3.3b: Rulers Dictate Many Properties of Segments and Rays

produced by Mark Barsamian, 2021.02.25

for Ohio University MATH 3110/5110 College Geometry

Topics

- **Coordinate Description of Rays and Segments**
- **Vector Description of Rays and Segments in the Euclidean plane**
- **Revisiting Theorem 3.3.4 about Equality of Rays**
- **The Congruent Segment Construction Theorem**
- **Midpoints**

Reading: Section 3.3 Line Segments and Rays, p 52 - 58 in *Geometry: A Metric Approach with Models, Second Edition* by Millman & Parker

Homework: Section 3.3 # 2, 3, 9, 11, 12

Important Stuff from Previous Sections

Using Vectors to Describe Lines and Rulers

Proposition 3.1.2 Using Vectors to Describe Cartesian Lines

Given two distinct points $A, B \in \mathbb{R}^2$ line $\overleftrightarrow{AB}$ can be described using vectors as follows:

$$L_{AB} = \{X \in \mathbb{R}^2 | X = A + t(B - A) \text{ for some } t \in \mathbb{R}\}$$

Observe that the use of the letter X is not really necessary.

$$L_{AB} = \{A + t(B - A) | t \in \mathbb{R}\}$$

Proposition 3.1.4 Using Vectors to Describe Rulers in the Euclidean Plane

If L_{AB} is a cartesian line, then $f: L_{AB} \to \mathbb{R}$ defined by

$$f\big(A + t(B - A)\big) = t\|B - A\|$$

is a ruler for the line L_{AB} in the *Euclidean plane*.

Remark: And we know that the ruler described in Proposition 3.1.4 is a special ruler for points A and B. That is has A as origin and B positive. (That is, $f(A) = 0$ and $f(B) > 0$.)

Definition of Betweenness for Points in a Metric Geometry

Symbol: $A - B - C$

Spoken: B *is between* A *and* C.

Usage: A, B, C are points in a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$.

Meaning: the following two things are both true

- A, B, C are distinct and collinear
- $d(A, C) = d(A, B) + d(B, C)$ That is, $AC = AB + BC$

Theorem 3.2.3 Betweenness of Points is Related to Betweenness of Coordinates

Given: Collinear points A, B, C on line l with ruler f in a metric geometry

Claim: The following are equivalent (*TFAE*)

(i) $A - B - C$ (betweenness of *points*)

(ii) $f(A) * f(B) * f(C)$ (betweenness of *coordinates*)

Theorem 3.2.6 Existence of Points With Certain Betweenness Relationships

Given: Distinct points A, B in a *metric geometry*

Claim:

(i) There exists a point C with $A - C - B$

(ii) There exists a point D with $A - B - D$

Definition of Betweenness for Four Points in a Metric Geometry

Symbol: $A - B - C - D$

Meaning: $A - B - C$ and $A - B - D$ and $A - C - D$ and $B - C - D$.

Fact about Betweenness involving Four Points (from Exercise 3.2#7)

If $A - B - C$ and $B - C - D$ then $A - B - D$ and $A - C - D$.

Remark: Since all four relationships are true, we write $A - B - C - D$.

Stuff from Section 3.3 Introduced in Previous Video

Definitions and Basic Examples

Definition of Line Segment

Symbol: $\overline{AB}$

Spoken: *segment* $A\ B$.

Usage: A, B are distinct points in a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$.

Meaning: the set

$$\overline{AB} = \{C \in \mathcal{P} | C = A \text{ or } A - C - B \text{ or } C = B\}$$

Additional Terminology

The **end points** (or **vertices**) of $\overline{AB}$ are the points A and B.

The **interior of the segment** is the set of all points of the segment that are *not* endpoints:

$$\text{int}(\overline{AB}) = \overline{AB} - \{A, B\} = \{C \in \mathcal{P} | A - C - B\}$$

The **length** of segment $\overline{AB}$ is the number AB. That is, the length is the number $d(A, B)$.

Definition of Ray

Symbol: $\overrightarrow{AB}$

Spoken: *ray* $A\ B$.

Usage: A, B are distinct points in a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$.

Meaning: the set

$$\overrightarrow{AB} = \{C \in \mathcal{P} | C = A \text{ or } A - C - B \text{ or } C = B \text{ or } A - B - C\}$$
$$= \overline{AB} \cup \{C \in \mathcal{P} | A - B - C\}$$

Additional Terminology

The **initial point** (or **vertex**) of $\overrightarrow{AB}$ is the point A.

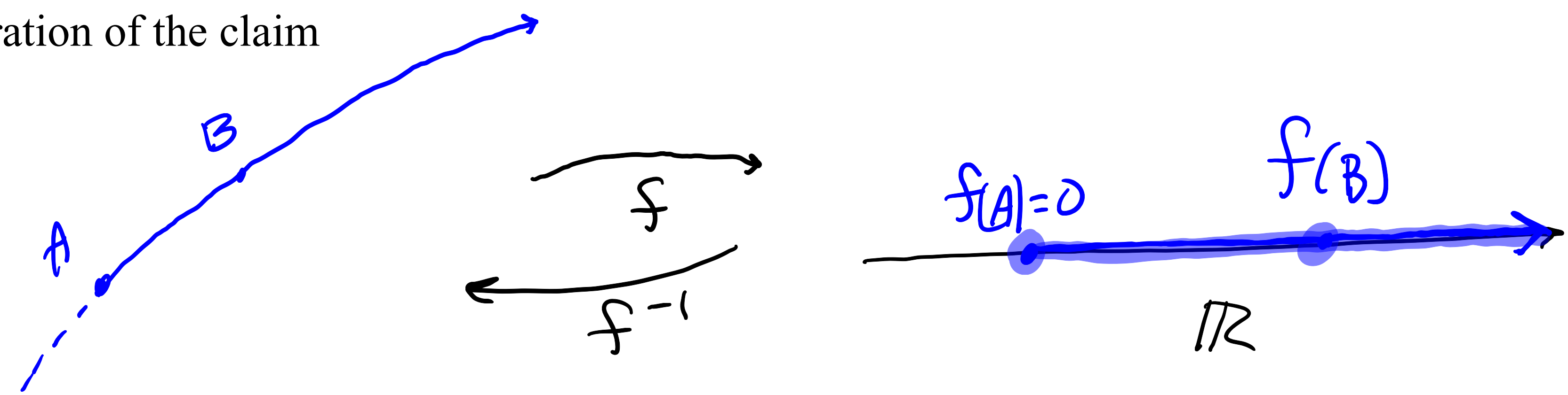
The **interior of the ray** is the set of all points of the ray except the initial point:

$$\text{int}\left(\overrightarrow{AB}\right) = \overrightarrow{AB} - \{A\} = \{C \in \mathcal{P} | A - C - B \text{ or } C = B \text{ or } A - B - C\}$$

New Material from Section 3.3 Coordinate Description of Rays and Segments

> **Theorem 3.3.5 The Coordinate Description of Rays (and segments)**
>
> **Given:** Distinct points A, B in a *metric geometry*
>
> **Claim:** There exists a ruler $f: \overleftrightarrow{AB} \to \mathbb{R}$ such that
>
> $$\overrightarrow{AB} = \{X \in \overleftrightarrow{AB} | f(X) \geq 0\} = f^{-1}([0, \infty))$$
>
> **(Missing Claim)** For the same ruler f, it will be true that
>
> $$\overline{AB} = \{X \in \overleftrightarrow{AB} | 0 \leq f(X) \leq f(B)\} = f^{-1}([0, f(B)])$$

Illustration of the claim



There is a proof of Theorem 3.3.5 in the book. The structure of that proof is perhaps not as clear as it could be. Here are the headings for the major parts.

Part I: Prove that if $X \in \overleftrightarrow{AB}$ and $f(X) \geq 0$ then $X \in \overrightarrow{AB}$.

Part II: Prove that if $X \in \overleftrightarrow{AB}$ and $f(X) < 0$ then $X \notin \overrightarrow{AB}$.

Vector Description of Rays and Segments in the Euclidean plane

Proposition 3.3.3 The Coordinate Description of Rays and Segments in the Euclidean Plane

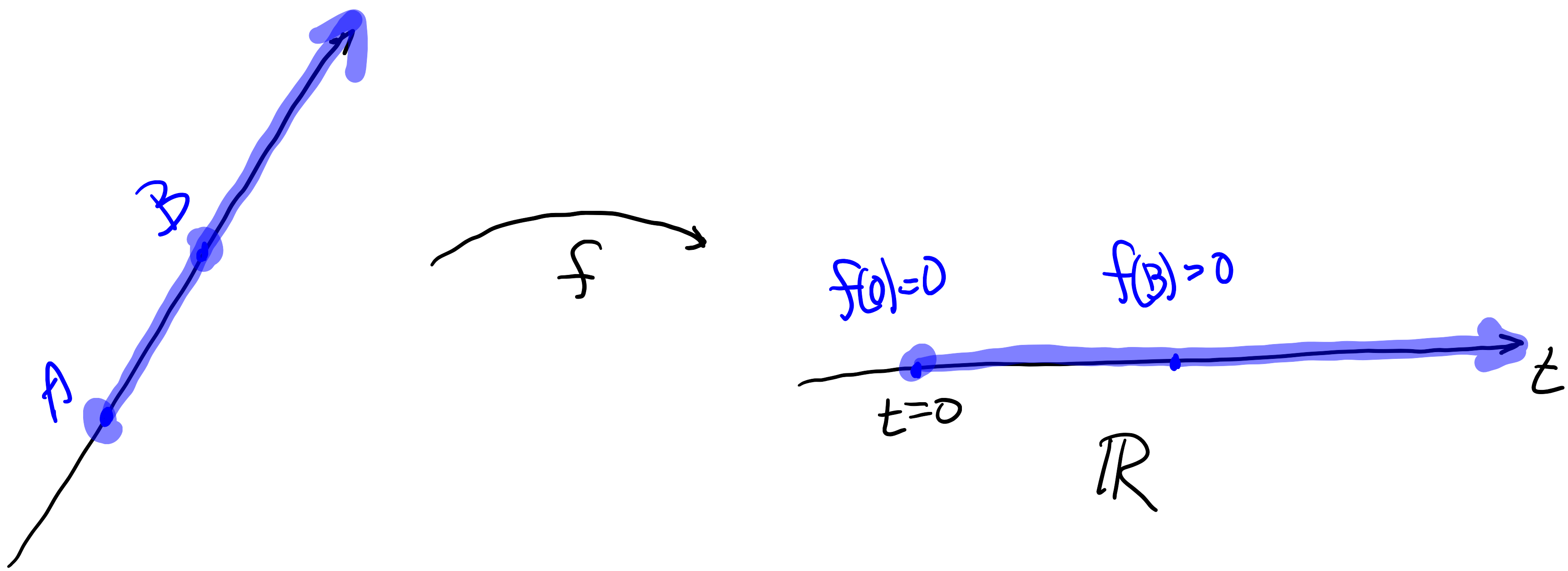
Given: Distinct points A, B in the *Euclidean plane*

Claim: Ray $\overrightarrow{AB}$ and segment $\overline{AB}$ can be described using vectors as follows

$$\overrightarrow{AB} = \{A + t(B - A) \text{ for some } t \in \mathbb{R} \text{ such that } 0 \leq t\}$$

$$\overline{AB} = \{A + t(B - A) \text{ for some } t \in \mathbb{R} \text{ such that } 0 \leq t \leq 1\}$$

Illustration of the Statement of the Proposition



Observe the book's ordering and proofs of Proposition 3.3.3, Theorem 3.3.4, and Theorem 3.3.5.

- First: Prop. 3.3.3 (about the coordinate description of rays and segments)
- Second: Thm 3.3.4 (about the vector description of rays and segments in the Euclidean plane). (Proof of Prop. 3.3.4 does not use Prop. 3.3.3 in any way.)
- Third: Theorem 3.3.5 The Coordinate Description of Rays (and segments) (Proof of Thm. 3.3.5 does not use Prop. 3.3.3 or Theorem 3.3.4 in any way.)

I would have been better for the authors to put Theorem 3.3.5 first, because it can be used to make the proof of Prop 3.3.3 and Thm. 3.3.4 very simple. I will use the following ordering in my notes.

- First: Theorem 3.3.5 The Coordinate Description of Rays (and segments)
- Second: Prop. 3.3.3 (about the coordinate description of rays and segments) (Proof of Prop. 3.3.3 uses Theorem 3.3.5)
- Third: Thm. 3.3.4 (about the vector description of rays and segments in the Euclidean plane). (Proof of Thm.. 3.3.4 uses Theorem 3.3.5)

Proof of Proposition 3.3.3 (Proof Uses Theorem 3.3.5)

Proof Part 1: Prove the claim about $\overrightarrow{AB}$.

(1) Let $\overrightarrow{AB}$ be a ray in the *Euclidean plane*.

(2) By Proposition 3.1.4, the function $f: L_{AB} \to \mathbb{R}$ defined by the following vector equation

$$f\big(A + t(B - A)\big) = t\|B - A\|$$

is a ruler for the line L_{AB} in the *Euclidean plane*.

(3) f is a special ruler with A as origin and B positive. (by the result of Exercise 3.1#5)

(4) By **Theorem 3.3.5**, ray $\overrightarrow{AB}$ can be described in terms of its coordinates as follows.

$$\overrightarrow{AB} = \left\{X \in \overleftrightarrow{AB} \middle| f(X) \geq 0\right\} = f^{-1}\big([0, \infty)\big)$$

(5) But by Proposition 3.1.2, the set of $X \in \overleftrightarrow{AB}$ that have $f(X) \geq 0$ will be the following set:

$$\left\{X \in \overleftrightarrow{AB} \middle| f(X) \geq 0\right\} = \{A + t(B - A) | t \in \mathbb{R} \text{ and } t \geq 0\}$$

In other words, $\overrightarrow{AB} = \{A + t(B - A)$ for some $t \in \mathbb{R}$ such that $0 \leq t\}$

End of Proof Part 1

Proof Part 2: Prove the claim about $\overline{AB}$:

This proof will be very similar to Proof Part 1.

Using Coordinate Descriptions of Rays and Segments

Now that we have learned about the *Coordinate Description of Rays and Segments*, we can use those descriptions to prove some important facts. But first, before proving those important facts, it we should take a moment to revisit a theorem that was introduced in the previous video, but whose proof was postponed.

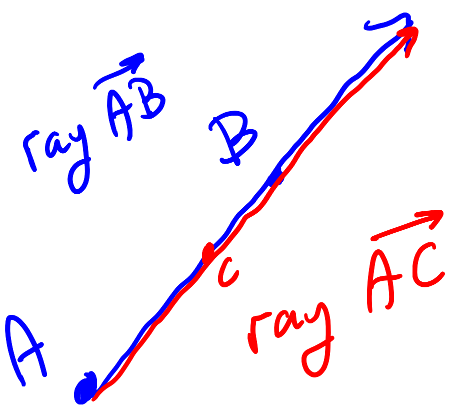
Revisiting Theorem 3.3.4 about Equality of Rays

In the previous video, I discussed Theorem 3.3.4, about some subtlety in the notation for a ray.

Theorem 3.3.4 Subtlety in the Notation for a Ray

(i) (Different symbols that represent the same ray.) If $C \in \overrightarrow{AB}$ and $C \neq A$, then $\overrightarrow{AC} = \overrightarrow{AB}$.

(ii) (If two rays are equal then their initial points are equal.) If $\overrightarrow{AB} = \overrightarrow{CD}$, then $A = C$.



In the previous video, Video 3.3a, we saw that the proof of statement (ii) of the theorem was easy, using the concept of extreme points of a ray. But in that video I postponed the proof of statement (i), saying that its proof would be very difficult using just the techniques that we knew at the time. Now that we know about the coordinate description of rays, the proof is fairly straightforward.

Proof of Theorem 3.3.4(i) (Proof Uses Theorem 3.3.5)

(1) Suppose $C \in \overrightarrow{AB}$ and $C \neq A$.

(2) Then $A - C - B$ or $C = B$ or $A - B - C$. (by (1) and definition of *ray*)

(3) Let $f: \overleftrightarrow{AB} \to \mathbb{R}$ be a special ruler for line $\overleftrightarrow{AB}$ with A as origin and B positive. (We know that such a ruler exists by **Theorem 2.3.2**, the **Ruler Placement Theorem**.)

(4) Then $f(A) * f(C) * f(B)$ or $f(C) = f(B)$ or $f(A) * f(B) * f(C)$ (by (2) and Thm 3.2.3 about the relationship between the *betweenness of points* and *betweeness of their coordinates*.)

(5) But then $0 < f(C) < f(B)$ or $f(C) = f(B)$ or $0 < f(B) < f(C)$ (by (3) and (4)).

(6) Observe that in every case of statement (5), $f(C)$ is positive. Also note that $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ are the same line. So the ruler f introduced in step (3) could be thought of as a ruler for line $\overleftrightarrow{AC}$ with A as origin and C positive. Realize that f is also the ruler mentioned in **Theorem 3.3.5**. That is,

(7) $\overrightarrow{AB} = \{X \in \overleftrightarrow{AB} | f(X) \geq 0\} = \{X \in \overleftrightarrow{AC} | f(X) \geq 0\} = \overrightarrow{AC}$

End of proof

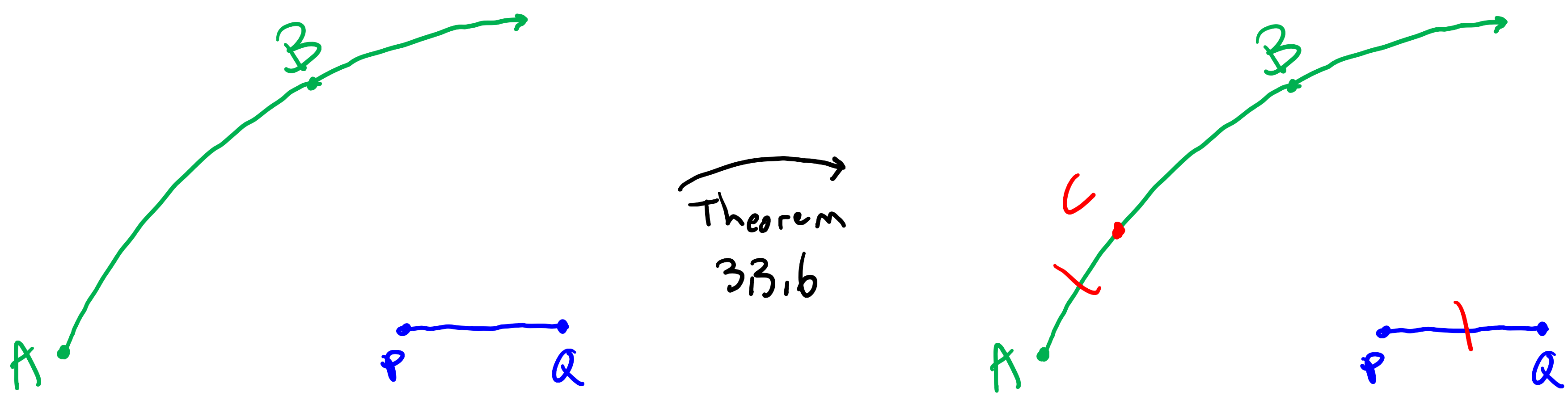
As mentioned three pages ago, now that we have learned about the *Coordinate Description of Rays and Segments*, we can use those descriptions to prove some important facts. The first of these important facts is the following theorem.

Theorem 3.3.6 The Congruent Segment Construction Theorem

Given: ray $\overrightarrow{AB}$ and segment $\overline{PQ}$

Claim: There exists a unique point $C \in \overrightarrow{AB}$ such that $\overline{AC} \simeq \overline{PQ}$.

Illustration of the Statement of the Theorem



There is a nice proof of this theorem in the book. It uses the *Coordinate Description of Rays*.

[Example 1] Example Illustrating Theorem 3.3.6 (Congruent Segment Construction Theorem)

In the Poincaré Plane, let $P = (-15,7)$ and $Q = (-15,14)$ and $A = (4,13)$ and $B = (16,5)$.

(a) Find the point $C \in \overrightarrow{AB}$ such that $\overline{AC} \simeq \overline{PQ}$.

Solution:

Get distance PQ.

P and Q have the same x coordinate, so Poincaré line $\overleftrightarrow{PQ}$ is the *type I line* ${}_{-15}L$.

Therefore, distance PQ will be given by the simple formula

$$PQ = d_H(P,Q) = d_H\big((-15,7),(-15,14)\big) = \left|\ln\left(\frac{14}{7}\right)\right| = \ln(2)$$

Get a special ruler for Poincaré line $\overleftrightarrow{AB}$

A and B do not have the same x coordinate, so Poincaré line $\overleftrightarrow{AB}$ is a *type II line*.

Because of this, we will need to find need c, r for the line in order to write its coordinate function

$$c = \frac{{y_2}^2 - {y_1}^2 + {x_2}^2 - {x_1}^2}{2(x_2 - x_1)} = \frac{5^2 - 13^2 + 16^2 - 4^2}{2(16-4)} = 4$$

$$r = \sqrt{(x_1 - c)^2 + {y_1}^2} = \sqrt{(4-4)^2 + 13^2} = 13$$

Using these, the *standard ruler* for Poincaré line $\overleftrightarrow{AB}$ is

$$f(x,y) = \ln\left(\frac{x-c+r}{y}\right) = \ln\left(\frac{x-4+13}{y}\right) = \ln\left(\frac{x+9}{y}\right)$$

We need to figure out if the *standard ruler* is a *special ruler*.

Get the coordinate $f(A)$ of point A on Poincaré line $\overleftrightarrow{AB}$.

$$f(A) = f(4,13) = \ln\left(\frac{4+9}{13}\right) = \ln(1) = 0$$

Get the coordinate $f(B)$ of point B on Poincaré line $\overleftrightarrow{AB}$.

$$f(B) = f(16,5) = \ln\left(\frac{16+9}{5}\right) = \ln(5)$$

So notice that using the coordinate function f, the coordinate of A is zero and the coordinate of B is positive. That is, f *is* a *special ruler*.

By Theorem 3.3.5 (the Coordinate Description of Rays and Segments), we know that all of the points C on ray $\overrightarrow{AB}$ will have coordinate $f(C) \geq 0$.

Set up equations that point $C = (x, y)$ must satisfy, and then solve them.

Point $C = (x, y)$ must satisfy the equation $PQ = AC$ and $f(C)$ must be positive.

That is, $PQ = AC = d_H(A, C) \underset{\substack{Ruler\\Equation}}{=} \left| \underbrace{f(A)}_{zero} - \underbrace{f(C)}_{positive} \right| = f(C)$

Using the known value of PQ and the known formula for the coordinate function f, we obtain,

$$\ln(2) = \ln\left(\frac{x+9}{y}\right)$$

$$2 = \frac{x+9}{y}$$

$$2y = x - 1 \quad \text{(coordinate equation)}$$

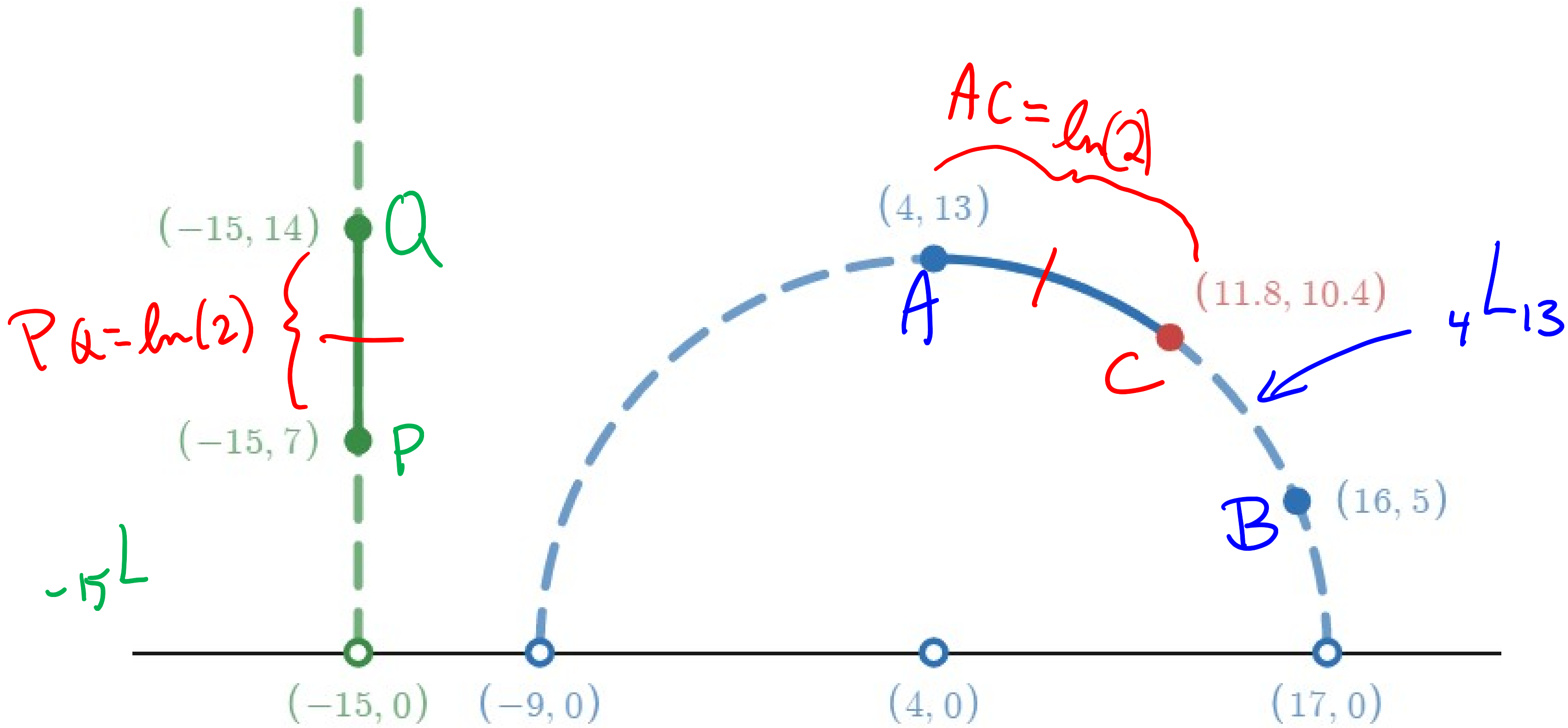
Point C must also satisfy the *circle equation* $(x - c)^2 + y^2 = r^2$, with $y > 0$, and with $c = 4$ and $r = 13$.

So we have two equations in the two unknowns x, y

$$\begin{cases} 2y = x + 9 \quad \text{(coordinate equation)} \\ (x-4)^2 + y^2 = 13^2 = 169 \;\; with\; y > 0 \quad \text{(circle equation)} \end{cases}$$

Solving these equations, we find $C = \left(\frac{59}{5}, \frac{52}{5}\right)$.

(b) Illustrate your solution to **(a)**.

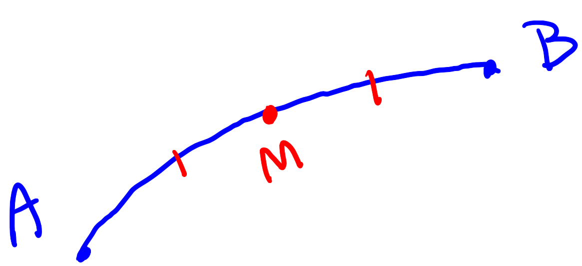


End of [Example 1]

Midpoints

As mentioned many pages ago, now that we have learned about the *Coordinate Description of Rays and Segments*, we can use those descriptions to prove some important facts. The second of these important facts has to do with *midpoints* of segments.

> **Definition of Midpoint of a Segment**
>
> **Words:** *M is a midpoint of* $\overline{AB}$
>
> **Usage:** A, B are distinct points in a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$.
>
> **Meaning:** M is a point that satisfies both of these requirements:
>
> - $M \in \overleftrightarrow{AB}$
> - $AM = MB$



Observe that I have drawn the midpoint M so that $A - M - B$. In one of your homework exercises, you will prove the following

$$\text{If } M \text{ is a midpoint of } \overline{AB}, \text{ then } A - M - B.$$

The key to proving this fact is to consider possible locations of M on line $\overleftrightarrow{AB}$ relative to A and B.

There are five possibilities:

$$M - A - B \quad \text{or} \quad M = A \quad \text{or} \quad A - M - B \quad \text{or} \quad M = B \quad \text{or} \quad A - B - M$$

Show that in any of the following four cases

$$M - A - B \quad \text{or} \quad M = A \quad \text{or} \quad M = B \quad \text{or} \quad A - B - M$$

the resulting distances AM and MB cannot be equal. (Use a ruler for line $\overleftrightarrow{AB}$ to make this argument.)

Therefore, the only possiblity left is $A - M - B$.

You will also prove the following Theorem in another homework exercise.

Theorem: Existence and Uniqueness of the Midpoint of a Segment

If A, B are distinct points in a metric geometry, then segment $\overline{AB}$ has exactly one midpoint.

Here are some hints for proving the theorem:

- Let f be a ruler for line $\overleftrightarrow{AB}$ with A as origin and B positive. That is, $f(A) = 0$ and $f(B) > 0$.
- Guess what the coordinate $f(M)$ would have to be.
- Check to see that if M has that coordinate, then M will indeed have the qualifications to be called a midpoint. This will prove that a midpoint exists.

End of Video