*

(a) Find the point P on the *vertical line* L_3 that has *coordinate* $\lambda = 5$ in the *standard ruler* for that line in the *Euclidean plane*.

Solution: Since P is on the *vertical line* L_3, we know that P must be of the form $P = (3, y)$.

The standard ruler for the *vertical line* L_a in the *Euclidean plane* is the function

$$\lambda = f(a, y) = y$$

We are given $a = 3$ and $\lambda = 5$. Therefore, $y = \lambda = 5$. Thus, P is the point $P = (3,5)$.

(b) Find the point P on the *non-vertical line* $L_{3,-4}$ that has *coordinate* $\lambda = 5$ in the *standard ruler* for that line in the *Euclidean plane*.

Solution: The standard ruler the *non-vertical line* $L_{3,-4}$ in the *Euclidean plane* is the function

$$\lambda = f(x, y) = x\sqrt{1 + m^2} = x\sqrt{1 + 3^2} = x\sqrt{10}$$

We are given $\lambda = 5$. So, $\lambda = 5 = x\sqrt{10}$. Solving for x, we obtain $x = \frac{5}{\sqrt{10}}$.

Since P is on the *non-vertical line* $L_{3,-4}$ we know y is obtained from the equation $y = 3x - 4$.

Therefore, $y = 3\left(\frac{5}{\sqrt{10}}\right) - 4 = \frac{15}{\sqrt{10}} - 4$. So $P = \left(\frac{5}{\sqrt{10}}, \frac{15}{\sqrt{10}} - 4\right) \approx (1.581,0.743)$.

End of [Example 5]

[Example 6] Finding point with specified coordinate on lines in the *taxicab plane*

(a) Find the point P on the *vertical line* L_3 that has *coordinate* $\lambda = 5$ in the *standard ruler* for that line in the *taxicab plane*.

Solution: Since P is on the *vertical line* L_3, we know that P must be of the form $P = (3, y)$.

The standard ruler for the *vertical line* L_a in the *taxicab plane* is the function

$$\lambda = f(a, y) = y$$

We are given $a = 3$ and $\lambda = 5$. Therefore, $y = \lambda = 5$. Thus, P is the point $P = (3,5)$.

(b) Find the point P on the *non-vertical line* $L_{3,-4}$ that has *coordinate* $\lambda = 5$ in the *standard ruler* for that line in the *taxicab plane*.

Solution: The standard ruler the *non-vertical line* $L_{3,-4}$ in the *taxicab plane* is the function

$$\lambda = f(x, y) = x(1 + |m|) = x(1 + |3|) = 4x$$

We are given $\lambda = 5$. So, $\lambda = 5 = 4x$. Solving for x, we obtain $x = \frac{5}{4}$.

Since P is on the *non-vertical line* $L_{3,-4}$ we know y is obtained from the equation $y = 3x - 4$.

Therefore, $y = 3\left(\frac{5}{4}\right) - 4 = \frac{15}{4} - 4 = -\frac{1}{4}$. So $P = \left(\frac{5}{4}, -\frac{1}{4}\right) = (1.25, -0.25)$.

End of [Example 6]

[Example 7] Finding point with specified coordinate on lines in the *Poincaré plane*

(a) Find the point P on the *type I line* ${}_3L$ that has *coordinate* $\lambda = 5$ in the *standard ruler* for that line in the *Poincaré plane*.

Solution: Since P is on the *type I line* ${}_3L$, we know that P must be of the form $P = (3, y)$.

The standard ruler for the *vertical line* L_a in the *Poincaré plane* is the function

$$\lambda = f(a, y) = \ln(y)$$

We are given $a = 3$ and $\lambda = 5$. Therefore, $\lambda = \ln(y) = \cancel{\ln(5)}$ 5. Solving this equation for y, we obtain $y = e^{(5)}$. Thus, P is the point $P = \left(3, e^{(5)}\right)$.

(b) Find the point P on the *type II line* ${}_2L_{\sqrt{3}}$ that has *coordinate* $\lambda = \ln(5)$ in the *standard ruler* for that line in the *Poincaré plane*.

Solution: The standard ruler the *type II line* ${}_2L_{\sqrt{3}}$ in the *Poincaré plane* is the function

$$\lambda = f(x, y) = \ln\left(\frac{x - c + r}{y}\right) = \ln\left(\frac{x - 2 + \sqrt{3}}{y}\right)$$

We are given $\lambda = \ln(5)$. That is,

$$\ln\left(\frac{x - 2 + \sqrt{3}}{y}\right) = \ln(5)$$

Solving this ~~eqution~~ equation for x in terms of y, we obtain

$$x = 5y + 2 - \sqrt{3}$$

This is what we will call the *coordinate equation*.

Because P lies on the line ${}_2L_{\sqrt{3}}$, we know that (x, y) must also satisfy the *circle equation*

$$(x-2)^2 + y^2 = \left(\sqrt{3}\right)^2 = 3$$

So we have two equations in x, y

$$\begin{cases} x = 5y + 2 - \sqrt{3} & \textit{coordinate equation} \\ (x-2)^2 + y^2 = \left(\sqrt{3}\right)^2 = 3 & \textit{circle equation} \end{cases}$$

We can solve for y by substituting the *coordinate equation* into the *circle equation*.

$$\begin{aligned} 3 &= (x-2)^2 + y^2 \\ &= \left(\left(5y + 2 - \sqrt{3}\right) - 2\right)^2 + y^2 \\ &= \left(5y - \sqrt{3}\right)^2 + y^2 \\ &= 25y^2 - 10y\sqrt{3} + 3 + y^2 \\ &= 26y^2 - 10y\sqrt{3} + 3 \end{aligned}$$

Subtracting 3 from both sides and factoring the right side, we obtain

$$\begin{aligned} 0 &= 26y^2 - 10y\sqrt{3} \\ &= 2y(13y - 5\sqrt{3}) \end{aligned}$$

The solutions to this equation are $y = 0$ and $y = \frac{5\sqrt{3}}{13}$. But $y = 0$ is not allowed, because our point must be in the upper half plane. (And notice that the original coordinate equation

$$\ln\left(\frac{x - 2 + \sqrt{3}}{y}\right) = \ln(5)$$

is not even defined when $y = 0$!!) So we conclude that $y = \frac{5\sqrt{3}}{13}$.

We find the corresponding x value by using the value of y in the *coordinate equation*.

$$x = 5y + 2 - \sqrt{3} = 5\left(\frac{5\sqrt{3}}{13}\right) + 2 - \sqrt{3} = \cdots = \frac{26 + 12\sqrt{3}}{13}$$

So point P has coordinates

$$P = (x, y) = \left(\frac{26 + 12\sqrt{3}}{13}, \frac{5\sqrt{3}}{13}\right) \approx (3.599, 0.666)$$

Remarks:

(1) One can get this approximate result from Wolfram Alpha by typing

Solve x=5y+2-sqrt(3) and (x-2)^2+y^2=3

(2) One can check the result most easily by substituting the decimal approximation

$$P = (x, y) = \left(\frac{26+12\sqrt{3}}{13}, \frac{5\sqrt{3}}{13}\right) \approx (3.599, 0.666)$$

into the special *ruler function*

$$f(x, y) = \ln\left(\frac{x-c+r}{y}\right) = \ln\left(\frac{x-2+\sqrt{3}}{y}\right)$$

The result is

$$f(P) = f(3.599, 0.666) = \ln\left(\frac{3.599-2+\sqrt{3}}{0.666}\right) \approx 1.609$$

By comparison,

$$\lambda = \ln(5) \approx 1.609$$

End of [Example 7]

End of Video