<u>3.3a: Introduction to Segments and Rays</u>

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for Ohio University MATH 3110/5110 College Geometry

Topics

- **Definitions of Segment and Ray and Basic Examples**
- **Equality of Segments and Equality of Rays**
- **Relationships Among Segments, Rays, and Lines**
- **Passing Points and Extreme Points**
- **Another Look at Equality of Segments and Equality of Rays**
- **Congruence of Segments**
- **The Segment Addition and Segment Subtraction Theorems (Corollaries, really)**

Reading: Section 3.3 Line Segments and Rays, p 52 - 58 in *Geometry: A Metric Approach with Models, Second Edition* by Millman & Parker

Homework: Section 3.3 # 4, 5, 6, 13, 14, 15

Important Stuff from Previous Sections

Definition of Betweenness for Points in a Metric Geometry

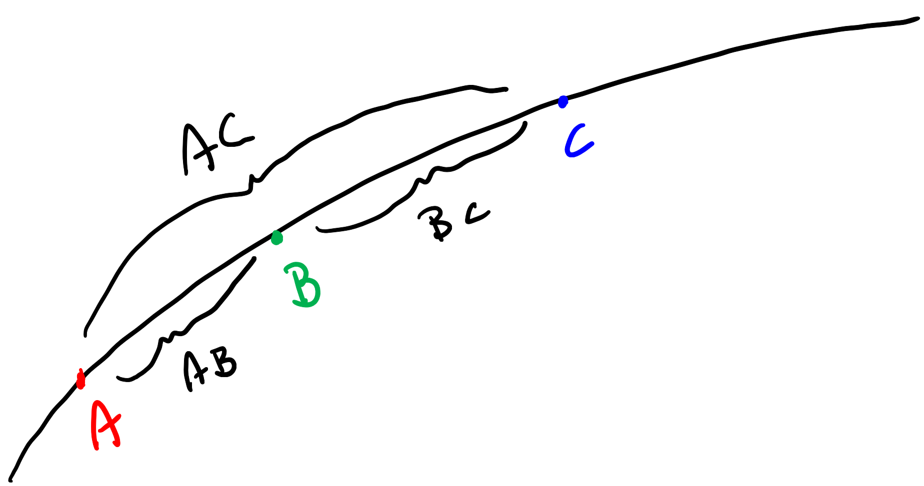
Symbol: $A - B - C$

Spoken: B *is between* A *and* C.

Usage: A, B, C are points in a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$.

Meaning: the following two things are both true

- A, B, C are distinct and collinear
- $d(A, C) = d(A, B) + d(B, C)$ That is, $AC = AB + BC$



Outside Points Can be Reversed in the Symbol for Betweenness

Theorem 3.2.2 (Really a Corollary of the Definition)

Given: Points A, B, C in a metric geometry

Claim: The following are equivalent (*TFAE*)

(i) $A - B - C$

(ii) $C - B - A$

But the Middle Point Cannot Be Swapped in the Symbol for Betweenness

Corollary 3.2.4 Fact about Three Distinct Collinear Points in a Metric Geometry

Given: Three distinct collinear points P, Q, R in a metric geometry

Claim: Exactly one of the points is between the other two.

Theorem 3.2.6 Existence of Points With Certain Betweenness Relationships

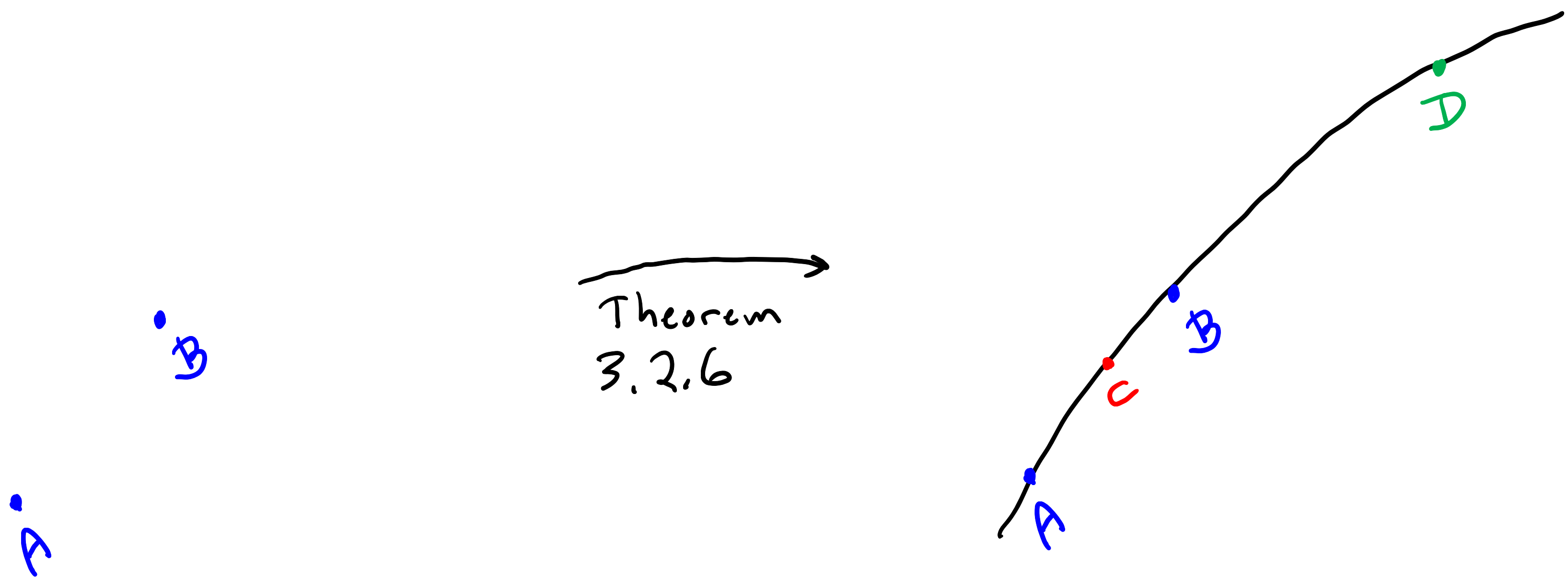
Given: Distinct points A, B in a *metric geometry*

Claim:

(i) There exists a point C with $A - C - B$

(ii) There exists a point D with $A - B - D$

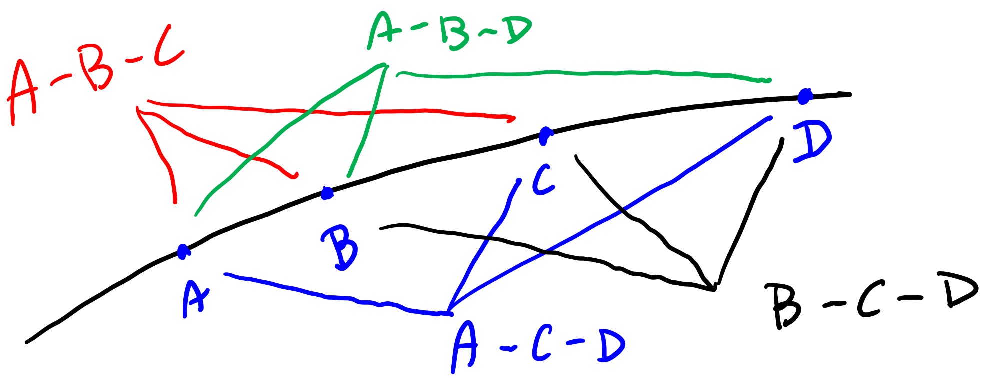
Illustration of the Statement of the Theorem



Definition of Betweenness for Four Points in a Metric Geometry

Symbol: $A - B - C - D$

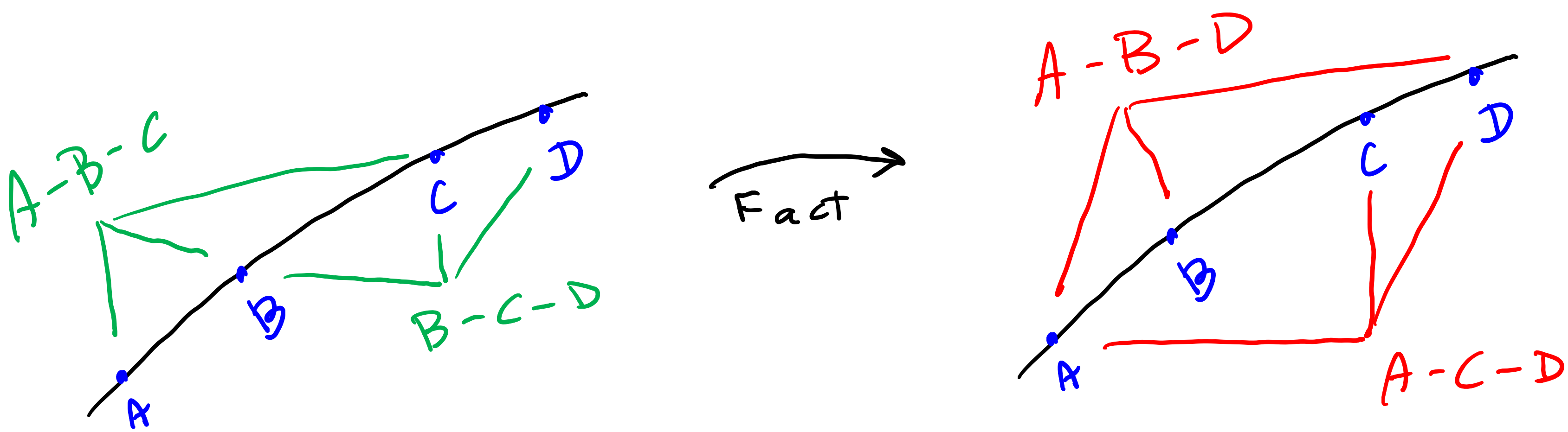
Meaning: $A - B - C$ and $A - B - D$ and $A - C - D$ and $B - C - D$.



Fact about Betweenness involving Four Points (from Exercise 3.2#7)

If $A - B - C$ and $B - C - D$ then $A - B - D$ and $A - C - D$.

Remark: Since all four relationships are true, we write $A - B - C - D$.

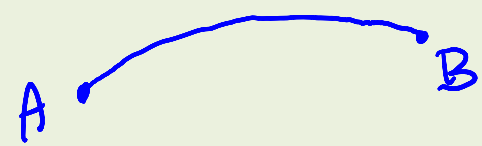


Section 3.3

Definitions of Segment and Ray and Basic Examples

Definition of Segment

Symbol: $\overline{AB}$



Spoken: *segment* $A\ B$.

Usage: A, B are distinct points in a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$.

Meaning: the set

$$\overline{AB} = \{C \in \mathcal{P} | C = A \text{ or } A - C - B \text{ or } C = B\}$$

Additional Terminology

The **end points** (or **vertices**) of $\overline{AB}$ are the points A and B.

The **interior of the segment** is the set of all points of the segment that are *not* endpoints:

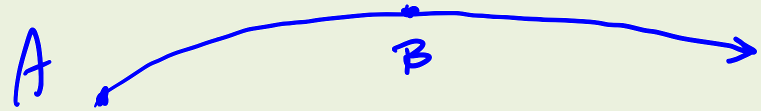
$$\text{int}(\overline{AB}) = \overline{AB} - \{A, B\} = \{C \in \mathcal{P} | A - C - B\}$$

Symbol: length$(\overline{AB})$

Spoken: the **length** of segment $\overline{AB}$

Meaning: the number AB. That is, the length is the number $d(A, B)$.

Definition of Ray



Symbol: $\overrightarrow{AB}$

Spoken: *ray A B*.

Usage: A, B are distinct points in a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$.

Meaning: the set

$$\overrightarrow{AB} = \{C \in \mathcal{P} | C = A \text{ or } A - C - B \text{ or } C = B \text{ or } A - B - C\}$$
$$= \overline{AB} \cup \{C \in \mathcal{P} | A - B - C\}$$

Additional Terminology

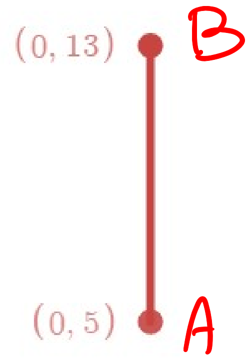
The **initial point** (or **vertex**) of $\overrightarrow{AB}$ is the point A.

The **interior of the ray** is the set of all points of the ray except the initial point:

$$\text{int}\left(\overrightarrow{AB}\right) = \overrightarrow{AB} - \{A\} = \{C \in \mathcal{P} | A - C - B \text{ or } C = B \text{ or } A - B - C\}$$

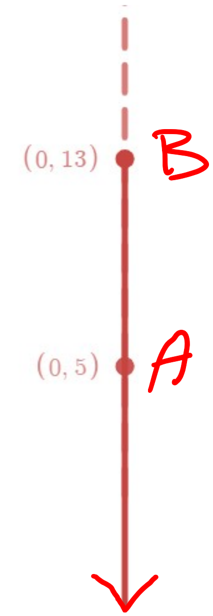
[Example 1] Let $A = (0,5)$ and $B = (0,13)$ and $C = (12,5)$

Euclidean segment $\overline{BA}$ is shown.

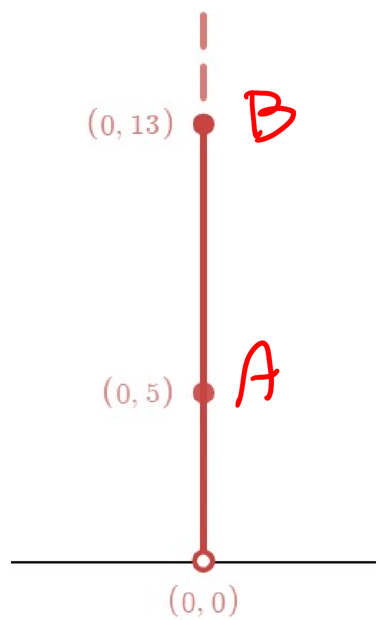


This is the same set of points as *Poincaré* segment $\overline{BA}$.

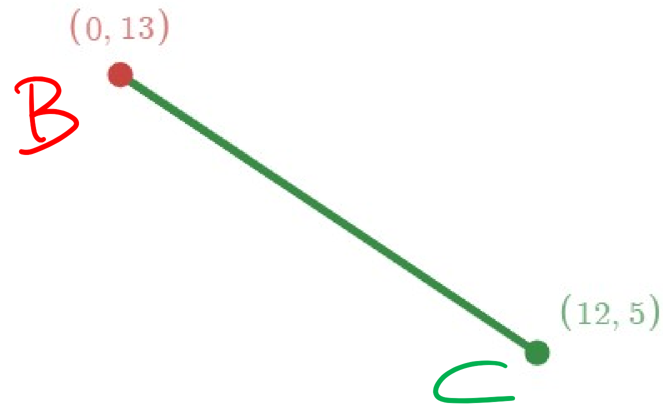
Euclidean ray $\overrightarrow{BA}$ is shown. Observe that it is not possible to draw the whole ray. The arrowhead conveys that the ray continues forever off-screen.



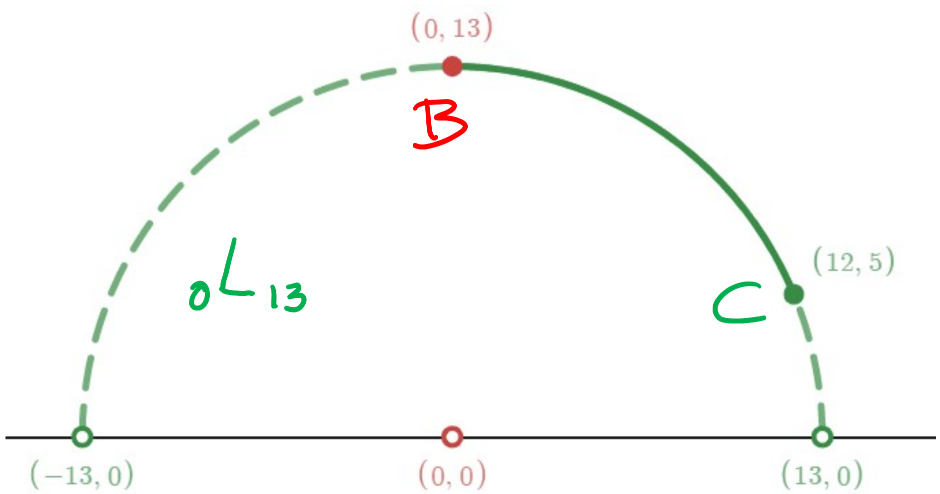
Poincaré ray $\overrightarrow{BA}$ is shown. Observe that it is *not* the same set of points as *Euclidean* ray $\overrightarrow{BA}$. Also observe that it is possible to draw the whole ray. The open circle at (0,0) denotes the missing endpoint of the ray.



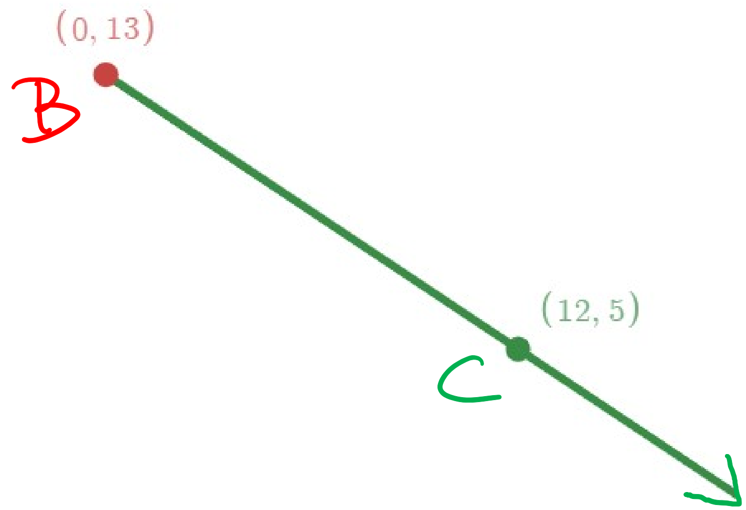
Euclidean segment $\overline{BC}$ is shown.



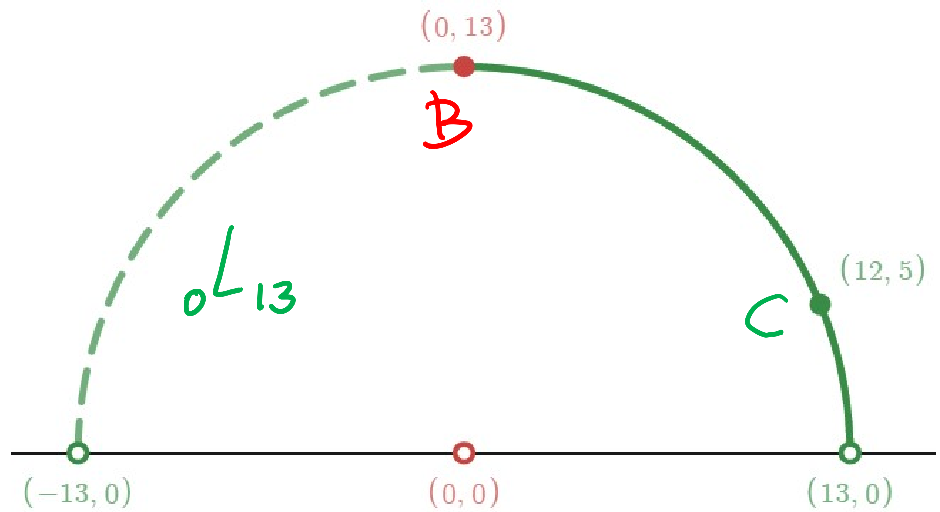
Poincaré segment $\overline{BC}$ is shown. Observe that it is *not* the same set of points as *Euclidean* segment $\overline{BC}$. Also note that my drawing of the segment includes a drawing of *Poincaré* line $\overleftrightarrow{BC}$, with the center of the semicircle and missing endpoints of the semicircle shown with their (x, y) coordinates, and with the line $\overleftrightarrow{BC}$ labeled.



Euclidean ray $\overrightarrow{BC}$ is shown.



Poincaré ray ~~$\overline{BA}$~~ $\overrightarrow{BC}$ is shown.



End of [Example 1]

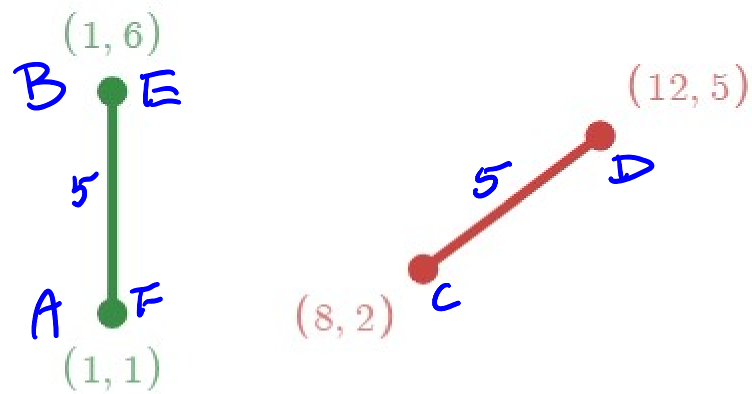
Equality of Segments and Equality of Rays

Note that *segments* and *rays* are defined as *sets of points* that satisfy certain requirements. Therefore, *equality of segments* and *equality of rays* refers to *equality of sets*. That is, to say that *two segments are equal* means that *they are the same set of points*.

[Example 2] Consider segments

$$\overline{AB}, \overline{CD}, \overline{EF}$$

shown at right.



Segments $\overline{AB}$ and $\overline{CD}$ are *not* equal, because they are not the same set of points. (They happen to have the same length, but they are not the same set of points.) We write $\overline{AB} \neq \overline{CD}$.

Segments $\overline{AB}$ and $\overline{EF}$ *are* equal, because they are the same set of points. We write $\overline{AB} = \overline{EF}$. It is important to notice, however, that $A \neq E$ and $B \neq F$.

End of [Example 2]

Relationships Among Segments, Rays, and Lines

[Example 3] Prove the following fact about the symbol for *segment*.

> **Order of Endpoints Does Not Matter in the Symbol for Segment**
>
> **Given:** Distinct points A, B in a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$
>
> **Claim:** $\overline{AB} = \overline{BA}$

This is a claim about equality of sets, so its proof must have the structure of a proof of set equality. But there is more than one style of proof of set equality. I will do two different styles of proof.

Proof #1

Proof Part 1: Show that $\overline{AB} \subset \overline{BA}$.

(1) Suppose that $C \in \overline{AB}$.

(2) Then $C = A$ or $A - C - B$ or $C = B$. (by (1) and definition of $\overline{AB}$.)

(3) So $C = B$ or $B - C - A$ or $C = A$. (by (2) and *commutativity of "or"* and Theorem 3.2.2)

(4)Therefore $C \in \overline{BA}$. (by (3) and definition of $\overline{BA}$.)

End of Proof Part 1

Proof Part 2: Show that $\overline{BA} \subset \overline{AB}$. Steps analogous to (1) – (4).

End of Proof #1

Proof #2

$$\overline{AB} = \{C \in \mathcal{P} | C = A \text{ or } A - C - B \text{ or } C = B\} \text{ (by definition of } \overline{AB}\text{)}$$

$$= \{C \in \mathcal{P} | C = B \text{ or } A - C - B \text{ or } C = A\} \text{ (by commutativity of the } \textit{or} \text{ operation)}$$

$$= \{C \in \mathcal{P} | C = B \text{ or } B - C - A \text{ or } C = A\} \text{ (by Theorem 3.2.2)}$$

$$= \overline{BA} \text{ (by definition of } \overline{BA}\text{)}$$

End of Proof #2

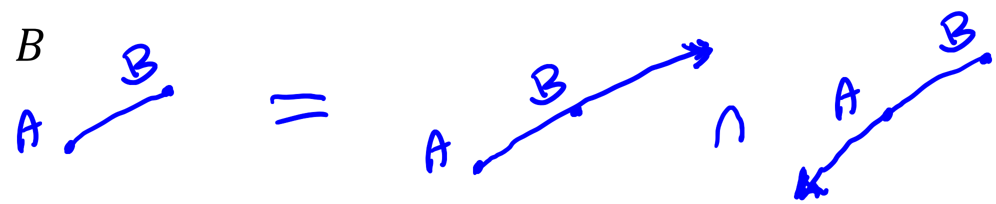
End of [Example 3]

Notice that the second proof is more concise and does not need two parts. For the proof of the statement $\overline{AB} = \overline{BA}$, both proof styles are easy enough. But for proofs of more complicated set relationships involving segments, rays, and lines, the second style of proof is much easier. The reason is that the first style of proof would involve cases, while the second style of proof does not.

[Example 4] Prove the following

Given: points A, B in a metric geometry, with $A \neq B$

Claim: $\overline{AB} = \overrightarrow{AB} \cap \overrightarrow{BA}$



Proof

$$\begin{aligned}\overrightarrow{AB} \cap \overrightarrow{BA} &= (\overline{AB} \cup \{C \in \mathcal{P} | A - B - C\}) \cap (\overline{BA} \cup \{C \in \mathcal{P} | B - A - C\}) \\ &= (\overline{AB} \cup \{C \in \mathcal{P} | A - B - C\}) \cap (\overline{AB} \cup \{C \in \mathcal{P} | B - A - C\}) \\ &= (\overline{AB} \cap \overline{AB}) \cup (\overline{AB} \cap \{C \in \mathcal{P} | B - A - C\}) \cup (\{C \in \mathcal{P} | A - B - C\} \cap \overline{AB}) \\ &\cup (\{C \in \mathcal{P} | A - B - C\} \cap \{C \in \mathcal{P} | B - A - C\}) \\ &= \overline{AB} \cup \phi \cup \phi \cup \phi \\ &= \overline{AB}\end{aligned}$$

End of Proof

End of [Example 4]

Passing Points and Extreme Points

Definition of Passing Point and Extreme Point of a Subset in a Metric Geometry

Words: B is a *passing point* of $\mathcal{A}$.

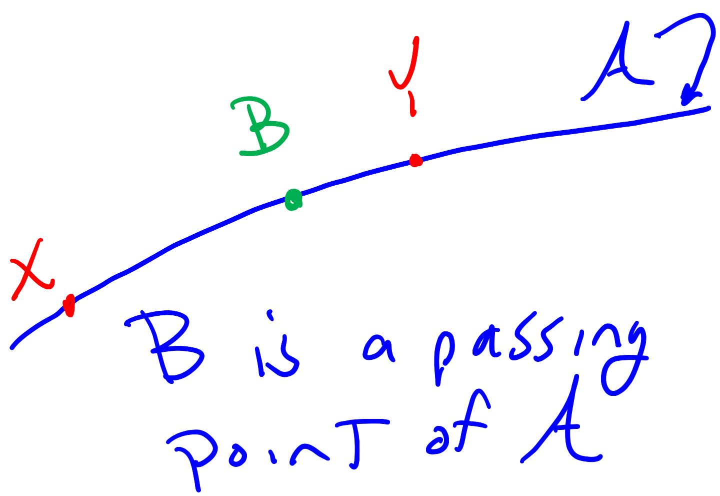
Usage: A metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$ in the discussion, and $\mathcal{A} \subset \mathcal{P}$ and $B \in \mathcal{A}$

Meaning: There exist points $X, Y \in \mathcal{A}$ such that $X - B - Y$

Words: B is an *extreme point* of $\mathcal{A}$.

Usage: Same usage

Meaning: B is *not* a *passing point* of $\mathcal{A}$.



Theorem 3.3.2 The Extreme Points of a Line Segment are the Endpoints

Given: distinct points A, B in a *metric geometry*

Claim: The only extreme points of $\overline{AB}$ are endpoints A, B; All other points are passing points.

(**Corollary:** If $\overline{AB} = \overline{CD}$, then $\{A, B\} = \{C, D\}$.

Proof

(1) Given distinct points A, B in a *metric geometry*.

Part I Show that elements of $\overline{AB}$ that are not endpoints are *passing points*.

(2) Suppose $C \in \overline{AB}$ and C is not one of the endpoints A, B.

(3) Then $A - C - B$ (by (2) and definition of segment $\overline{AB}$.)

(4) So C is a passing point of $\overline{AB}$ (by (3) and definition of *passing point*.

Part II Show that A is an *extreme point*. (Proof by contradiction.)

(5) Suppose that A is *not* an *extreme point* of $\overline{AB}$.That is, suppose that A is a *passing point* of $\overline{AB}$.

(assumption for proof by contradiction.)

(6) There exist points $X, Y \in \overline{AB}$ such that $X - A - Y$. (by (5) and definition of *passing point*.)

(7) Either $Y = B$ or $A - Y - B$ (by (6) and definition of segment)

(8) **(Case 1)** Suppose $Y = B$.

(9) Then $X - A - B$ (by (6) and (8))

(10) Point X cannot be an element of segment $\overline{AB}$. (by (9) and definition of *segment*)

(11) **Statement (10) contradicts statement (6).** Therefore, our assumption in step (5) was wrong. A cannot be a passing point. in this case.

(12) **(Case 2)** Suppose $A - Y - B$.

(13) Then $X - A - B$ (by (6) and (12) and result of Exercise 3.2#7)

(14) Point X cannot be an element of segment $\overline{AB}$. (by (13) and definition of *segment*)

(15) **Statement (14) contradicts statement (6).** Therefore, our assumption in step (5) was wrong. A cannot be a passing point. in this case.

(16) **(Conclusion of Cases)** Therefore A cannot be a passing point of $\overline{AB}$. (Because it is true in each case.) In other words, A must be an extreme point.

Part III Show that *B* is an *extreme point*.

Steps identical to the steps that show that A is an extreme point.

End of Proof

Missing Theorem: The Only Extreme Point of a Ray is the Initial Point

Given: distinct points A, B in a *metric geometry*

Claim: The only extreme point of $\overrightarrow{AB}$ is A; All other points are passing points.

(**Corollary:** If $\overrightarrow{AB} = \overrightarrow{CD}$, then $A = C$.

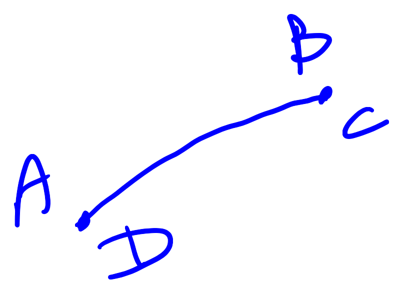
This theorem would be proven in the same way as Theorem 3.3.2.

I will leave it to you to do this proof.

Another Look at Equality of Segments and Equality of Rays

Subtlety in the Notation for a Segment

If two segments are equal then their endpoints are equal.) If $\overline{AB} = \overline{CD}$, then $\{A, B\} = \{C, D\}$.



The proof is straightforward if we use the notion of *extreme points*.

Proof

(1) Suppose $\overline{AB} = \overline{CD}$.

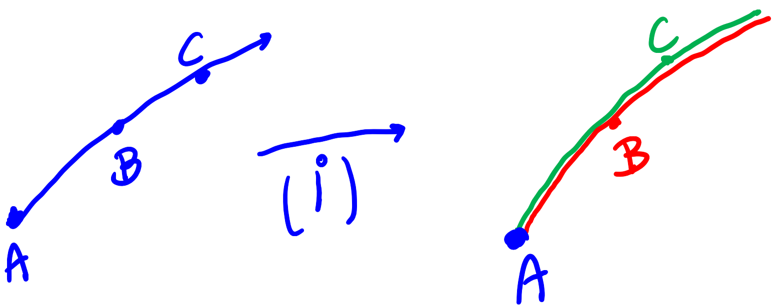
(2) Then the set of extreme points of $\overline{AB}$ is the same as the set of extreme points of $\overline{CD}$.

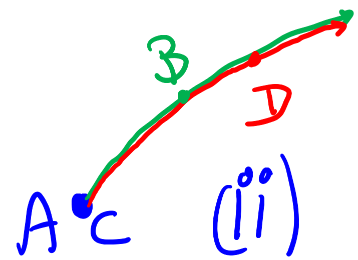
(3) Therefore, $\{A, B\} = \{C, D\}$.

End of Proof

There is also Subtlety in the Notation for a Ray

> **Theorem 3.3.4 Subtlety in the Notation for a Ray**
>
> **(i) (Different symbols that represent the same ray.)** If $C \in \overrightarrow{AB}$ and $C \neq A$, then $\overrightarrow{AC} = \overrightarrow{AB}$.
>
> **(ii) (If two rays are equal then their initial points are equal.)** If $\overrightarrow{AB} = \overrightarrow{CD}$, then $A = C$.





The proof of (ii) is straightforward if we use the notion of *extreme points*. I won't do the proof here.

But the proof of (i) would be very difficult if we just consider the definitions of the sets $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Many cases would be involved, and the proof would be unbearably tedious to read. I will postpone the proof until the next video, where we study the use of *rulers* and *coordinates* in describing segments and rays.

Congruence of Line Segments

Definition of Segment Congruence

Symbol: $\overline{AB} \simeq \overline{CD}$

Words: *Segment A B is congruent to segment C D.*

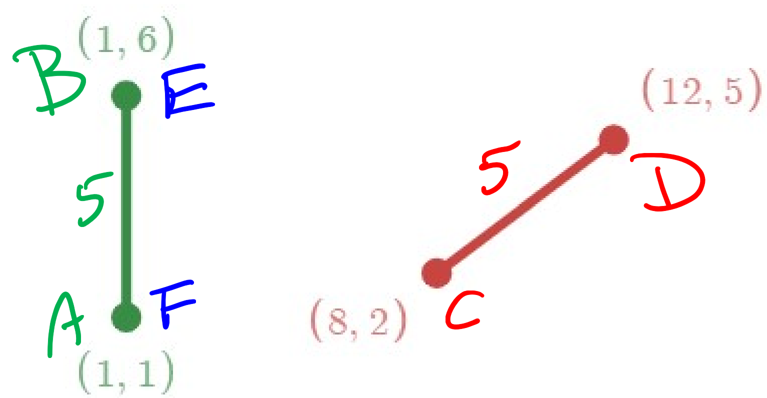
Usage: A metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$ in the discussion, and $A, B, C, D \in \mathcal{P}$.

Meaning: $\text{length}(\overline{AB}) = \text{length}(\overline{CD})$

[Example 2, revisited] Consider segments

$$\overline{AB}, \overline{CD}, \overline{EF}$$

shown at right.



Observe that $\overline{AB} \neq \overline{CD}$ but $\overline{AB} \simeq \overline{CD}$.

Observe that $\overline{AB} = \overline{EF}$ and $\overline{AB} \simeq \overline{EF}$.

End of [Example 2, revisited]

Segment Congruence is an Equivalence Relation

Observe that in a metric geometry, segment congruence is a relation on the set of all line segments in that geometry. You should be able to prove that segment congruence is an *equivalence relation*. I will remind you of some of the proof structure, but you should be able to supply the remaining structure (for the proof of Transitivity) and the details of the proof.

Proof that Segment Congruence is Reflexive

(1) Suppose that $\overline{AB}$ is a segment.

*** some statements here ***

(*) Conclude that $\overline{AB}$ is congruent to itself.

End of proof of reflexivity

Proof that Segment Congruence is Symmetric

(1) Suppose that $\overline{AB}$ is congruent to $\overline{CD}$.

*** some statements here ***

(*) Conclude that $\overline{CD}$ is congruent to $\overline{AB}$.

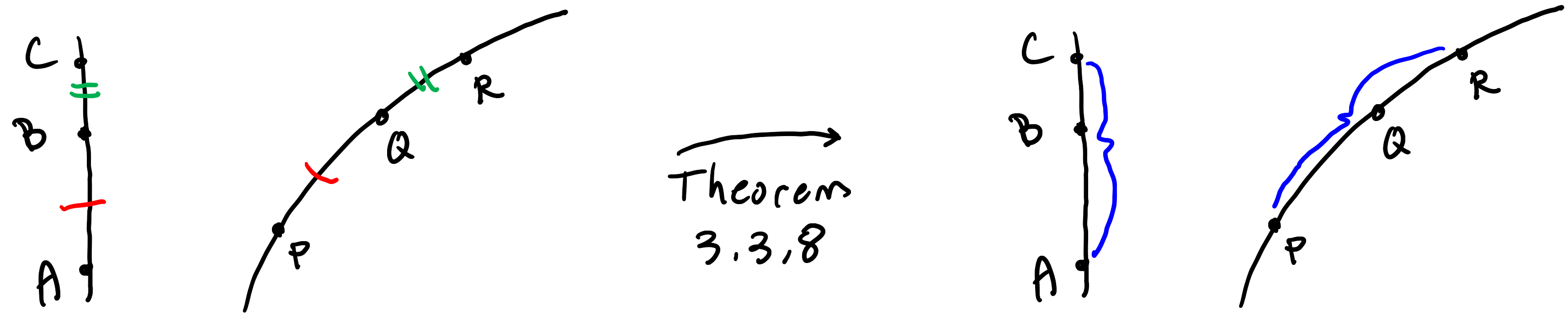
End of proof of reflexivity

Segment Addition and Segment Subtraction Theorems

Theorem 3.3.8: If $A-B-C$ and $P-Q-R$ and $\overline{AB} \simeq \overline{PQ}$ and $\overline{BC} \simeq \overline{QR}$, then $\overline{AB} \simeq \overline{PR}$.

Theorem 3.3.9: If $A-B-C$ and $P-Q-R$ and $\overline{AB} \simeq \overline{PQ}$ and $\overline{AC} \simeq \overline{PR}$, then $\overline{BC} \simeq \overline{QR}$.

Illustration of the Statement of Theorem 3.3.8



I will leave it to you to come up with an illustration of the statement of Theorem 3.3.9.

You are assigned to prove these theorems in a homework exercises. You should find that after setting up the correct proof *frame* and then recognizing *defined terms* in the proof frame that need to be *unpacked* or *packed up*, the proofs will not require any clever steps to bridge the gap in the middle. The proofs are so simple that these *theorems* should really just be called *corollaries* of the definitions of *betweenness* and segment *congruence*.

End of Video