Video 4.1: Introduction to Plane Separation

produced by Mark Barsamian, 2021.03.03

for Ohio University MATH 3110/5110 College Geometry

Topics

- **Observations about the way that lines split a drawing.**
- **Partition of a Set**
- **Convex Sets**
- **Proving Statements about Convexivity**
- **The Plane Separation Axiom**
- **Proving Statements Using Given Conditional Statements and Their Contrapositives**
- **Proving a Fact about Half Planes in a Metric Geometry**

Reading: Section 4.1 The Plane Separation Axiom, p 63 - 68 in *Geometry: A Metric Approach with Models, Second Edition* by Millman & Parker

Homework: Section 4.1 # 1, 2, 4, 5, 6, 8, 9, 10, 11, 13

Recall Definitions of Abstract Geometry and Incidence Geometry from Section 2.1

Definition of Abstract Geometry

An ***abstract geometry*** $\mathcal{A}$ is an ordered pair $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ where $\mathcal{P}$ denotes a set whose elements are called **points** and $\mathcal{L}$ denotes a non-empty set whose elements are called **lines**, which are **sets of points** satisfying the following two requirements, called ***axioms***:

(i) For every two distinct points $A, B \in \mathcal{P}$, there exists at least one line $l \in \mathcal{L}$ such that $A \in l$ and $B \in l$.

(ii) For every line $l \in L$ there exist at least two distinct points that are elements of the line.

Definition of Incidence Geometry

An ***incidence geometry*** $\mathcal{A}$ is an *abstract geometry* $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ that satisfies the following two additional requirements, called ***axioms***:

(i) For every two distinct points $A, B \in \mathcal{P}$, there exists **exactly one** line $l \in \mathcal{L}$ such that $A \in l$ and $B \in l$.

(ii) There exist (at least) three *non-collinear* points.

Recall this theorem from Section 3.2

Theorem 3.2.6 Existence of Points with Certain Betweenness Relationships

Given: Distinct points A, B in a *metric geometry*

Claim:

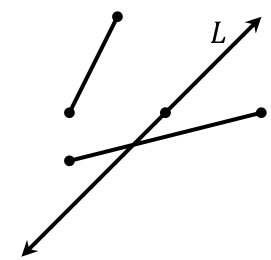
(i) There exists a point C with $A - C - B$

(ii) There exists a point D with $A - B - D$

Chapter 4 Plane Separation

Introduction: In the previous chapters, we saw that there is a notion of distance in our metric geometry that agrees with our notions about distance in drawings. In chapter 4, we are interested in other kinds of behavior of drawings. Here are four examples of familiar behavior of drawings:

Example #1: Consider the way a drawn line L "splits" the plane of a drawing.



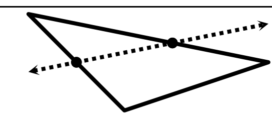
Notice three things:

(1) Any point must be either on line L or on one side of it or the other.

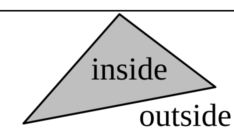
(2) If two points are on the same side of line L, then the segment connecting those two points also lies on the same side of L and does not intersect L.

(3) If two points are on opposite sides of line L, then the segment connecting those two points will intersect line L.

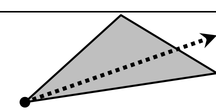
Example #2: In a drawing, any line that intersects a side of a triangle at a point that is not a vertex must also intersect at least one of the opposite sides.



Example #3: Drawn triangles have an "inside" and an "outside".



Example #4: In a drawing, any ray drawn from a vertex into the inside of a triangle must hit the opposite side of the triangle somewhere and go out.



Section 4.1 The Plane Separation Axiom

In Chapter 4, we will see the introduction of a new axiom, called the *Plane Separation Axiom (PSA)*, that will insure that our axiomatic geometry will exhibit behavior analogous to the behavior of drawings described above.

In order to understand the statement of the Plane Separation Axiom, we need to first discuss the concept of a *partition of a set* and the concept of a *convex set*.

Partition of a Set

The *Plane Separation Axiom (PSA)* uses the terminology of *partition of a set*. Here's the definition.

Definition of Partition of a Set

Words: $\{A_1, A_2, A_3, \ldots\}$ *is a partition of set* A.

Meaning: The following three requirements are all satisfied.

- Each of the A_i is a non-empty subset of A.
- A is the union of all the A_i. That is,

$$A = \bigcup_i A_i$$

- The sets $A_1, A_2, A_3, \ldots$ are mutually disjoint. That is,

$$\text{If } i \neq j \text{ then } A_i \cap A_j = \phi$$

[Example 1] Examples of Partitions

(a) The set $\{even\ integers, odd\ integers\}$ is a partition of $\mathbb{Z}$.

$$\{\dots, -4, -2, 0, 2, 4, \dots\} \cup \{\dots, -3, -1, 1, 3, \dots\} = \{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\} = \mathbb{Z}$$

(b) The set $\{\mathbb{R}^+, \mathbb{R}^-\}$ is not a partition of $\mathbb{R}$. because $\mathbb{R}^+ \cup \mathbb{R}^-$ is not all of $\mathbb{R}$

$\mathbb{R}^- \cup \mathbb{R}^+ \neq \mathbb{R}$

missing 0!!

(c) The set $\{\mathbb{R}^+, \mathbb{R}^-, \{0\}\}$ is a partition of $\mathbb{R}$.

$$\mathbb{R}^- \cup \{0\} \cup \mathbb{R}^+ = \mathbb{R}$$

(d) The set $\{rational\ numbers, irrational\ numbers\}$ is a partition of $\mathbb{R}$.

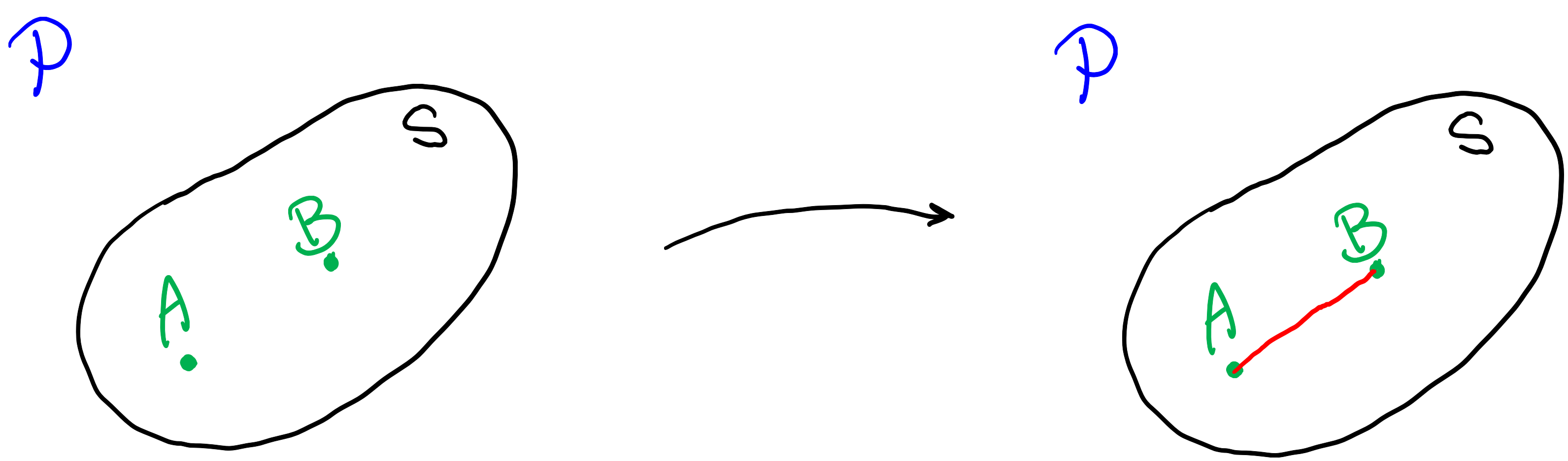
End of [Example 1]

Convex Sets

Another term used in the *Plane Separation Axiom (PSA)* is *convex set*.

Definition of Convex

- **Words:** S is convex
- **Usage:** A metric geometry $(\mathcal{P}, \mathcal{L}, d)$ is given, and $S \subset \mathcal{P}$ is a set of points.
- **Meaning:** for every two distinct points $A, B \in S$, the segment $\overline{AB} \subset S$.
- **Quantified version:** $\forall A, B \in S, A \neq B(\overline{AB} \subset S)$.
- **Universal Conditional Version:** $\forall A, B \in \mathcal{P}, A \neq B$(If $A, B \in S$ then $\overline{AB} \subset S$)



How do we prove that a set is convex?

The statement of convexity is a universal statement, so we must do a general proof.

[Example 2] In the Euclidean plane metric geometry, $(\mathbb{R}^2, \mathcal{L}_E, d_E)$, define $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^2$ by

$$\mathcal{P}_1 = \{(x,y) \in \mathbb{R}^2 | 3 < x < 5\} \quad \text{and} \quad \mathcal{P}_2 = \{(x,y) \in \mathbb{R}^2 | 3 < y < 5\}$$

We claim that $\mathcal{P}_1$ and $\mathcal{P}_2$ are convex.

Proof that $\mathcal{P}_1$ is convex.

(1) Suppose that $A, B \in \mathcal{P}_1$ and that $A \neq B$

(2) Then $A = (x_A, y_A)$ where $3 < x_A < 5$ (by (1) and definition of set $\mathcal{P}_1$.

(3) And $B = (x_B, y_B)$ where $3 < x_B < 5$ (by (1) and definition of set $\mathcal{P}_2$.

(4) Suppose $C \in \overline{AB}$.

(5) Then $C = A$ or $C = B$ or $A - C - B$. (by (4) and definition of *segment*)

(6) **(Cases (i) and (ii))** Clearly, if $C = A$ or $C = B$, then $C \in \mathcal{P}_1$.

(7) **(Case (iii))** Also, if $A - C - B$, then the x coordinate of C will be between x_A and x_B. So the x coordinate of C will be somewhere between 3 and 5. So $C \in \mathcal{P}_1$ in this case as well.

(8) Conclude that $C \in \mathcal{P}_1$ (because it is true in every case).

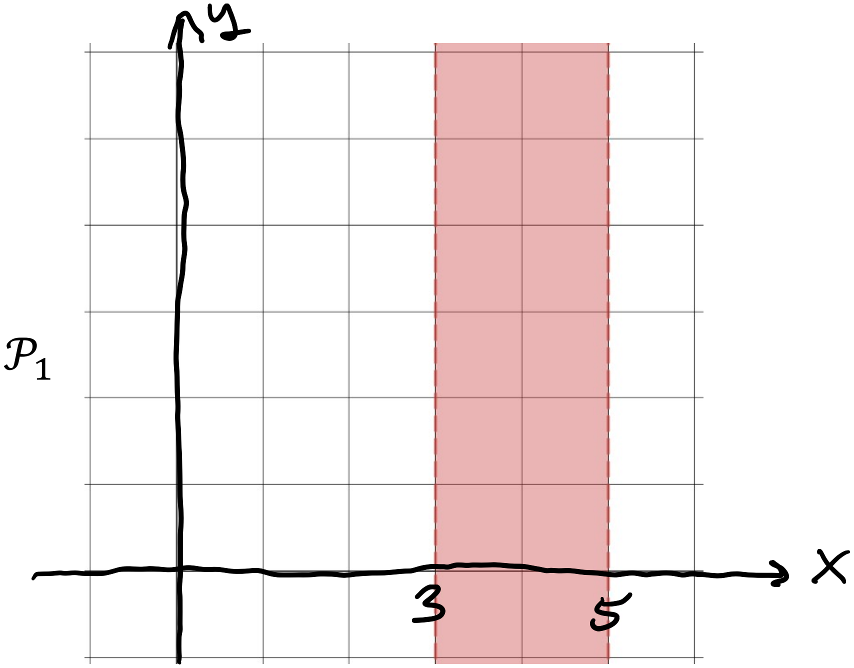
(9) Therefore, $\overline{AB} \subset \mathcal{P}_1$ (by (4),(8), and definition of *subset*)

(10) Conclude that $\mathcal{P}_1$ is convex (by (1), (9), and definition of convex)

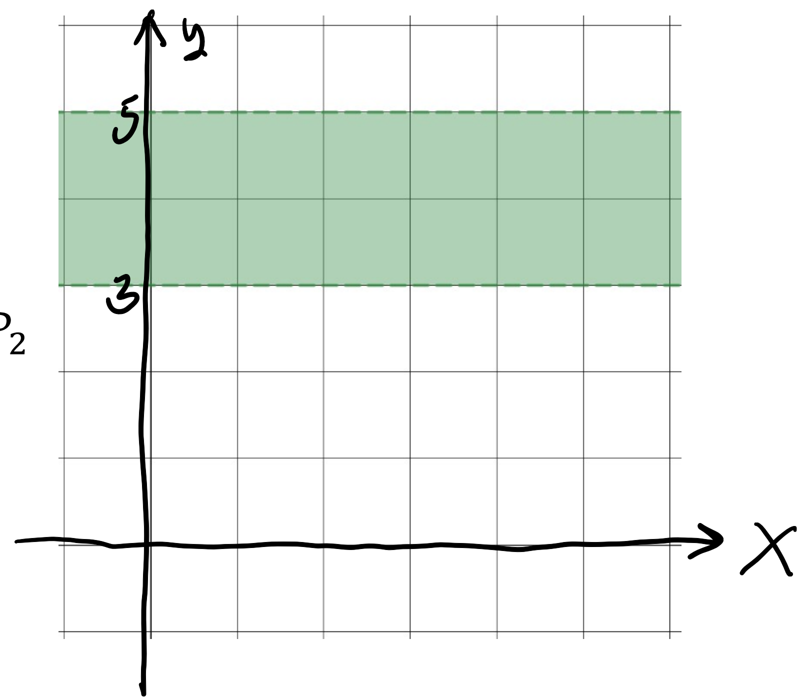
End of proof

The Proof that $\mathcal{P}_2$ is convex is similar

Picture of set $\mathcal{P}_1$



Picture of set $\mathcal{P}_2$



End of [Example 2]

[Example 3] More abstract proof about convexivity.

Let $\mathcal{P}_1 \subset \mathcal{P}$ and $\mathcal{P}_2 \subset \mathcal{P}$ be sets of points in a metric geometry.

Prove or disprove: If $\mathcal{P}_1$ and $\mathcal{P}_2$ are convex, then $\mathcal{P}_1 \cap \mathcal{P}_2$ is convex.

Solution: The statement doesn't mention anything specific about the metric geometry, so the claim is actually *implicitly quantified.* That is, it is a *universal claim* about *all metric geometries*. That is,

$$\forall \text{metric geometry}(\mathcal{P}, \mathcal{L}, d)\left(\forall \mathcal{P}_1, \mathcal{P}_2 \subset \mathcal{P}(\text{If } \mathcal{P}_1, \mathcal{P}_2 \text{ are convex, then } \mathcal{P}_1 \cap \mathcal{P}_2 \text{ is convex})\right)$$

Proof: Direct Proof

(1) Suppose $(\mathcal{P}, \mathcal{L}, d)$ is a metric geometry, and that $\mathcal{P}_1, \mathcal{P}_1$ are *convex* subsets of $\mathcal{P}$

(2) Suppose that $P, Q \in \mathcal{P}_1 \cap \mathcal{P}_2$ and that $P \neq Q$.

(3) $P, Q \in \mathcal{P}_1$ and $P, Q \in \mathcal{P}_2$ (by (2) and definition of intersection.)

(4) $\overline{PQ} \subset \mathcal{P}_1$ and $\overline{PQ} \subset \mathcal{P}_2$ (by (1), (3), and definition of convex)

(5) Then $\overline{PQ} \subset \mathcal{P}_1 \cap \mathcal{P}_2$. (by (4)) and definition of intersection)

(6) We have shown that $\mathcal{P}_1 \cap \mathcal{P}_2$ is convex. (by (2),(5) and definition of convex)

End of Proof

End of [Example 3]

How do we prove that a set is not convex?

Must first determine the negation of the statement that the set is convex.

$$\begin{aligned}
\text{Convex} &\equiv \forall P, Q \in S, P \neq Q\left(\overline{PQ} \subset S\right) \\
NOT(\text{Convex}) &\equiv NOT\left(\forall P, Q \in S, P \neq Q\left(\overline{PQ} \subset S\right)\right) \\
&\equiv \exists P, Q \in S, P \neq Q\left(NOT\left(\overline{PQ} \subset S\right)\right) \\
&\equiv \exists P, Q \in S, P \neq Q\left(\overline{PQ} \not\subset S\right)
\end{aligned}$$

Observe that the statement of *not convex* is an existential statement. Therefore, if the goal is to prove that a set is *not convex*, then one must produce an *example* of distinct points $P, Q \in S$ such that $\overline{PQ} \not\subset S$.

[Example 2] Let $\mathcal{P}_1 \subset \mathcal{P}$ and $\mathcal{P}_2 \subset \mathcal{P}$ be sets of points in a metric geometry.

Prove or disprove: If $\mathcal{P}_1$ and $\mathcal{P}_2$ are convex, then $\mathcal{P}_1 \cup \mathcal{P}_2$ is convex.

Solution: As with the previous example, the claim is *implicitly quantified.* That is, the claim is

$$\forall \text{metric geometry} (\mathcal{P}, \mathcal{L}, d) \big(\forall \mathcal{P}_1, \mathcal{P}_2 \subset \mathcal{P} (\text{If } \mathcal{P}_1, \mathcal{P}_2 \text{ are convex, then } \mathcal{P}_1 \cup \mathcal{P}_2 \text{ is convex}) \big)$$

In this case, the statement is *false*. To prove that the statement is *false*, we need to prove that its negation is *true*. In order to know how to do that, we need to first correctly write the negation. It is extremely important that you remember that the negation of a *conditional statement* is *not* another conditional statement!

Negating a Conditional Statement

Incorrect negation: $NOT(\text{If } A \text{ then } B) \equiv \text{If } A \text{ then } NOT(B)$

Correct negation: $NOT(\text{If } A \text{ then } B) \equiv A \text{ and } NOT(B)$

So the negation of the original claim is the following:

$$\exists \text{metric geometry} (\mathcal{P}, \mathcal{L}, d) \big(\exists \mathcal{P}_1, \mathcal{P}_2 \subset \mathcal{P} (\mathcal{P}_1, \mathcal{P}_2 \text{ are convex and } \mathcal{P}_1 \cup \mathcal{P}_2 \text{ is not convex}) \big)$$

We see that the negation is an *existential* statement. Remember that one must prove an *existential* statement by providing an *example*.

In the special case that one is *disproving a universal statement* by providing an example to show that the negation is true, the example is called a *counterexample*.

So our job is to produce an example consisting of a metric geometry $(\mathcal{P}, \mathcal{L}, d)$ and two sets of points, $\mathcal{P}_1, \mathcal{P}_2 \subset \mathcal{P}$, such that $\mathcal{P}_1$ and $\mathcal{P}_2$ are convex and $\mathcal{P}_1 \cup \mathcal{P}_2$ is not convex.

Consider the Euclidean plane metric geometry, $(\mathbb{R}^2, \mathcal{L}_E, d_E)$ and sets $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^2$ defined by

$$\mathcal{P}_1 = \{(x, y) \in \mathbb{R}^2 | 3 < x < 5\}$$

$$\mathcal{P}_2 = \{(x, y) \in \mathbb{R}^2 | 3 < y < 5\}$$

We observed in **[Example 2]** that $\mathcal{P}_1$ and $\mathcal{P}_2$ are convex.

However, notice that $\mathcal{P}_1 \cup \mathcal{P}_2$ is not convex.

We need to be very clear about the claim that $\mathcal{P}_1 \cup \mathcal{P}_2$ is not convex. That is, we must provide an example of points $A, B \in \mathcal{P}_1 \cup \mathcal{P}_2$ such that $\overline{AB} \not\subset \mathcal{P}_1 \cup \mathcal{P}_2$.

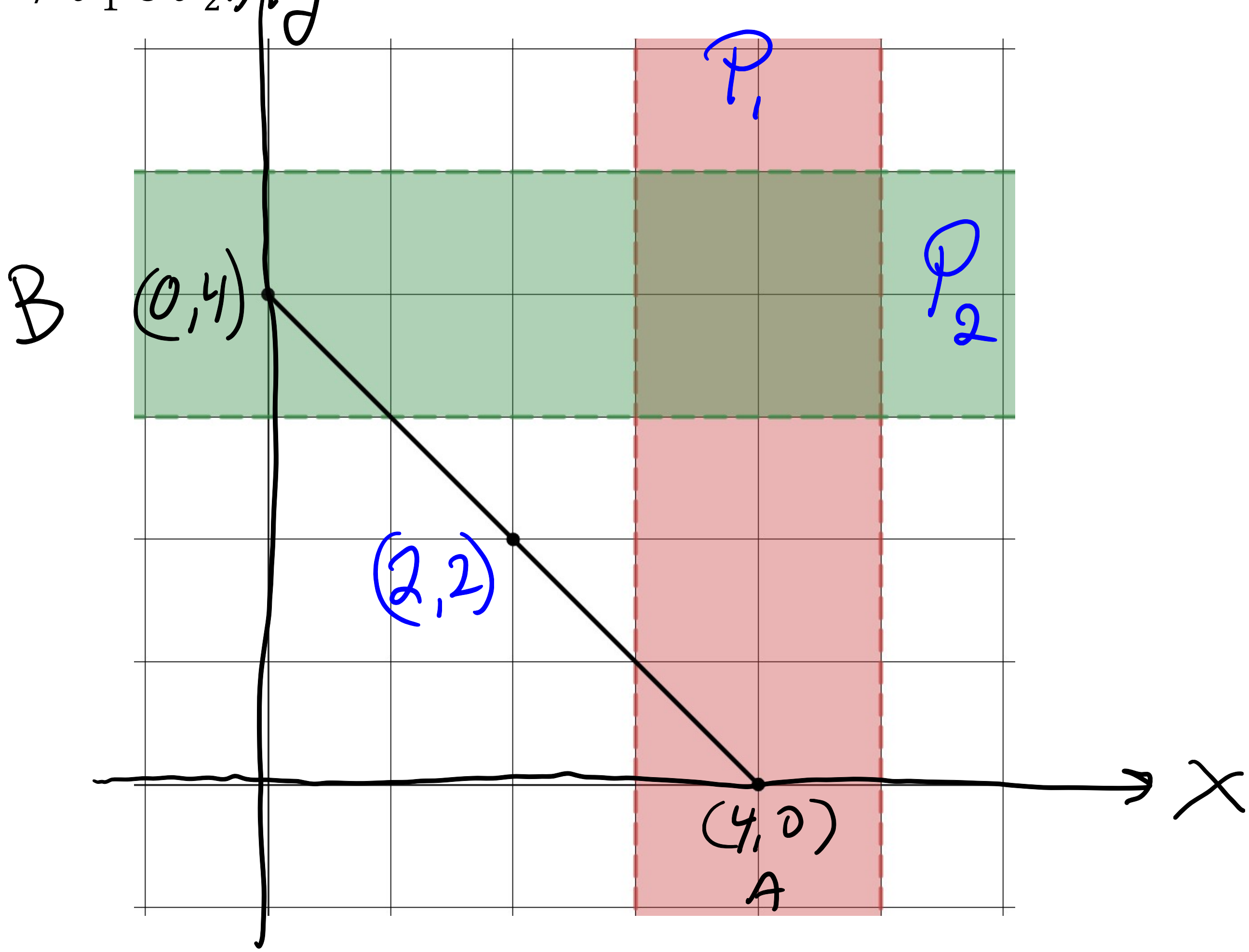
So let $\cancel{A = (3,0)}$ and $B \cancel{= (0,3)}$. $A = (4,0)$ $B = (0,4)$

Then $A \in \mathcal{P}_1$, so $A \in \mathcal{P}_1 \cup \mathcal{P}_2$

And $B \in \mathcal{P}_2$, so $B \in \mathcal{P}_1 \cup \mathcal{P}_2$

Now observe that $(2,2) \in \overline{AB}$ but $(2,2) \notin \mathcal{P}_1 \cup \mathcal{P}_2$.

This shows that $\overline{AB} \not\subset \mathcal{P}_1 \cup \mathcal{P}_2$.



End of [Example 4]

The Plane Separation Axiom

With the terminology of partition of a set and convex set, we are now able to understand the wording of the Plane Separation Axiom.

Definition: The Plane Separation Axiom (PSA) (**My version of the definition**)

- **Words:** A metric Geometry $(\mathcal{P}, \mathcal{L}, d)$ satisfies the **plane separation axiom** (PSA)
- **Meaning:** For every line $l \in \mathcal{L}$, there are two associated sets of points called *half planes*, denoted H_1 and H_2, with the following properties:

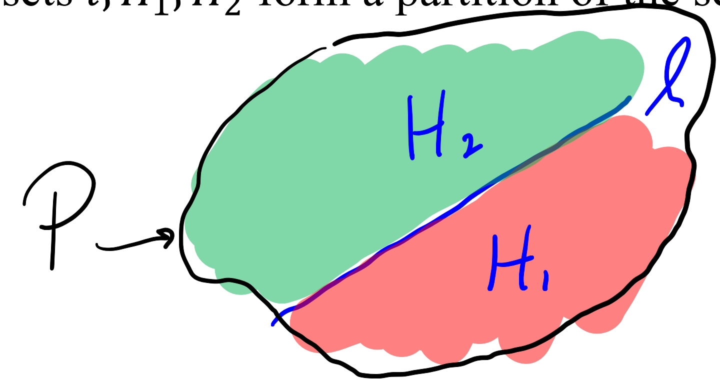
  (i) The three sets l, H_1, H_2 form a partition of the set $\mathcal{P}$ of all points.

  (ii) Each of the *half planes* is convex.

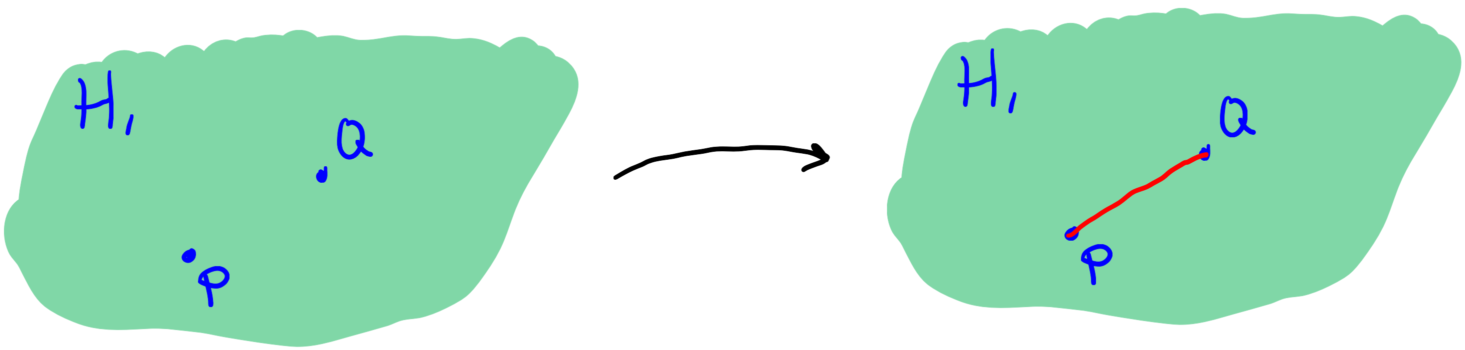
  (iii) If $A \in H_1$ and $B \in H_2$, then $\overline{AB}$ *intersects* line l.
- **Additional Terminology:**
  - Line l is called the ***edge*** of *half planes* H_1 and H_2.
  - **Words:** *Points A, B lie on the* ***same side*** *of line l.*
  - **Meaning:** Points A, B are elements of the same half plane associated to l.
  - **Words:** *Points A, B lie on* ***opposite sides*** *of line l.*
  - **Meaning:** Points A, B are elements of different half planes associated to l.

Illustrations of the **Plane Separation Axiom (PSA)**

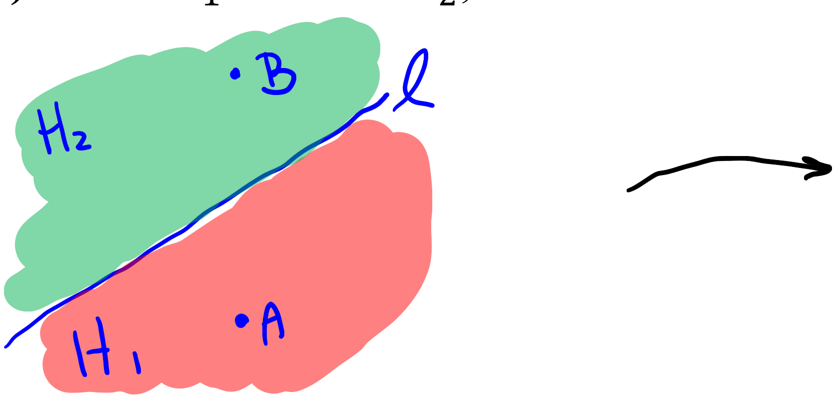
PSA (i) The three sets l, H_1, H_2 form a partition of the set $\mathcal{P}$ of all points.

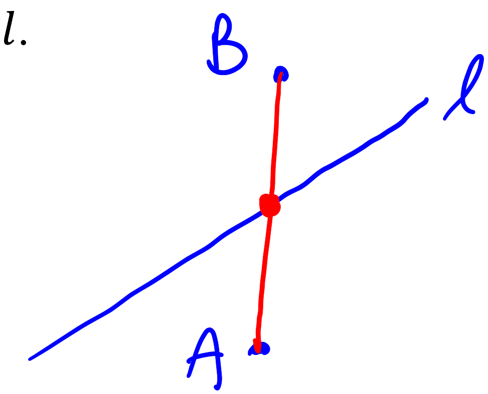


PSA (ii) Each of the *half planes* is convex.



PSA (iii) If $A \in H_1$ and $B \in H_2$, then $\overline{AB}$ *intersects* line l.





It is worth comparing the wording of my version of the PSA with the book's version on page 64.

Definition: The Plane Separation Axiom (PSA) (**Book version of the definition**)

A metric Geometry $\{\mathcal{P}, \mathcal{L}, d\}$ satisfies the **plane separation axiom** (*PSA*) if for every line $l \in \mathcal{L}$, there are two subsets H_1 and H_2 of $\mathcal{P}$ (called **half planes determined by $\boldsymbol{l}$**) such that

(i) $\mathcal{P} - l = H_1 \cup H_2$.

(ii) H_1 and H_2 are disjoint and each is convex.

(iii) If $A \in H_1$ and $B \in H_2$, then segment $\overline{AB} \cap l \neq \phi$.

Observe that the overall meaning of PSA described by the two versions of the definition is the same. But I have three complaints about the book's version of the definition.

- The book's version of the definition does not use the terminology of a *partition of a set*. This is silly, because that should be standard terminology for a course at this level.
- Furthermore, in the book's definition, the fact that the three sets l, H_1, H_2 form a partition of the set $\mathcal{P}$ of all points is conveyed partly by (i) and partly by the start of (ii). This is silly. The fact that those three sets form a partition should all be part of just one statement.
- The book's definition does not use the terminology of a segment and line *intersecting*.

Because of these complaints, I will hereafter refer only to *my* version of the *PSA*.

To actually use the **PSA**, it will be crucial to be able to do three things:

(1) Understand the interpretations of **PSA (ii)** and **(iii)** as conditional statements.

(2) Understand the associated *contrapositive* for each conditional statements.

(3) Know how to use a given conditional statement and its contrapositive to prove new statements.

Interpretations of PSA (ii) and (iii) as Conditional Statements, and their Contrapositives

Consider **PSA (ii):** Each of the *half planes* is convex.

Using the definition of *convex*, we can restate PSA (ii) as a conditional statement, and state its contrapositive:

PSA (ii): If distinct points P, Q are in the same *half plane*, then $\overline{PQ}$ does not intersect line l.
PSA (ii) (contrapositive): If $\overline{PQ}$ does intersect line l, then P, Q are *not* in the same *half plane*.

Now consider PSA (iii), which already has the form of a conditional statement. We can state it in a slightly different version, and state its contrapositive:

PSA (iii) If P, Q are not in the same *half plane*, then $\overline{PQ}$ intersects line l.
PSA (iii) (contrapositive) If $\overline{PQ}$ does not intersect line l, then P, Q are distinct points in the same *half plane*.

Proving Statements Using Given Conditional Statements and their Contrapositives

Suppose that two axioms (or theorems) are stated in the form of conditional statements, as follows.

- **Axiom <100>:** If the dog is blue, then the car is red.
- **Axiom <101>:** If the car is red, then the bear is hungry.

The contrapositives of these two axioms would be the following statements:

- **Axiom <100> (contrapositive):** If the car is not red, then the dog is not blue.
- **Axiom <101> (contrapositive):** If the bear is not hungry, then the car is not red.

Remember that the contrapositive statements are logically equivalent to the original statements.

For example, suppose that we wanted to prove that **the car is red**.

We would have to use **Axiom <100>.** Our strategy would be to

- First prove somehow that the dog is blue.
- Then use **Axiom <100>** to say that the car is red.

Note that we would *not* use **Axiom <101>** to prove that the car is red. **Axiom <101>** tells us something about the situation where we *already know* that the car is red. (It tells us that in this situation, the bear is hungry.)

Now suppose that we want to prove that **the car is not red.**

It is important to realize that **Axiom <100>** does not help us in this case! To prove that the car is not red, we must use **Axiom <101> (contrapositive)**. Our strategy would be to

- First prove somehow that the bear is not hungry.
- Then use **Axiom <101> (contrapositive)** to say that the car is not red.

The discussion above is relevant to your use of **PSA (ii)** and **PSA (iii)** in proofs. Since we will be so often referring to PSA (ii) and (iii) and their contrapositives, it is worthwhile to present them in a nice green box.

PSA (ii) and (iii) and their Contrapositives

PSA (ii): If distinct points P, Q are in the same *half plane*, then $\overline{PQ}$ does not intersect line l.

PSA (ii) (contrapositive): If $\overline{PQ}$ does intersect line l, then P, Q are *not* in the same *half plane*.

PSA (iii) If P, Q are not in the same *half plane*, then $\overline{PQ}$ intersects line l.

PSA (iii) (contrapositive) If $\overline{PQ}$ does not intersect line l, then P, Q are distinct points in the same *half plane*.

For instance, suppose that know that distinct points P and Q are not on some line l, and you want to prove that they are in the same half plane of l.

You should *not* use **PSA (ii)**. That statement says something about the situation where you *already know* that points P and Q are in the same half plane. (It says that in that situation, segment $\overline{PQ}$ does not intersect line l.)

Rather, you should use **PSA (iii) (contrapositive)**. Your strategy should be to

- Prove somehow that segment $\overline{PQ}$ does not intersect line l
- Then use **PSA (iii) (contrapositive)** to say that points P and Q are in the same half plane.

Or instead, suppose that know that distinct points P and Q are not on some line l, and you want to prove that they are *not* in the same half plane of L.

You should *not* use *PSA (iii)*. That statement says something about the situation where you *already know* that points P and Q are not in the same half plane. (It says that in that situation, segment $\overline{PQ}$ does intersect line l.)

Rather, you should use **PSA (ii) (contrapositive)**. Your strategy should be to

- Prove somehow that segment $\overline{PQ}$ does intersect line l.
- Then use **PSA (ii) (contrapositive)** to say that points P and Q are not in the same half plane.

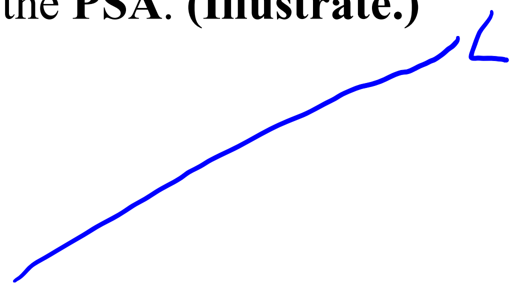
[Example 3] Prove the following:

Given: a line L in a metric geometry that satisfies the **Plane Separation Axiom (PSA)**

Claim: Each of the *half planes* determined by L contains a point.

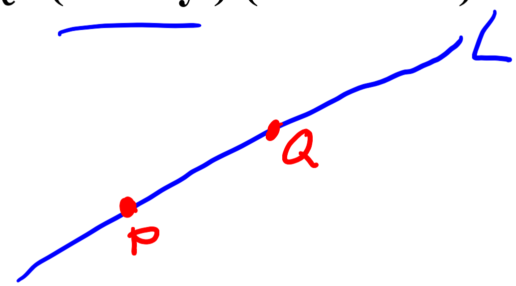
Proof

(1) Suppose L is a line in a metric geometry that satisfies the **PSA**. **(Illustrate.)**



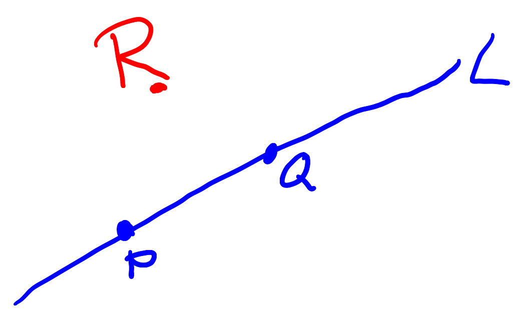
(2) There exist two distinct points on L. Call them P and Q. **(Justify.) (Illustrate.)**

Abstract geometry axiom (ii)
Says every line has at least
two points.



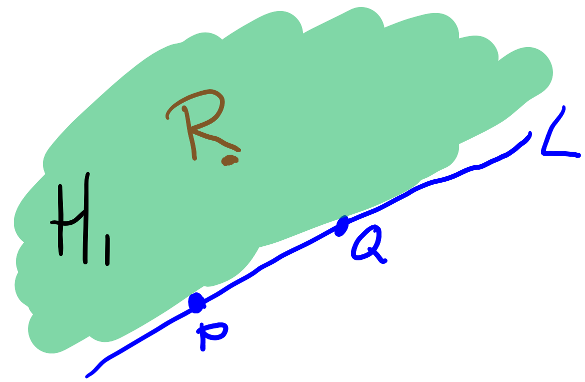
(3) There exists a point not on L. Call it R. **(Justify.) (Illustrate.)**

Incidence geometry axiom (ii)
Says that there are at least
three non-collinear points.



(4) Point R lies in one of the two *half planes* determined by line L. **(Justify.)** Call that *half plane* H_1. **(Illustrate.)**

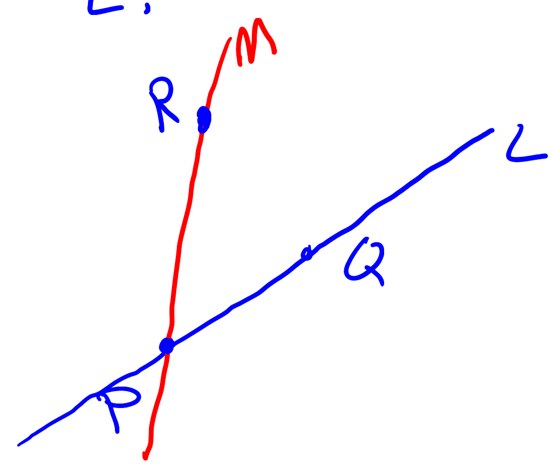
By PSA (i), R must lie in exactly one of the sets L, H_1, H_2
But R is not on line L
So, R is in H_1 or H_2



(5) There exists a unique line passing through P and R. **(Justify.)**

By incidence axiom (i)

(6) The line passing through P and R is not L. **(Justify.)** So it must be a new line. Call it M. **(Illustrate.)** Because we know that R is not on L.



(7) There exists a point such that $R - P - point$. **(Justify.)**

By theorem 3.2.6(i) Given distinct points A, B, there exists a point C such that A-B-C

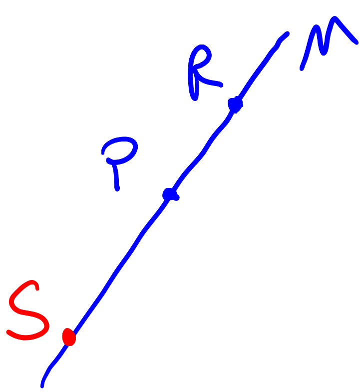
applied to given points R, P

(8) This *point* cannot be the same as any of our previous three points. **(Justify.)** So it must be a *new point*. Call it S. So $R - P - S$. **(Illustrate.)**

We know point ≠ P because R-P-point tells us that the three points are distinct.

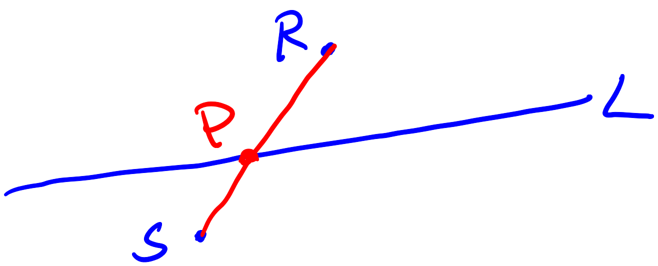
we also know point ≠ R by the same reasoning.

We know point ≠ Q because if point = Q, then R-P-Q would be true, which would mean that R, P, Q are collinear. But we know that is impossible, because R is not on line L.



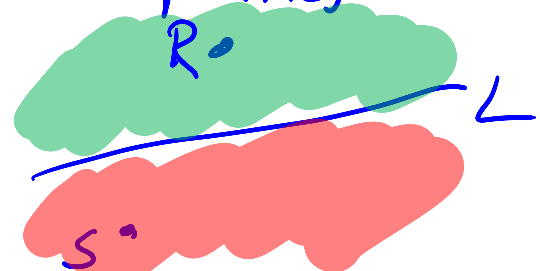
(9) Point S lies in the other *half plane* determined by line L. **(Justify.)** Call that *half plane* H_2.

Observe that segment $\overline{RS}$ intersects line L at point P that is between R and S

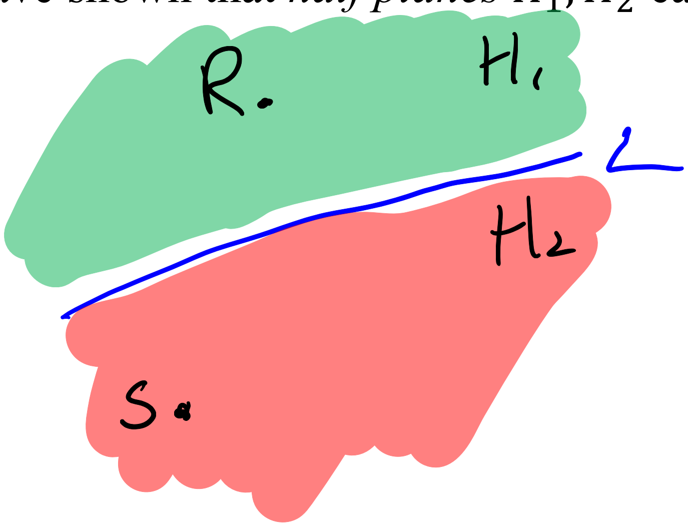


PSA(ii) contrapositive

so R and S are in different half planes



(10) We have shown that *half planes* H_1, H_2 each contain a point. **(Illustrate.)**



End of Proof

End of [Example 3]

End of Video