3.2b:Properties of Betweenness

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for Ohio University MATH 3110/5110 College Geometry

Topics:

- **Outside Points Can be Reversed in the Symbol for Betweenness**
- **Betweenness is Related to Coordinates**
- **Fact about Three Distinct Collinear Points in a Metric Geometry**
- **Betweenness of Points Expressed Using the Vector Description of a Line**
- **Existence of Points with Certain Betweenness Relationships**
- **Betweenness for Four Points in a Metric Geometry**

Reading: pages 49 - 51 of Section 3.2 Betweenness, in the book

Geometry: A Metric Approach with Models, Second Edition by Millman & Parker

Homework: Section 3.2 # 1, 2, 5, 7, 8, 9

Recall Important Stuff From Section 3.1

Using Vectors to Describe Lines and Rulers

Proposition 3.1.2 Using Vectors to Describe Cartesian Lines

Given two distinct points $A, B \in \mathbb{R}^2$ line $\overleftrightarrow{AB}$ can be described using vectors as follows:

$$L_{AB} = \{X \in \mathbb{R}^2 | X = A + t(B - A) \text{ for some } t \in \mathbb{R}\}$$

Observe that the use of the letter X is not really necessary.

$$L_{AB} = \{A + t(B - A) | t \in \mathbb{R}\}$$

Proposition 3.1.4 Using Vectors to Describe Rulers in the Euclidean Plane

If L_{AB} is a cartesian line, then $f: L_{AB} \to \mathbb{R}$ defined by

$$f\big(A + t(B - A)\big) = t\|B - A\|$$

is a ruler for the line L_{AB} in the *Euclidean plane*.

Recall Important Stuff From Section 3.2 Introduced in Video 3.2a

Definition of Betweenness for Real Numbers

Symbol: $x * y * z$

Spoken: y *is between* x *and* z.

Usage: $x, y, z \in \mathbb{R}$

Meaning: $x < y < z$ or $z < y < x$

Remark: It is a property of real numbers that for given any three distinct real numbers, one is smallest, one is largest, and the other is between them.

Lemma: Betweenness for Real Numbers is Related to the Distance Between Them

Given: distinct real numbers x, y, z

Claim: The following are equivalent (*TFAE*)

(a) $x * y * z$ (That is, y *is between* x *and* z.)

(b) $d_{\mathbb{R}}(x,z) = d_{\mathbb{R}}(x,y) + d_{\mathbb{R}}(y,z)$ (That is, $|x - z| = |x - y| + |y - z|$)

Definition of Betweenness for Points in a Metric Geometry

Symbol: $A - B - C$

Spoken: $B \text{ is between } A \text{ and } C.$

Usage: A, B, C are points in a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$.

Meaning: the following two things are both true

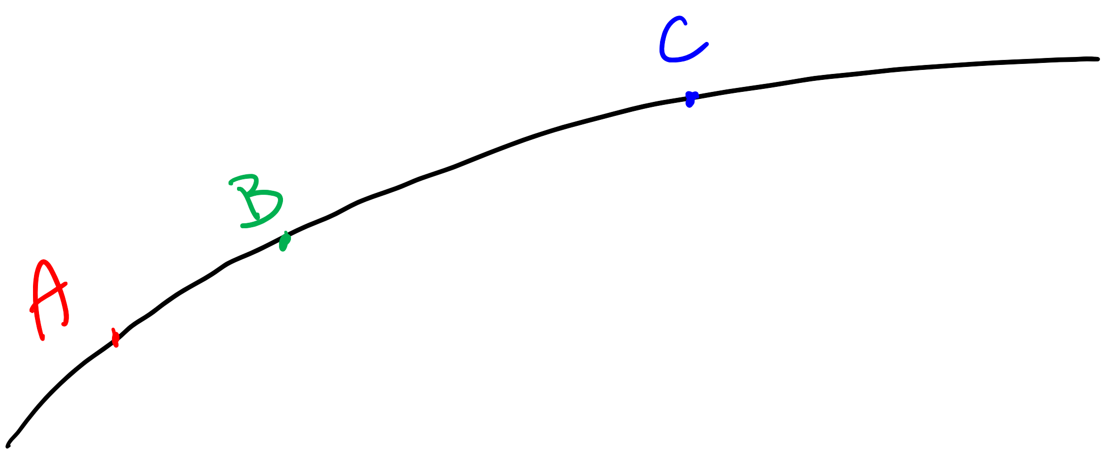
- A, B, C are distinct and collinear
- $d(A, C) = d(A, B) + d(B, C)$ That is, $AC = AB + BC$

Video 3.2b Properties of Betweenness

Outside Points Can be Reversed in the Symbol for Betweenness

> **Theorem 3.2.2 (Really a Corollary of the Definition)**
>
> **Given:** Points A, B, C in a metric geometry
>
> **Claim:** The following are equivalent (*TFAE*)
>
> (i) $A - B - C$
>
> (ii) $C - B - A$

The proof is very simple. See the book page 49.



Betweenness is Related to Coordinates

Theorem 3.2.3 Betweenness of Points is Related to Betweenness of Coordinates

Given: Collinear points A, B, C on line l with ruler f in a metric geometry

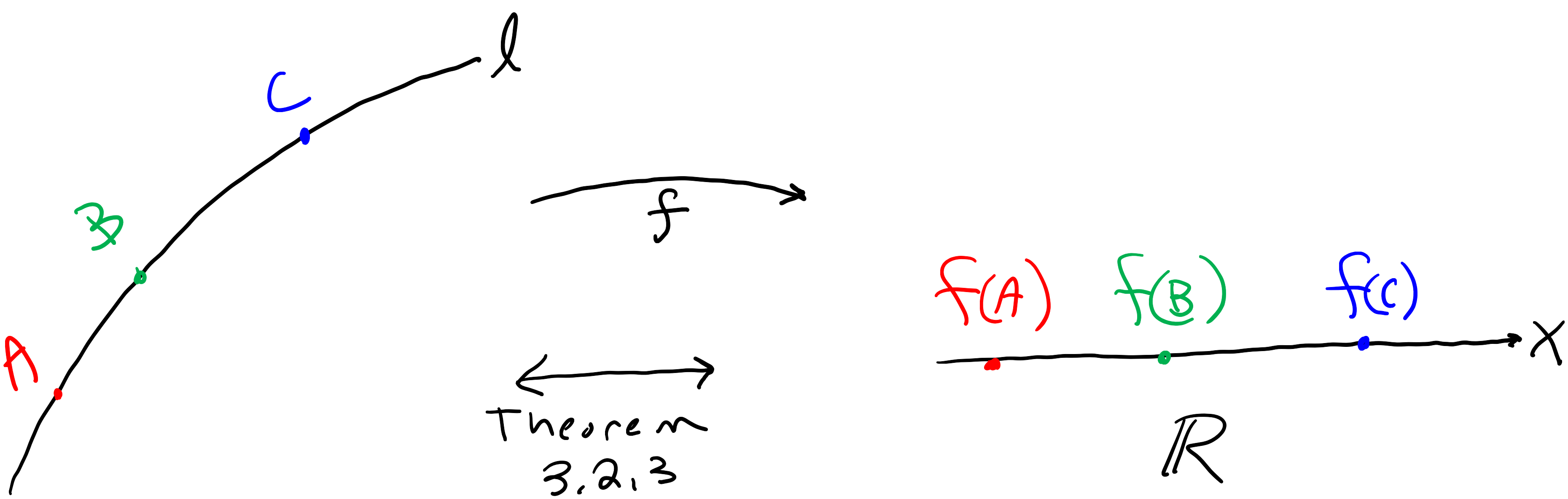
Claim: The following are equivalent (*TFAE*)

(i) $A - B - C$ Betweenness of Points

(ii) $f(A) * f(B) * f(C)$ Betweenness of their coordinates

There is a nice proof of this theorem in the book on pages 49 – 50, but there are no drawings.

It is helpful to at least illustrate the statement of the theorem.



Corollary about Three Distinct Collinear Points in a Metric Geometry

Corollary 3.2.4 Fact about Three Distinct Collinear Points in a Metric Geometry

Given: Three distinct collinear points P, Q, R in a metric geometry

Claim: Exactly one of the points is between the other two.

Proof

(1) Suppose P, Q, R are distinct collinear points in a metric geometry.

(2) Let f be a ruler for the line that they lie on. (In a metric geometry, every line has a ruler.)

(3) Then $f(P), f(Q), f(R)$ are distinct real numbers (because f is injective, because it is a ruler.)

(4) Therefore, one of the three numbers is between the other two numbers (by the ordering of $\mathbb{R}$.)

(5) The corresponding points is between the other two points. (by (4) and Thm 3.2.3 (ii) ➔ (i)

End of Proof

Theorem 3.2.3 can be used to formulate criteria for determining when collinear points A, B, C have the betweenness relationship $A - C - B$ on certain kinds of lines.

[Example 1] Betweenness Condition for Points on a *Poincaré type I line* ${}_aL$.

Consider points $A = (x_A, y_A), B = (x_B, y_B), C = (x_C, y_C)$ on a *Poincaré type I line*.

We ask when the betweenness relationship $A - C - B$ will be true for points A, B, C.

By Theorem 3.2.3, the betweenness relationship $A - C - B$ for points A, B, C will be true exactly when the betweenness relationship $f(A) * f(C) * f(B)$ is true for numbers $f(A), f(C), f(B)$.

That is, when $f(A) < f(C) < f(B)$ or $f(B) < f(C) < f(A)$.

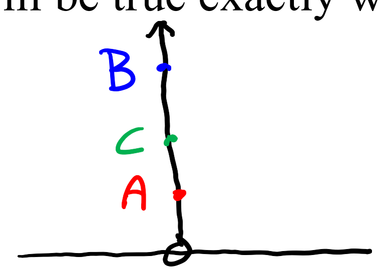
The standard ruler for a *type I line* is the function $f\colon {}_aL \to \mathbb{R}$ defined by $f(x, y) = \ln(y)$.

With this ruler, the coordinates are $f(A) = \ln(y_A)$ and $f(B) = \ln(y_B)$ and $f(C) = \ln(y_C)$.

So we are interested in the case when $\ln(y_A) < \ln(y_C) < \ln(y_B)$ or $\ln(y_B) < \ln(y_C) < \ln(y_A)$.

But the function $\ln(y)$ is an increasing function, so these inequalities will be true exactly when

$$y_A < y_C < y_B \text{ or } y_B < y_C < y_A$$



End of [Example 1]

Betweenness of Points Expressed Using the Vector Description of a Line

Proposition 3.2.5 Betweenness of Points Expressed Using the Vector Description of a Line

Given: points A, B, C in the *Euclidean plane*, with A, C distinct points.

Claim: The following are equivalent (*TFAE*)

(i) $A - B - C$

(ii) There exists a number t with $0 < t < 1$ such that $B = A + t(C - A)$

Here is **Part I** of the proof. You will write **Part II** in a homework exercise.

Proof Part I: Show that (i) ➔ (ii)

(1) Suppose that A, B, C are points in the Euclidean plane and that $A - B - C$ is true.

(2) Then A, B, C must be distinct, collinear points (by (1) and definition of betweenness)

(3) (By Proposition 3.1.2) Line L_{AC} in the *Euclidean plane* can be described using vectors as

$$L_{AC} = \{A + t(C - A) | t \in \mathbb{R}\}$$

In this description, the point A corresponds to $t = 0$ and the point C corresponds to $t = 1$.

(4) The function $f: L_{AC} \to \mathbb{R}$ defined by

$$f\big(A + t(C - A)\big) = t\|C - A\|$$

is a *ruler* for the line L_{AC} in the *Euclidean plane*. (by **Proposition 3.1.4**) Furthermore, by the discussion at the end of Video 3.1 (the subject of exercise 3.1#5), it is a **special ruler with *A* as origin and *C* positive**. Indeed, we see that since point A corresponds to $t = 0$ and the point C corresponds to $t = 1$, their coordinates will be

$$f(A) = 0\|C - A\| = 0$$

$$f(C) = 1\|C - A\| = \|C - A\|$$

positive

(5) $f(A) * f(B) * f(C)$ must be true. (by Theorem 3.2.3 (i) ➔ (ii) and statement (1))

(6) Therefore $f(A) < f(B) < f(C)$ or $f(C) * f(B) < f(A)$. (by (5) and definition of betweenness for real numbers)

(7) It must be that $f(A) < f(B) < f(C)$. (by (4) and (6)

(8) $0 < f(B) < \|C - A\|$ (by (4) and (7)

(9) Therefore, $f(B)$ must be of the form $f(B) = t\|C - A\|$ for some t with $0 < t < 1$.

(10) Therefore, the point B must be of the form $B = A + t(C - A)$ for some t with $0 < t < 1$. (by (9) and definition of how the ruler f works.) therefore, (ii) is true

End of Proof Part I

Proof Part I: Show that (ii) ➔ (i) You will do this in a Homework Exercise

Remark on How an Equivalence Theorem Gets Used in the Proof:

Notice how Theorem 3.2.3 gets used in the proof of Theorem 3.2.5.

Realize that Theorem 3.2.3 does not state that either (i) or (ii) is true. Rather, it only states that the two statements always have the same truth value, either true or false. So, one can only use Theorem 3.2.3 in situations where one *already knows* that *one* of the statements is *true*. In that situation, Theorem 3.2.3 allows you to say that the *other* statement is *also true*.

When using Theorem 3.2.3 (or any other **Equivalence Theorem**) in a proof, one should be very clear about which statement is *already known to be true* and which statement Theorem 3.2.3 allows us to say is *also true*.

Look again at the way I wrote the justification for Statement (5). The justification makes clear what statement was *already known to be true*, and how Theorem 3.2.3 was used to justify saying that the *other* statement is *also true*.

Existence of Points with Certain Betweenness Relationships

> **Theorem 3.2.6 Existence of Points with Certain Betweenness Relationships**
>
> **Given:** distinct points A, B in a *metric geometry*
>
> **Claim:** (i) There exists a point C with $A - B - C$
>
> (ii) There exists a point D with $A - D - B$

Here is a Proof. You will provide Justifications and Illustrations in a Homework Exercise.

(1) Given distinct points A, B in a metric geometry. **(Illustrate)**

(2) Line $\overleftrightarrow{AB}$ exists. **(Justify) (Illustrate)**

(3) There exists a special ruler f for $\overleftrightarrow{AB}$ with A as origin and B positive **(Justify) (Illustrate)**

(4) There exist real numbers $z = f(B) + 1$ and $w = \frac{f(A)+f(B)}{2}$. **(Illustrate)**

(5) Observe that $f(A) < w < f(B) < z$.

(6) Therefore, $f(A) * w * f(B)$ and $f(A) * f(B) * z$. (**Justify**)

(7) Let $C = f^{-1}(z)$ and $D = f^{-1}(w)$. **(Illustrate)**

(8) Observe $f(A) * f(B) * f(C)$ and $f(A) * f(D) * f(B)$.

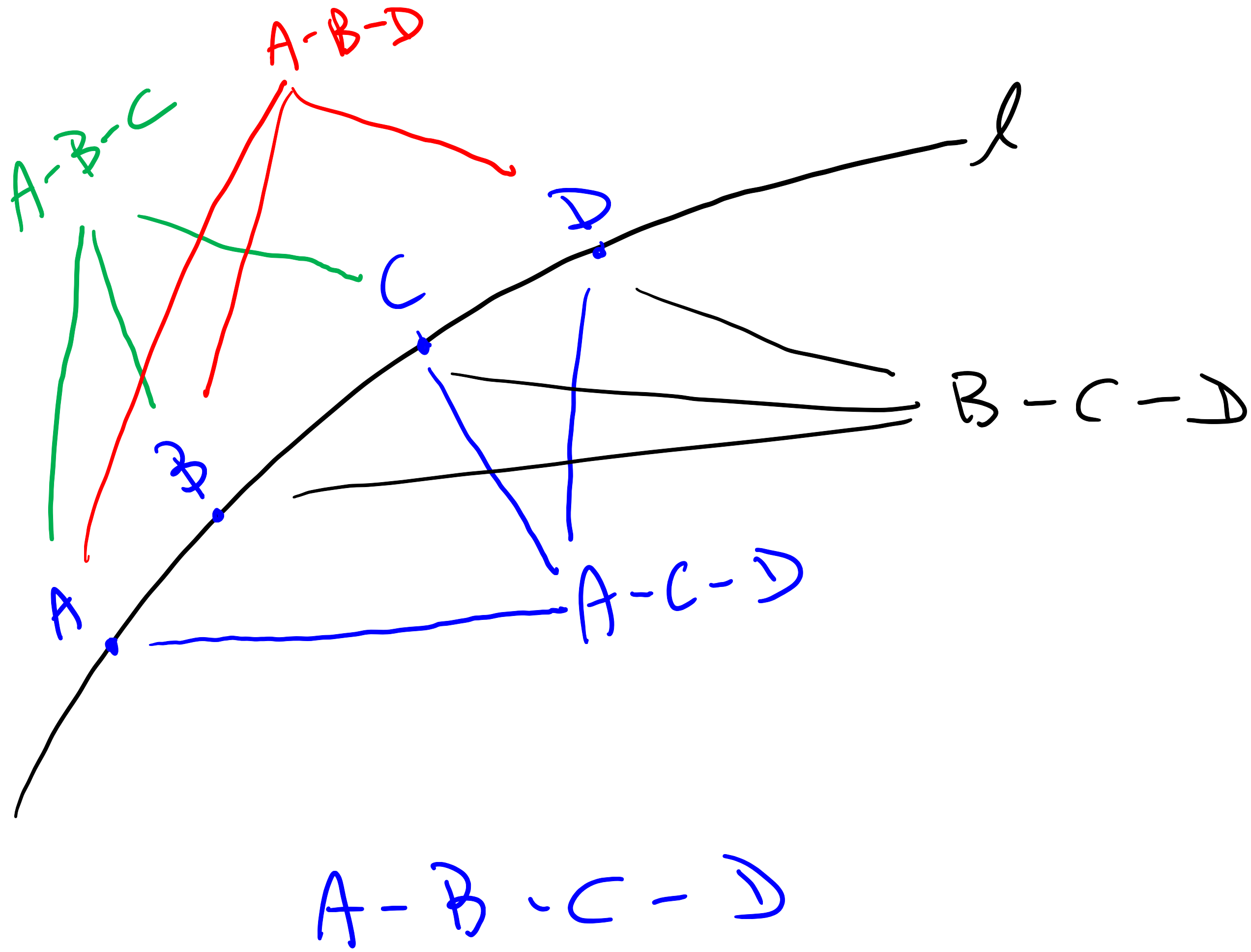
(9) Therefore $A - B - C$ and $A - D - B$. **(Justify)**

End of Proof

Betweenness for Four Points in a Metric Geometry

> **Definition of Betweenness for Four Points in a Metric Geometry**
>
> **Symbol:** $A - B - C - D$
>
> **Meaning:** $A - B - C$ and $A - B - D$ and $A - C - D$ and $B - C - D$.

Illustration:

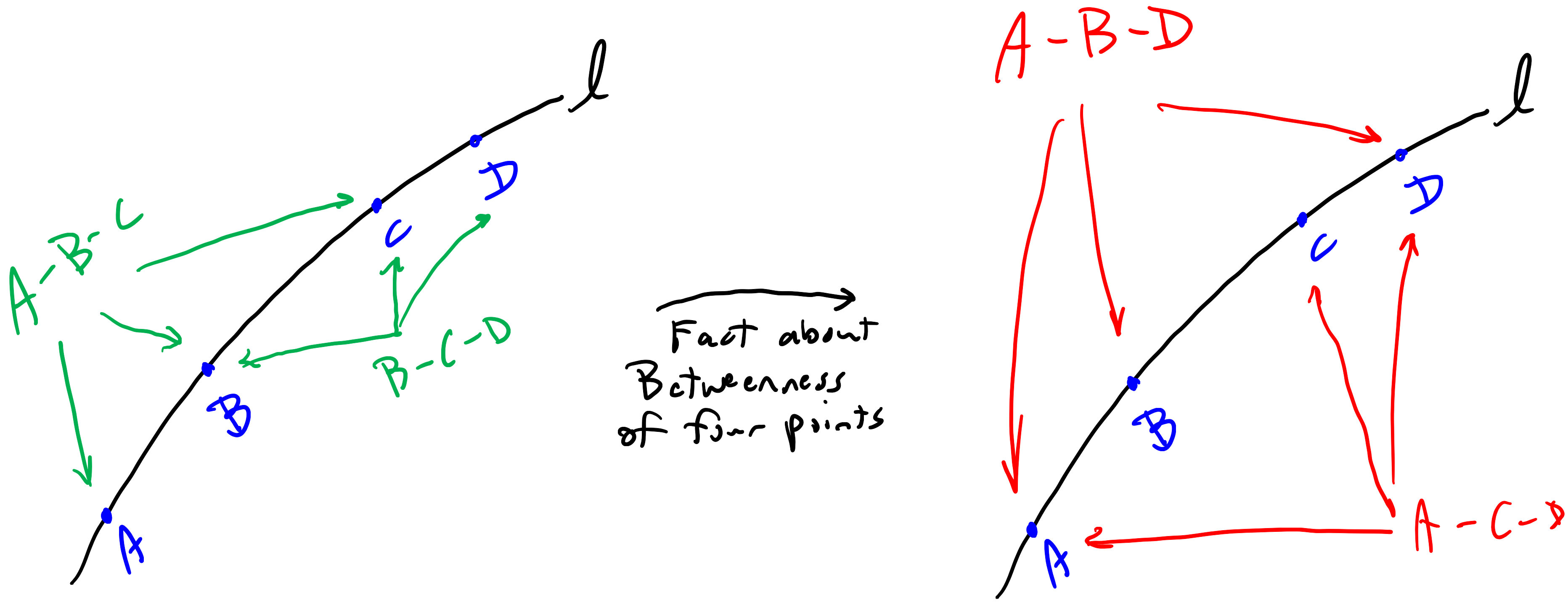


Fact about Betweenness involving Four Points

If $A - B - C$ and $B - C - D$ then $A - B - D$ and $A - C - D$.

Remark: Since all four relationships are true, we write $A - B - C - D$.

Illustration of the Statement



Proof

(1) Suppose $A - B - C$ and $B - C - D$.

(2) A, B, C must be distinct, collinear points in a metric geometry (by definition of $A - B - C$) and B, C, D must also be distinct and collinear in that same geometry (by definition of $B - C - D$), so we conclude that A, B, C, D must all be collinear.

(3) Let f be a ruler for the line that all four points lie on.

(4) Then $f(A) * f(B) * f(C)$ (by (1) and **Theorem 3.2.3 (i) ➔ (ii)**)

(5) Then $\left(f(A) < f(B) < f(C) \text{ } or \text{ } f(A) < f(B) < f(A)\right)$ by (4) and definition of betweenness for real numbers.

(6) and $f(B) * f(C) * f(D)$. (by (1) and **Theorem 3.2.3 (i) ➔ (ii)**)

(7) and $\left(f(B) < f(C) < f(D) \text{ } or \text{ } f(D) < f(C) < f(B)\right)$ by (4) and definition of betweenness for real numbers.

Case 1

(8) Suppose $f(A) < f(B) < f(C)$

(9) Then of the two possibilities in statement 7, it must be $f(B) < f(C) < f(D)$ that is true.

(10) Therefore $f(A) < f(B) < f(C) < f(D)$ (by (8) and (9))

(11) So $f(A) < f(B) < f(D)$ and $f(A) < f(C) < f(D)$ (by 10).

(12) So $A - B - D$ and $A - C - D$ in this case. (by (11) and **Theorem 3.2.3 (ii) ➔ (i)**)

Case 2

(13) Suppose $f(C) < f(B) < f(D)$.

(14),(15),(16),(17) Steps similar to (9),(10),(11),(12) will show that $A - B - D$ and $A - C - D$ in this case in this case as well.

Conclusion of Cases

(18) Therefore $A - B - D$ and $A - C - D$ (because it is true in both cases.)

End of Proof

Remark on How an Equivalence Theorem Gets Used in the Proof:

Notice how Theorem 3.2.3 (which is an Equivalence Theorem) gets used in the proof of this fact. In particular, look again at the way I wrote the justifications for Statement (4) and for Statement (12). They make clear what statement was *already known to be true*, and how Theorem 3.2.3 was used to justify saying that the *other* statement is *also true*. This is similar to the way I used Theorem 3.2.3 in my proof of Theorem 3.2.5.

End of Video