<u>3.2a: Introduction to Betweenness</u>

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for Ohio University MATH 3110/5110 College Geometry

Topics:

- Betweenness for Real Numbers
- Betweenness for Points in a Metric Geometry
- Basic Examples

Reading: pages 47 - 48 of Section 3.2 Betweenness, in the book

Geometry: A Metric Approach with Models, Second Edition by Millman & Parker

Homework: Section 3.2 # 3, 10, 13

Recall Important Concepts from Section 3.1 the *triangle inequality for distance functions*

Definition of the Triangle Inequality for Distance Functions

Words: Distance function d on set $\mathcal{P}$ satisfies the triangle inequality.

Meaning: For all $A, B, C \in \mathcal{P}$, the inequality $d(A, C) \leq d(A, B) + d(B, C)$ is true

In Symbols: $\forall A, B, C \in \mathcal{P}\big(d(A, C) \leq d(A, B) + d(B, C)\big)$

It is important to remember that not all distance functions satisfy the *triangle inequality for distance functions*.

In the previous video, we reviewed the *Absolute Value Distance Function on* $\mathbb{R}$

Definition of the Absolute Value Distance Function on $\mathbb{R}$

symbol: $d_{\mathbb{R}}$

meaning: the function $d_{\mathbb{R}}: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $d_{\mathbb{R}}(x, y) = |x - y|$

and we showed that it does satisfy the *triangle inequality for distance functions*.

For all $x, y, z \in \mathbb{R}$, the inequality $d_{\mathbb{R}}(x, z) \leq d_{\mathbb{R}}(x, y) + d_{\mathbb{R}}(y, z)$ is true.

In other words,

For all $x, y, z \in \mathbb{R}$, the inequality $|x - z| \leq |x - y| + |y - z|$ is true.

Also discussed in Section 3.1:

Proposition 3.1.6

The *Euclidean distance function* satisfies the *triangle inequality for distance functions*

For all $A, B, C \in \mathbb{R}^2$, the inequality $d_E(A, C) \leq d_E(A, C) + d_E(A, C)$ is true

.

When Does the Triangle Inequality For The Absolute Value Distance Become an Equality?

An obvious question: When do distinct real numbers x, y, z cause the triangle inequality to become an equality? That is when do distinct real numbers x, y, z cause these equations to be true?

$$d_{\mathbb{R}}(x,z) = d_{\mathbb{R}}(x,y) + d_{\mathbb{R}}(y,z)$$

$$|x - z| = |x - y| + |y - z|$$

To answer that, let's consider some cases. There are six possibilities

Case (i): $x < y < z$

Case (ii): $z < y < x$

Case (iii): $y < x < z$

Case (iv): $z < x < y$

Case (v): $x < z < y$

Case (vi): $y < z < x$

Let's see what the triangle inequality becomes in each of those six cases.

Case (i): When $x < y < z$

We can do a trick: subtract and add something to the quantity $z - x$

$$z - x = z - y + y - x$$

We can add parentheses to this equation without changing anything:

$$(z - x) = (z - y) + (y - x)$$

Because $y < x < z$, we know that each of the quantities in parenthese is positive.

So each of the quantities in parenthese is the same as its absolute value.

$$|z - x| = |z - y| + |y - x|$$

We can change the order of the letters inside the absolute values without changing the result.

$$|x - z| = |y - z| + |x - y|$$

And we can change the order of the addition on the right side, and we see that in this case, the $\leq$ is actually an **equality**.

$$|x - z| = |x - y| + |y - z|$$

Case (ii): When $z < y < x$

We do steps similar to **Case (ii)** and find that in this case, the inequality is again an **equality**:

$$|x - z| = |x - y| + |y - z|$$

Case (iii): When $y < x < z$

We can again do a trick: subtract and add something to the quantity $z - y$

$$z - x + x - y = z - y$$

We can add parentheses to this equation without changing anything:

$$(z - x) + (x - y) = (z - y)$$

Because $y < x < z$, we know that each of the quantities in parenthese is positive.

So each of the quantities in parenthese is the same as its absolute value.

$$|z - x| + |x - y| = |z - y|$$

We can change the order of the letters inside the absolute values without changing the result.

$$|x - z| + |x - y| = |y - z|$$

And we can subtract the quantity $|x - y|$ from both sides

$$|x - z| = |y - z| - |x - y|$$

And notice that if we add something positive to the right side, we get a strict inequality

$$|x - z| = |y - z| - |x - y| < |y - z| - |x - y| + 2|x - y| = |x - y| + |y - z|$$

We have shown that in this case, the inequality is a **strict inequality**:

$$|x - z| < |x - y| + |y - z|$$

Case (iv): $\boldsymbol{z < x < y}$ **Case (v):** $\boldsymbol{x < z < y}$ **Case (vi):** $\boldsymbol{y < z < x}$	In each of these three cases, we do steps similar to **Case (iii)** and find that in each case, the inequality is again a **strict inequality**: $$|x - z| < |x - y| + |y - z|$$

We summarize these results in a nice green box

Cases of the Triangle Inequality for Distinct Real Numbers $\boldsymbol{x, y, z}$

The Triangle Inequality is the inclusive inequality

$$|x - z| \leq |x - y| + |y - z|$$

Case (i): When $x < y < z$, the *TE* becomes an *equality* $|x - z| = |x - y| + |y - z|$.

Case (ii): $z < y < x$, the *TE* becomes an *equality* $|x - z| = |x - y| + |y - z|$.

Case (iii): $y < x < z$, the *TE* becomes a *strict inequality* $|x - z| < |x - y| + |y - z|$.

Case (iv): $z < x < y$, the *TE* becomes a *strict inequality* $|x - z| < |x - y| + |y - z|$.

Case (v): $x < z < y$, the *TE* becomes a *strict inequality* $|x - z| < |x - y| + |y - z|$.

Case (vi): $y < z < x$, the *TE* becomes a *strict inequality* $|x - z| < |x - y| + |y - z|$.

Betweenness for Real Numbers

A central feature of metric geometry is the existence of a *ruler* for every *line*. This has many consequences. Because of *rulers*, all features of the real number line will have analogues on lines in a metric geometry.

One important feature of the real number line that we will want to use in metric geometry is the concept of *betweenness*. We will define and explore *Betweenness for Real Numbers* now, in preparation for later defining *Betweenness for Points in a Metric Geometry*.

Definition of Betweenness for Real Numbers

Symbol: $x * y * z$

Spoken: *y is between x and z.*

Usage: $x, y, z \in \mathbb{R}$

Meaning: $x < y < z$ or $z < y < x$

Remark: It is a property of real numbers that for given any three distinct real numbers, one is smallest, one is largest, and the other is between them.

Betweenness for Real Numbers is Related to the Distance Between Them

Recognize that the inequalities in the definition of Betweenness for Real Numbers showed up previously, in the green box two pages back, involving cases where the triangle inequality becomes either an *equality* or a *strict inequality*. The results in the green box can be used to prove this nice lemma about **betweenness of real numbers**.

Lemma: Betweenness for Real Numbers is Related to the Distance Between Them

Given: distinct real numbers x, y, z

Claim: The following are equivalent (*TFAE*)

(a) $x * y * z$ (That is, y *is between* x *and* z.)

(b) $d_{\mathbb{R}}(x,z) = d_{\mathbb{R}}(x,y) + d_{\mathbb{R}}(y,z)$ (That is, $|x - z| = |x - y| + |y - z|$)

Remarks on The Statement of the Claim and on Proof Strategy

Most books state this sort of theorem as

(a) if and only if (b)

I prefer to use the terminology of *equivalent statements*. That is, the statements always have the same truth value. Either they are both true, or they are both false.

The most common proof strategy is

Proof Part I: Prove $(a) \rightarrow (b)$.

Proof Part II: Prove $(b) \rightarrow (a)$.

But I find it easier to prove the contrapositive in Part II

Proof Part I: Prove $(a) \rightarrow (b)$.

Proof Part II: Prove $NOT(a) \rightarrow NOT(b)$.

Proof Part I: Prove $(a) \to (b)$.

(1) Suppose (a) is true. That is, suppose $x * y * z$

(2) Then $x < y < z$ or $z < y < x$ (by (1) and definition of *betweenness*)

(3) Then $|x - z| = |x - y| + |y - z|$ (by Cases (i),(ii) of the Triangle Inequality, from the green box four pages ago.) So (b) is true.

End of Proof Part I

Proof Part II: Prove $NOT(a) \to NOT(b)$.

(1) Suppose (a) is false. That is, suppose $x * y * z$ is false.

(2) Then neither $x < y < z$ nor $z < y < x$ is true (by (1) and definition of *betweenness*) so it must be that $y < x < z$ or $z < x < y$ or $x < z < y$ or $y < z < x$.

(3) Then $|x - z| < |x - y| + |y - z|$ (by Cases (iii),(iv),(v),(vi) of the Triangle Inequality, from the green box four pages ago.) So (b) is false.

End of Proof Part I

Some New Notation

Having defined and explored *Betweenness for Real Numbers*, we are now ready *Betweenness for Points in a Metric Geometry*. But first, we review some useful notation and introduce some more..

Recall the notation for the unique line containing two given points in an *incidence geometry*.

> **Definition of Notation for the Unique Line Containing Two Given Points**
>
> **Symbol:** $\overleftrightarrow{AB}$
>
> **Spoken:** *line A B*
>
> **Usage:** There is an *incidence geometry* $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ in the discussion and $A, B \in \mathcal{P}$ are two distinct points
>
> **Meaning:** the unique line $l \in \mathcal{L}$ such that $A \in l$ and $B \in l$.

There is some new notation for the distance between two given points in a metric geometry

> **Definition of Notation for the Distance Between Two Given Points**
>
> **Symbol:** AB
>
> **Usage:** There is a *metric geometry* $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$ in the discussion and $A, B \in \mathcal{P}$ are two points, not necessarily distinct.
>
> **Meaning:** $d(A, B)$

Betweenness for Points in a Metric Geometry

Definition of Betweenness for Points in a Metric Geometry

Symbol: $A - B - C$

Spoken: B *is between* A *and* C.

Usage: A, B, C are points in a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$.

Meaning: the following two things are both true

- A, B, C are distinct and collinear
- $d(A, C) = d(A, B) + d(B, C)$ That is, $AC = AB + BC$

[Example 1] Let $A = (3,4)$ and $B = (6,5)$ and $D = (10,3)$ be points in the *Poincaré plane*.

(a) Show that $A - B - D$.

Solution:

Clearly, A, B, D are distinct.

And it is straightforward to show that they all lie on the line ${}_6L_5$, so they are collinear.

We also need to show that the distances add up.

$$d_H(A,B) = \left| \ln\left(\frac{\frac{x_A - c + r}{y_A}}{\frac{x_B - c + r}{y_B}} \right) \right| = \left| \ln\left(\frac{\frac{3-6+5}{4}}{\frac{6-6+5}{5}} \right) \right| = \left| \ln\left(\frac{1}{2}\right) \right| = |-\ln(2)| = \ln(2)$$

$$d_H(B,D) = \left| \ln\left(\frac{\frac{x_B - c + r}{y_B}}{\frac{x_D - c + r}{y_D}} \right) \right| = \left| \ln\left(\frac{\frac{6-6+5}{5}}{\frac{10-6+5}{3}} \right) \right| = \left| \ln\left(\frac{1}{3}\right) \right| = |-\ln(3)| = \ln(3)$$

$$d_H(A,D) = \left| \ln\left(\frac{\frac{x_A - c + r}{y_A}}{\frac{x_D - c + r}{y_D}} \right) \right| = \left| \ln\left(\frac{\frac{3-6+5}{4}}{\frac{10-6+5}{3}} \right) \right| = \left| \ln\left(\frac{\frac{1}{2}}{3} \right) \right| = \left| \ln\left(\frac{1}{6}\right) \right| |-\ln(6)| = \ln(6)$$

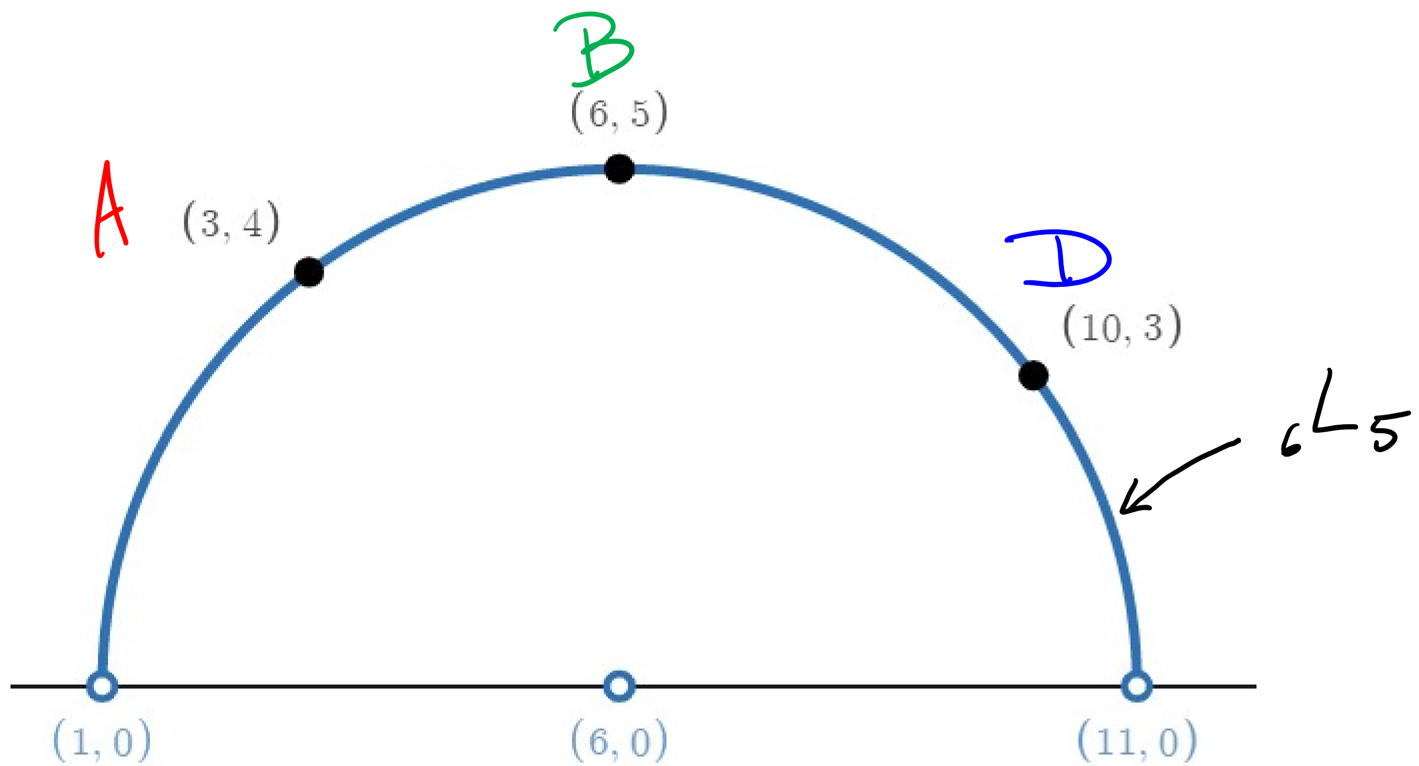
Observe that: $d_H(A,B) + d_H(B,D) = \ln(2) + \ln(3) = \ln(2 \cdot 3) = \ln(6) = d_H(A,D)$

↑ rule of logarithms $\ln(a) + \ln(b) = \ln(a \cdot b)$

We conclude that $A - B - D$ is true.

(b) Illustrate the result from (a)

Solution:



End of [Example 1]

The Importance of Collinearity in the Definition of Betweenness

Note that *both* requirements for $A - B - C$ are important.

- A, B, C are distinct and collinear
- $d(A, C) = d(A, B) + d(B, C)$ That is, $AC = AB + BC$

It is possible for the second requirement to be true without the first requirement being true.

[Example 2] (a) Let $A = (2,1)$ and $B = (6,3)$ and $C = (8,4)$ be points in the *Taxicab plane*.

Observe that A, B, C are collinear, because they all lie on the *non-vertical line* $y = \left(\frac{1}{2}\right) x$

Computing the distances, we find

$$d_T(A, B) + d_T(B, C) = (|2 - 6| + |1 - 3|) + (|6 - 8| + |3 - 4|) = 4 + 2 + 2 + 1 = 9$$

$$d_T(A, C) = |2 - 8| + |1 - 4| = 6 + 3 = 9$$

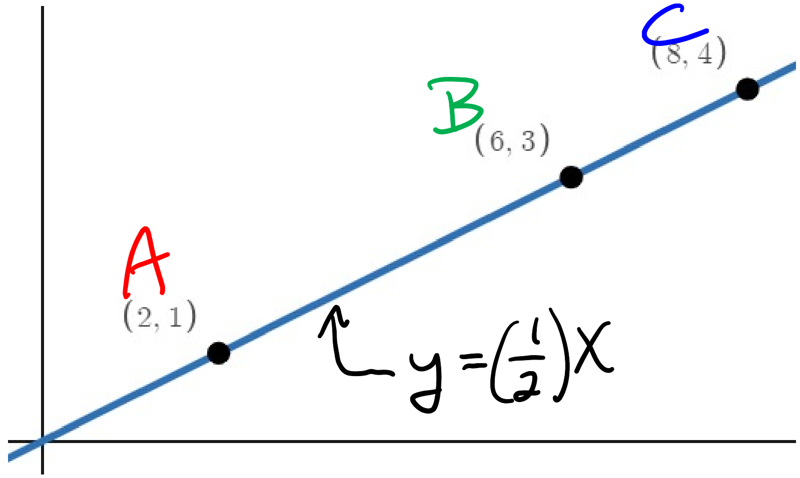
Observe that

$$d_T(A,B) + d_T(B,C) = 9 = d_T(A,C)$$

We conclude that $A - B - C$ is true.

The three points are plotted at right.

There is no surprise here.



(b) Let $A = (2,1)$ and $D = (8,1)$ and $C = (8,4)$ be points in the *Taxicab plane*.

Observe that A, D, C are *not collinear*, because

- Points A and C lie on the *non-vertical line* $y = \left(\frac{1}{2}\right)x$.
- Points A and D llie on the *non-vertical line* $y = 1$.
- Points C and D llie on the *vertical line* $x = 8$.

Computing the distances, we find

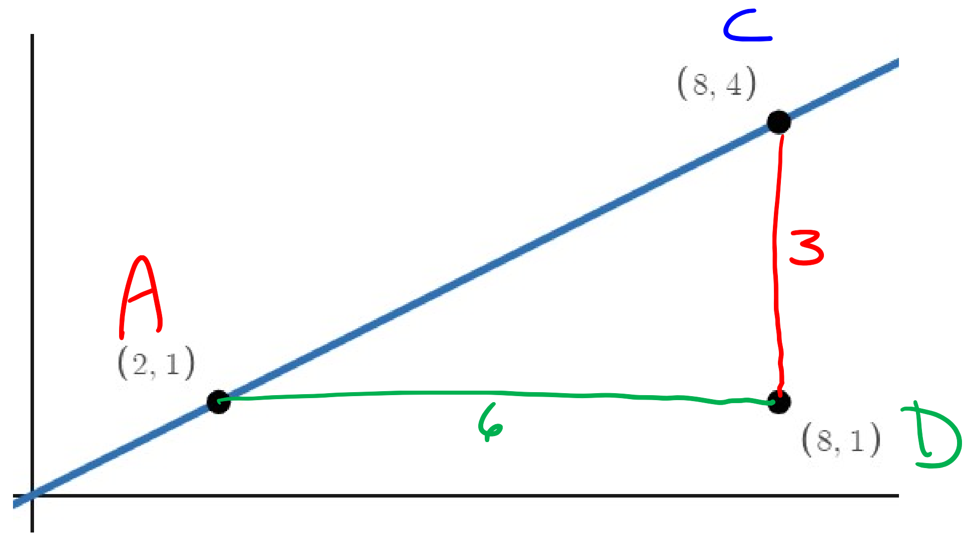
$$d_T(A,D) + d_T(D,C) = (|2-8| + |1-1|) + (|8-8| + |1-4|) = 6 + 3 = 9$$

$$d_T(A,C) = |2-8| + |1-4| = 6 + 3 = 9$$

Observe that $d_T(A,D) + d_T(D,C) = 9 = d_T(A,C)$

So although $d_T(A, D) + d_T(D, C) = d_T(A, C)$,
We conclude that D is not between A and C
because the points are *not collinear*. That is,
$A - D - C$ is *false*.

The three points are plotted at right.



End of [Example 2]

End of Video