2.3: Special Coordinate Systems

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for Ohio University MATH 3110/5110 College Geometry

Topics:

- Getting New Rulers from Old Rulers
  - Ruler Sliding
  - Ruler Flipping
- Special Rulers
  - The Ruler Placement Theorem
  - Procedure for Finding a Ruler with A as Origin and B Positive
  - Examples

Reading: Section 2.3 Special Coordinate Systems (p 37 – 40) in *Geometry: A Metric Approach with Models, Second Edition* by Millman & Parker (Springer, 1991, ISBN 3-540-97412-1)

Homework: Section 2.3 # 1, 2, 3, 4, 5, 6

Recall the definitions of Surjective and Injective from Chapter 1

Definition of Surjective Function

Words: f *is* ***surjective***, or f *is a* ***surjection***

Alternate Words: f *is* ***onto***

Usage: f is a function, $f: X \to Y$

Meaning: For every element in the range, there exists an element of the domain (***at least one***) that can be used as input to cause that element of the range to be output.

Meaning Written Formally: $\forall y \in Y \big(\exists x \in X(f(x) = y)\big)$

Definition of Injective Function

Words: f *is* ***injective***, or f *is an* ***injection***, or f is ***one-to-one***

Usage: f is a function, $f: X \to Y$

Meaning: If two inputs cause outputs that are equal, then the inputs must be equal.

Meaning Written Formally: $\forall x_1, x_2 \in X(If\ f(x_1) = f(x_2)\ then\ x_1 = x_2)$

Recall Important Things From Section 2.1

Definition of Abstract Geometry

An ***abstract geometry*** $\mathcal{A}$ is an ordered pair $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ where $\mathcal{P}$ denotes a set whose elements are called **points** and $\mathcal{L}$ denotes a non-empty set whose elements are called **lines**, which are **sets of points** satisfying the following two requirements, called ***axioms***:

(i) For every two distinct points $A, B \in \mathcal{P}$, there exists **at least one** line $l \in \mathcal{L}$ such that $A \in l$ and $B \in l$.

(ii) For every line $l \in L$ there exist at least two distinct points that are elements of the line.

Definition of Incidence Geometry

An ***incidence geometry*** $\mathcal{A}$ is an *abstract geometry* $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ that satisfies the following two additional requirements, called ***axioms***:

(i) For every two distinct points $A, B \in \mathcal{P}$, there exists **exactly one** line $l \in \mathcal{L}$ such that $A \in l$ and $B \in l$.

(ii) There exist (at least) three *non-collinear* points.

Procedure for Finding the *Cartesian* Line Passing through Two Distinct Points in $\mathbb{R}^2$

Suppose $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are any two distinct points of $\mathbb{R}^2$.

If $x_1 = x_2$ then let $a = x_1 = x_2$. In this case, $L_a \in \mathcal{L}_H$ and $P, Q \in L_a$.

If $x_1 \neq x_2$ then define constants m, b by the following formulas:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$b = y_2 - m x_2$$

Then $P, \in L_{m,b}$ and $Q \in L_{m,b}$.

Procedure for Finding the *Poincaré Line* Passing Through Two Distinct Points in $\mathbb{H}$

Suppose $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are any two distinct points of $\mathbb{H}$.

If $x_1 = x_2$ then let $a = x_1 = x_2$. In this case, ${}_aL \in \mathcal{L}_H$ and $P, Q \in {}_aL$.

If $x_1 \neq x_2$ then define constants c, r by the following formulas:

$$c = \frac{x_2^2 - x_1^2 + y_2^2 - y_1^2}{2(x_2 - x_1)}$$

$$r = \sqrt{(x_1 - c)^2 + y_1^2}$$

In this case, ${}_cL_r \in \mathcal{L}_H$ and $P, Q \in {}_cL_r$.

And important things from Section 2.2

Definition of Distance Function

words: d is a distance function on set S

meaning: d is a function $d: S \times S \to \mathbb{R}$ that satisfies these requirements

(i) $\forall P, Q \in S(d(P, Q) \geq 0)$

(ii) $d(P, Q) = 0$ if and only if $P = Q$

(iii) $d(P, Q) = d(Q, P)$

Definition of the Euclidean Distance Function on $\mathbb{R}^2$

symbol: d_E

meaning: the function $d_E: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$d_E\big((x_1, y_1), (x_2, y_2)\big) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Definition of the Taxicab Distance Function on $\mathbb{R}^2$

symbol: d_T

meaning: the function $d_T: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$d_T\big((x_1, y_1), (x_2, y_2)\big) = |x_1 - x_2| + |y_1 - y_2|$$

Definition of the Poincaré Distance Function on $\mathbb{H}$

symbol: d_H

meaning: the function $d_H: \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ defined by in the following way

Suppose $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are any two points of $\mathbb{H}$.

If $x_1 = x_2$ then compute the distance between them using the formula

$$d_H(P,Q) = \left|\ln\left(\frac{y_2}{y_1}\right)\right|$$

If $x_1 \neq x_2$ then compute the distance between them using the formula

$$d_H(P,Q) = \left|\ln\left(\frac{\frac{x_1 - c + r}{y_1}}{\frac{x_2 - c + r}{y_2}}\right)\right|$$

where c, r are the constants describing the *type II* line that passes through P and Q.

Definition of a Ruler for a Line

words: f *is a ruler for line* l

alternate words: f *is a coordinate system for line* l

alternate words: f *is a coordinate function for line* l

usage: There is an incidence geometry $(\mathcal{P}, \mathcal{L})$ in the discussion, and there is a distance function d on the set of points $\mathcal{P}$ in the discussion, and $l \in \mathcal{L}$.

meaning: f is a function $f: l \to \mathbb{R}$ that satisfies these requirements

(i) f is a bijection.

(ii) f "agrees with" the distance function d in the following way:

For each pair of points P and Q (not necessarily distinct) on line l, this equation is true:

$$|f(P) - f(Q)| = d(P, Q)$$

Additional Terminology:

The equation above is called the **Ruler Equation**.

The number $f(P)$ is called the **coordinate of P with respect to f**.

Definition of Metric Geometry

A ***metric geometry*** $\mathcal{M}$ is an ordered triple $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$ that satisfies the following:

- $(\mathcal{P}, \mathcal{L})$ is an *incidence geometry*.
- d is a *distance function* on the set of points $\mathcal{P}$
- Every line $l \in \mathcal{L}$ has a *ruler*. This is requirement is called the **Ruler Postulate**.

Proposition 2.2.4: The triple $(\mathbb{R}^2, \mathcal{L}_E, d_E)$ is consisting of the *Cartesian Plane* $\mathcal{C} = (\mathbb{R}^2, \mathcal{L}_E)$ *incidence geometry* along with the *Euclidean distance function* d_E satisfies the *Ruler Postulate*, so the triple $(\mathbb{R}^2, \mathcal{L}_E, d_E)$ is qualified to be called a *metric geometry*.

Definition: The **Euclidean Plane** $\mathcal{E}$ is defined to be the *metric geometry* $\mathcal{E} = (\mathbb{R}^2, \mathcal{L}_E, d_E)$.

Proposition 2.2.7: The triple $(\mathbb{R}^2, \mathcal{L}_E, d_T)$ is consisting of the *Cartesian Plane* $\mathcal{C} = (\mathbb{R}^2, \mathcal{L}_E)$ *incidence geometry* along with the *taxicab distance function* d_T satisfies the *Ruler Postulate*, so the triple $(\mathbb{R}^2, \mathcal{L}_E, d_T)$ is qualified to be called a *metric geometry*.

Definition: The **Taxicab Plane** $\mathcal{T}$ is defined to be the *metric geometry* $\mathcal{T} = (\mathbb{R}^2, \mathcal{L}_E, d_T)$.

Proposition 2.2.6 The triple $(\mathbb{H}, \mathcal{L}_H, d_H)$ is consisting of the *Poincaré Plane* $\mathcal{H} = (\mathbb{H}, \mathcal{L}_H)$ *incidence geometry* along with the *Poincaré distance function* d_H satisfies the *Ruler Postulate*, so the triple $(\mathbb{H}, \mathcal{L}_H, d_H)$ is qualified to be called a *metric geometry*.

Definition: The **Poincaré Plane** $\mathcal{H}$ is defined to be the *metric geometry* $(\mathbb{H}, \mathcal{L}_H, d_H)$.

Rulers for Some of our Geometries

Incidence Geometry	**Distance Function**	**Type of Line**	**Standard Ruler**
$\mathcal{C} = (\mathbb{R}^2, \mathcal{L}_E)$	d_E	$L_a = \{(a, y) \in \mathbb{R}^2\}$ $L_{m,b} = \{(x, y) \in \mathbb{R}^2 \mid y = mx + b\}$	$f(a, y) = y$ $f(x, y) = x\sqrt{1 + m^2}$
$\mathcal{C} = (\mathbb{R}^2, \mathcal{L}_E)$	d_T	$L_a = \{(a, y) \in \mathbb{R}^2\}$ $L_{m,b} = \{(x, y) \in \mathbb{R}^2 \mid y = mx + b\}$	$f(a, y) = y$ $f(x, y) = x(1 + \lvert m \rvert)$
$\mathcal{C} = (\mathbb{R}^2, \mathcal{L}_E)$	d_S	$L_a = \{(a, y) \in \mathbb{R}^2\}$ $L_{m,b} = \{(x, y) \in \mathbb{R}^2 \mid y = mx + b\}$	We won't discuss
$\mathcal{H} = (\mathbb{H}, \mathcal{L}_H)$	d_H	$L_a = \{(a, y) \in \mathbb{H}\}$ ${}_cL_r = \left\{(x, y) \in \mathbb{H} \mid (x - c)^2 + y^2 = r \quad 2\right\}$	$f(a, y) = \ln(y)$ $f(x, y) = \ln\left(\frac{x - c + r}{y}\right)$

Previously Overlooked Definition

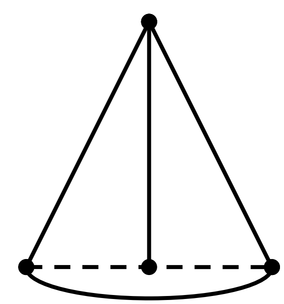
I overlooked a useful definition when we were discussing Abstract Geometry and Incidence Geometry.

Recall that in an *abstract geometry* $\mathcal{A} = (\mathcal{P}, \mathcal{L})$, for every two distinct points $A, B \in \mathcal{P}$, there exists **at least one** line $l \in \mathcal{L}$ such that $A \in l$ and $B \in l$. For two particular distinct points A, B it is possible that is *more than one line* $l \in \mathcal{L}$ such that $A \in l$ and $B \in l$.

We have seen examples of abstract geometries where this happens.

For example, it happens in the *abstract geometry* $(\mathcal{P}, \mathcal{L})$ with

- points $\mathcal{P} = \{A, B, C, D\}$
- lines $\mathcal{L} = \big\{\{A, B\}, \{A, C\}, \{A, D\}, \{B, C, D\}, \{B, D\}\big\}$

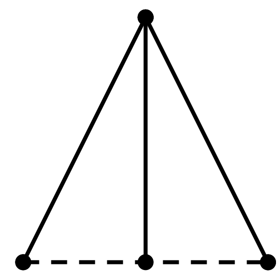


In this geometry, we cannot say the phrase

the line that both B and D lie on

because that phrase is ambiguous. There are *two* lines that both B and D lie on. More information would be needed to specify which line is being referred to.

But in an *incidence geometry* $\mathcal{A} = (\mathcal{P}, \mathcal{L})$, for, for every two distinct points $A, B \in \mathcal{P}$, there exists **exactly one** line $l \in \mathcal{L}$ such that $A \in l$ and $B \in l$.



For example, in the *incidence geometry* $(\mathcal{P}, \mathcal{L})$ with

- points $\mathcal{P} = \{A, B, C, D\}$
- lines $\mathcal{L} = \big\{\{A, B\}, \{A, C\}, \{A, D\}, \{B, C, D\}\big\}$

we *can* the phrase

the line that both B and D lie on

because there is no ambiguity. There is only *one* line that both B and D lie on. This is the idea behind the following notation

Definition of Notation for the Unique Line Containing Two Given Points

Symbol: $\overleftrightarrow{AB}$

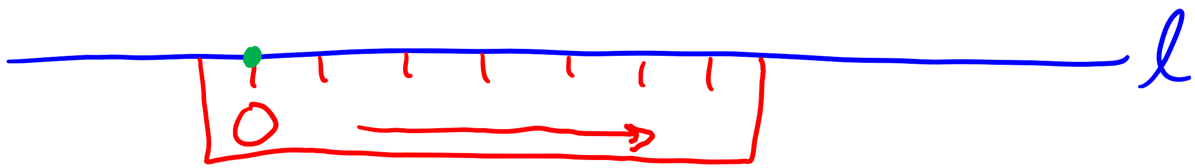
Spoken: *line A B*

Usage: There is an *incidence geometry* $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ in the discussion and $A, B \in \mathcal{P}$ are two distinct points

Meaning: the unique line $l \in \mathcal{L}$ such that $A \in l$ and $B \in l$.

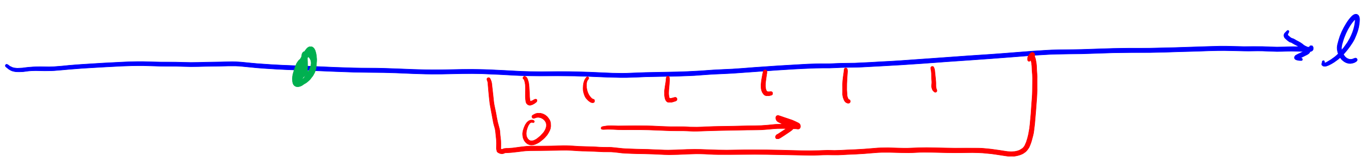
Getting New Rulers from Old Rulers

In a *drawing*, given a *ruler* lying along a *line*,

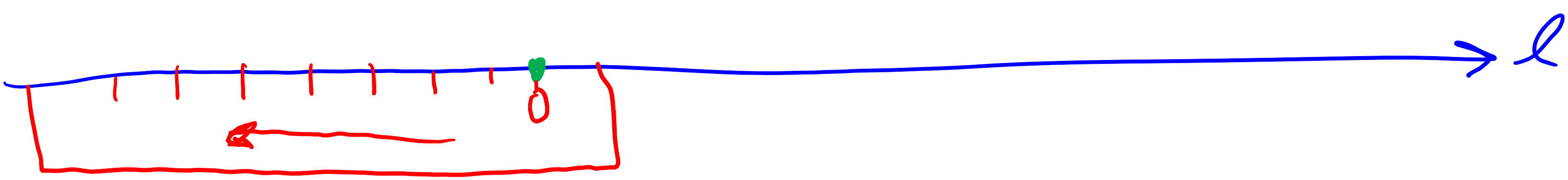


there are two simple operations that can be done to obtain a *new ruler*:

Sliding: Slide the ruler along the line.



Flipping: Keep the 0 of the ruler pinned to its current point on the line, but flip the ruler over so that it points in the other direction.



In a *metric geometry* (a geometry that has a *distance function* and in which every line has a ruler, we have analogous operations that can be done on a *given ruler* to obtain a *new ruler*.

Ruler Sliding Theorem

Given

- a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$
- a line $l \in \mathcal{L}$
- a ruler f for line l
- a number $a \in \mathbb{R}$

Claim: The function $g: l \to \mathbb{R}$ defined by

$$g(P) = f(P) - a$$

qualifies to be called a ruler for line l.

Remark: The new ruler g is analogous to the ruler that one would obtain by *sliding* a ruler f along a line l in a drawing.

Proof of Ruler Sliding Theorem

We must show that g fulfills the requirements of a ruler for line l.

Part 1: Prove that g is indeed a function with *domain* l and *range* $\mathbb{R}$.

First, note that for any $Q \in l$, the symbol $g(Q)$ does represent a real number.

$$g(Q) = f(Q) - a \in \mathbb{R}$$

So g is indeed a function with *domain* l and *range* $\mathbb{R}$. That is, the symbol $g: l \rightarrow \mathbb{R}$ is valid.

(Mathematicians would say that g is *well-defined*.)

Part 2: Prove that g is *surjective*.

Recall what it means to say that a function is *surjective*:

*For every element in the range, there exists an element of the domain (**at least one**) that can be used as input to cause that element of the range to be output.*

To prove that a function $g: l \to \mathbb{R}$ is surjective, we must show the following:

For every real number t, there exists a point P on line l (at least one) such that $g(P) = t$

Proof that g is *surjective*.

(1) Suppose $t \in \mathbb{R}$

(2) Then $t + a \in \mathbb{R}$ (because the real numbers are closed under addition)

(3) There exists a point $P \in l$ such that $f(P) = t + a$. (because f is known to be surjective, because f is a ruler)

(4) Then $g(P) = f(P) - a = (t + a) - a = t$. (by (3) and definition of g)

(6) We have shown that there exists a point $P \in l$ such that $g(P) = t$. (by 3,4)

(7) Thus, g is *surjective*. (by (1),(6), and definition of *surjective*)

End of proof

Part 3: Prove that *g* is *injective.*

Recall what it means to say that a function is *injective*:

If two inputs cause outputs that are equal, then the inputs must be equal.

To prove that a function $g: l \to \mathbb{R}$ is injective, we must show the following:

If $P, Q \in l$ *and* $g(P) = g(Q)$*, then* $P = Q$.

Proof that *g* is *injective*. Direct Proof

(1) Suppose $P, Q \in l$ and $g(P) = g(Q)$. unpack definition of g

(2) Then $f(P) - a = f(Q) - a$. (by (1) and definition of g)

(3) Therefore $f(P) = f(Q)$. (subtracted a from both sides)

(4) This tells us that $P = Q$. (because f is known to be *injective*, because f is a ruler)

(5) Thus, g is *injective*. (by (1),(4), and definition of *injective*)

End of proof that *g* is *injective*

Part 4: Prove that *g* satisfies the *Ruler Equation*.

Suppose $P, Q \in l$. Then

$$|g(P) - g(Q)| = |(f(P) - a) - (f(Q) - a)| = |f(P) - f(Q)| = d(P, Q)$$

by definition of g — arithmetic — f is a ruler so it satisfies the ruler equation

End of proof that *g* satisfies the Ruler Equation

End of Proof of Ruler Sliding Theorem

Ruler Flipping Theorem

Given

- a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$
- a line $l \in \mathcal{L}$
- a ruler f for line l

Claim: The function $g: l \to \mathbb{R}$ defined by

$$g(P) = -f(P)$$

qualifies to be called a ruler for line l.

Remark: The new ruler g is analogous to the ruler that one would obtain by *flipping* the direction of a ruler f on a line l in a drawing.

Outline of Proof of Ruler Flipping Theorem

See if you can provide the details of this proof. You must show that g fulfills the requirements of a ruler for line l.

Part 1: Prove that g is indeed a function with *domain* l and *range* $\mathbb{R}$.

Part 2: Prove that g is *surjective*.

Part 3: Prove that g is *injective*.

Part 4: Prove that g satisfies the *Ruler Equation*.

Combining Ruler Sliding and Flipping

The previous two theorems can be used to prove this theorem that is presented in the book

Theorem 2.3.1 (Ruler Sliding and Flipping Theorem)

Given

- a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$
- a line $l \in \mathcal{L}$
- a ruler f for line l
- a number $a \in \mathbb{R}$
- a number ε that is either 1 or -1

Claim: The function $h_{a,\varepsilon}: l \rightarrow \mathbb{R}$ defined by

$$h_{a,\varepsilon}(P) = \varepsilon(f(P) - a)$$

qualifies to be called a ruler for line l.

Remark: The new ruler $h_{a,\varepsilon}$ is analogous to the ruler that one would obtain by *sliding* a ruler f along on a line l in a drawing and then (if $\varepsilon = -1$) flipping the ruler.

Proof of Theorem 2.3.1 (Ruler Sliding and Flipping Theorem)

(1) Suppose all the given stuff is given

(2) **(Case 1)** Suppose $\varepsilon = 1$.

(3) Then $h_{a,\varepsilon}$ is the function $h_{a,1}: l \to \mathbb{R}$ defined by $h_{a,1}(P) = 1(f(P) - a) = f(P) - a$

This function $h_{a,\varepsilon}$ is qualified to be called a ruler, by the Ruler Sliding Theorem.

So the claim is true in this case.

(4) **(Case 2)** Suppose $\varepsilon = -1$.

(5) Let g be the function $g: l \to \mathbb{R}$ defined by $g(P) = f(P) - a$

This function g is qualified to be called a ruler, by the Ruler Sliding Theorem.

(6) Then $h_{a,\varepsilon}$ is the function $h_{a,-1}: l \to \mathbb{R}$ defined by

$$h_{a,-1}(P) = -1(f(P) - a) = -g(P)$$

This function $h_{a,\varepsilon}$ is qualified to be called a ruler, by the Ruler Flipping Theorem.

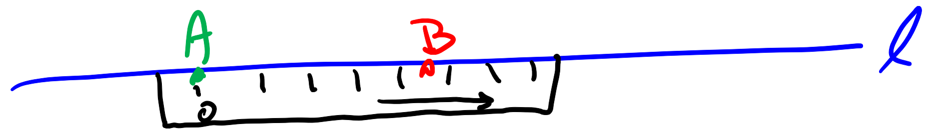
So the claim is true in this case, as well.

(7) **(Conclusion of Cases)** So $h_{a,\varepsilon}$ is qualified to be called a ruler. (because it is true in both cases)

End of Proof.

Special Rulers and the Ruler Placement Theorem

In a drawing, when using a ruler to measure the distance between two given drawn points A and B, it is most convenient to put the zero of the ruler on one of the points, with the ruler oriented so that it goes in the direction of the other point.



In axiomatic geometry, where *rulers* are *functions*, it is similarly often convenient to have a ruler that behaves in an analogous fashion. That is the idea behind the Ruler Placement ~~Thoerem~~ Theorem.

Theorem 2.3.2 (Ruler Placement Theorem)

Given

- a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$
- distinct points $A, B \in \mathcal{P}$

Claim: There exists a ruler g for line $\overleftrightarrow{AB}$ such that $g(A) = 0$ and $g(B) > 0$.

Terminology: Such a ruler g is called a ***ruler with A as origin and B positive***.

I won't discuss the proof of the Ruler Placement Theorem, because there is a nice proof in the book. But I will point out that the steps in the book's proof of Proposition 2.3.2 provide us with a procedure for finding a *ruler with A as origin and B positive* for any two given distinct points in a metric geometry. I will present the procedure here.

Procedure for Finding a *Ruler with A as Origin and B Positive* for Distinct Points A, B in a Metric Geometry

Suppose A and B are any two distinct points in a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$.

Then let f be a ruler for line $\overleftrightarrow{AB}$. (A ruler exists because $\mathcal{M}$ is a metric geometry.)

Use f to obtain a new ruler in the following way:

Define a real number ε by $\varepsilon = sgn\big(f(B) - f(A)\big)$

(That is, ε is the *sign* of $f(B) - f(A)$. In other words

$$\varepsilon = \begin{cases} +1 \text{ if } f(B) - f(A) > 0 \\ -1 \text{ if } f(B) - f(A) < 0 \end{cases}$$

Let $h_{f(A),\varepsilon}$ be the function $h_{a,\varepsilon}: \overleftrightarrow{AB} \to \mathbb{R}$ defined by

$$h_{f(A),\varepsilon}(P) = \varepsilon\big(f(P) - f(A)\big)\Big)$$

Then $h_{f(A),\varepsilon}$ is a *ruler with A as origin and B positive*.

[Example 1] In the *Euclidean plane*,

let $A = (x_A, y_A) = (4,3)$ and $B = (x_B, y_B) = (4,2)$ and $C = (x_C, y_C) = (-4,5)$

(a) Find a *ruler with A as origin and B positive.*

Solution:

Because $x_A = x_B = 4$, we know that line $\overleftrightarrow{AB}$ is the *vertical* L_4.

The *standard ruler* for line L_4 in the *Euclidean plane* is $f(x,y) = y$.

Using this ruler $f(A) = f(4,3) = 3$ and $f(B) = f(4,2) = 2$.

Therefore, $f(B) - f(A) = 2 - 3 = -1 =$ negative

Therefore, the value of ε will be $\varepsilon = -1$

Let g be the function $g: \overleftrightarrow{AB} \to \mathbb{R}$ defined by

$$g(x,y) = h_{f(A),\varepsilon}(x,y) = \varepsilon\big(f(x,y) - f(A)\big) = -(y-3) = 3 - y$$

Then g is a *ruler with A as origin and B positive.*

Check:

$$g(A) = g(4,3) = 3 - 3 = 0$$

$$g(B) = g(4,2) = 3 - 2 = 1 = \text{ positive}$$

(b) Find a *ruler with A as origin and C positive.* in the Euclidean Plane

Solution:

Because $x_A \neq x_c$, we know that line $\overleftrightarrow{AC}$ is a *non-vertical line*. $L_{m,b}$.

We need to find the *standard ruler* for that line. For that, we will need to find m.

$$m = \frac{y_C - y_A}{x_C - x_A} = \frac{5-3}{(-4)-4} = \frac{2}{-8} = -\frac{1}{4}$$

The *standard ruler* for line $\overleftrightarrow{AC}$ in the *Euclidean plane* will be

$$f(x,y) = x\sqrt{1+m^2} = x\sqrt{1+\left(-\frac{1}{4}\right)^2} = x\sqrt{1+\frac{1}{16}} = x\sqrt{\frac{17}{16}} = \frac{x\sqrt{17}}{4}$$

Using this ruler,

$$f(A) = f(4,3) = \frac{4\sqrt{17}}{4} = \sqrt{17} \approx 4.123$$

$$f(C) = f(-4,5) = \frac{(-4)\sqrt{17}}{4} = -\sqrt{17} \approx -4.123$$

Therefore, $f(C) - f(A)$ will be negative.

Therefore, the value of ε will be $\varepsilon = -1$

Let g be the function $g\colon \overleftrightarrow{AC} \to \mathbb{R}$ defined by

$$g(x,y) = h_{f(A),\varepsilon}(x,y) = \varepsilon\big(f(x,y) - f(A)\big) = -\left(\frac{x\sqrt{17}}{4} - \sqrt{17}\right) = \sqrt{17} - \frac{x\sqrt{17}}{4}$$

Then g is a *ruler with A as origin and C positive*.

Check:

$$g(A) = g(4,3) = \sqrt{17} - \frac{(4)\sqrt{17}}{4} = 0$$

$$g(C) = g(-4,5) = \sqrt{17} - \frac{(-4)\sqrt{17}}{4} = 2\sqrt{17} = \text{positive}$$

End of [Example 1]

[Example 2] In the *Taxicab plane*,

let $A = (x_A, y_A) = (4,3)$ and $B = (x_B, y_B) = (4,2)$ and $C = (x_C, y_C) = (-4,5)$

(a) Find a *ruler with A as origin and B positive.*

Solution:

As in **[Example 1](a)**, line $\overleftrightarrow{AB}$ is the *vertical* L_4.

The *standard ruler* for line L_4 in the *Taxicab plane* is $f(x,y) = y$.

This is the same as the *standard ruler* for that line in the *Euclidean plane*. Therefore, the solution will proceed exactly as in [Example 1](a), and we we will reach the same result:

Let g be the function $g: \overleftrightarrow{AB} \to \mathbb{R}$ defined by

$$g(x,y) = h_{f(A),\varepsilon}(x,y) = \varepsilon\big(f(x,y) - f(A)\big) = -(y-3) = 3 - y$$

Then g is a *ruler with A as origin and B positive.*

$$g(x,y) = 3 - y$$

(b) Find a *ruler with A as origin and C positive.* (in the taxicab plane)

Solution:

As in **[Example 1](b)**, line $\overleftrightarrow{AC}$ is a *non-vertical line* with slope $m = -\frac{1}{4}$.

The standard ruler for line $\overleftrightarrow{AC}$ in the Taxicab plane is

$$f(x,y) = x(1 + |m|) = x\left(1 + \left|-\frac{1}{4}\right|\right) = x\left(1 + \frac{1}{4}\right) = x\left(\frac{5}{4}\right)$$

Using this ruler, $f(A) = f(4,3) = (4)\left(\frac{5}{4}\right) = 5$ and $f(C) = f(-4,5) = (-4)\left(\frac{5}{4}\right) = -5$

Therefore, $f(C) - f(A)$ will be negative, and so the value of ε will be $\varepsilon = -1$

Let g be the function $g: \overleftrightarrow{AC} \to \mathbb{R}$ defined by

$$g(x,y) = h_{f(A),\varepsilon}(x,y) = \varepsilon\big(f(x,y) - f(A)\big) = -\left(x\left(\frac{5}{4}\right) - 5\right) = 5 - x\left(\frac{5}{4}\right)$$

Then g is a *ruler with A as origin and C positive.*

$$g(x,y) = 5 - x\left(\frac{5}{4}\right)$$

Check: $g(A) = g(4,3) = 5 - (4)\left(\frac{5}{4}\right) = 0$ and $g(C) = g(-4,5) = 5 - (-4)\left(\frac{5}{4}\right) = 10 =$ pos

End of [Example 2]

[Example 3] In the *Poincaré plane*,

let $A = (x_A, y_A) = (4,3)$ and $B = (x_B, y_B) = (4,2)$ and $C = (x_C, y_C) = (-4,5)$

(a) Find a *ruler with A as origin and B positive.*

Solution:

Because $x_A = x_B = 4$, we know that line $\overleftrightarrow{AB}$ is the *type I line* ${}_4L$.

The *standard ruler* for this line is $f(x,y) = \ln(y)$.

Using this ruler $f(A) = f(4,3) = \ln(3)$ and $f(B) = f(4,2) = \ln(2)$.

Therefore, $f(B) - f(A) = \ln(2) - \ln(3) = \ln\left(\frac{2}{3}\right) =$ negative because $0 < \frac{2}{3} < 1$

Therefore, the value of ε will be $\varepsilon = -1$

Let g be the function $g: \overleftrightarrow{AB} \to \mathbb{R}$ defined by

$$g(x,y) = h_{f(A),\varepsilon}(x,y) = \varepsilon\big(f(x,y) - f(A)\big) = -(\ln(y) - \ln(3)) = \ln(3) - \ln(y)$$

Then g is a *ruler with A as origin and B positive.*

$g(x,y) = \ln(3) - \ln(y)$

Check:

$$g(A) = g(4,3) = \ln(3) - \ln(3) = 0$$

$$g(B) = g(4,2) = \ln(3) - \ln(3) = \ln\left(\frac{3}{2}\right) \quad \text{this will be positive because } \frac{3}{2} > 1$$

$\ln(2)$

(b) Find a *ruler with A as origin and C positive.* in the Poincaré Plane

Solution:

Because $x_A \neq x_C$, we know that line $\overleftrightarrow{AC}$ is a *type II line.*

We need to find the *standard ruler* for that line. For that, we will need to find c and r.

$$c = \frac{x_C^2 - x_A^2 + y_C^2 - y_A^2}{2(x_C - x_A)} = \frac{(-4)^2 - 4^2 + 5^2 - 3^2}{2\big((-4) - 4\big)} = \frac{25 - 9}{2(-8)} = \frac{16}{2(-8)} = -1$$

$$r = \sqrt{(x_A - c)^2 + y_A^2} = \sqrt{\big(4 - (-1)\big)^2 + 3^2} = \sqrt{5^2 + 3^2} = \sqrt{34}$$

The *standard ruler* for line $\overleftrightarrow{AC}$ will be

$$f(x, y) = \ln\left(\frac{x - c + r}{y}\right) = \ln\left(\frac{x - (-1) + \sqrt{34}}{y}\right) = \ln\left(\frac{x + 1 + \sqrt{34}}{y}\right)$$

Using this ruler,

$$f(A) = f(4,3) = \ln\left(\frac{4 + 1 + \sqrt{34}}{3}\right) = \ln\left(\frac{5 + \sqrt{34}}{3}\right) \approx 1.284$$

$$f(C) = f(-4,5) = \ln\left(\frac{(-4) + 1 + \sqrt{34}}{5}\right) = \ln\left(\frac{-3 + \sqrt{34}}{5}\right) \approx -0.569$$

Therefore, $f(C) - f(A)$ will be negative.

Therefore, the value of ε will be $\varepsilon = -1$

Let g be the function $g\colon \overleftrightarrow{AC} \to \mathbb{R}$ defined by

$$\begin{aligned} g(x,y) &= h_{f(A),\varepsilon}(x,y) \\ &= \varepsilon\big(f(x,y) - f(A)\big)\Big) \\ &= -\left(\ln\left(\frac{x+1+\sqrt{34}}{y}\right) - \ln\left(\frac{5+\sqrt{34}}{3}\right)\right) \end{aligned}$$

$$g(x,y) = \ln\left(\frac{5+\sqrt{34}}{3}\right) - \ln\left(\frac{x+1+\sqrt{34}}{y}\right)$$

Then g is a *ruler with A as origin and C positive.*

Check: $g(A) = g(4,3) = \ln\left(\frac{5+\sqrt{34}}{3}\right) - \ln\left(\frac{4+1+\sqrt{34}}{3}\right) = 0$

$$g(C) = g(-4,5)$$

$$= \ln\left(\frac{5+\sqrt{34}}{3}\right) - \ln\left(\frac{(-4)+1+\sqrt{34}}{5}\right)$$

$$= \ln\left(\frac{5+\sqrt{34}}{3}\right) - \ln\left(\frac{-3+\sqrt{34}}{5}\right)$$

$$= \ln\left(\left(\frac{5+\sqrt{34}}{3}\right)\left(\frac{5}{-3+\sqrt{34}}\right)\right)$$

$$= \ln\left(\frac{5(5+\sqrt{34})}{3(-3+\sqrt{34})}\right)$$

≈ 1.853 positive

These numbers are all very ugly. But notice that

$$|f(A) - f(C)| \approx |1.284 - (-0.569)| = 1.853 = |g(A) - g(C)|$$

This is good, because the number 1.853 is an approximation for the distance $d_H(A, C)$.

End of [Example 3]

End of Video