4.2: Plane Separation Behavior of the Euclidean, Taxicab, and Poincaré Planes

produced by Mark Barsamian, 2021.03.05

for Ohio University MATH 3110/5110 College Geometry

Topics

- **Cartesian half planes**
- **Poincaré half planes**
- **The Euclidean plane satisfies PSA**
- **The Taxicab plane satisfies PSA**
- **The Poincaré plane satisfies PSA**
- **Drawings showing how Poincaré plane satisfies PSA**

Reading: Section 4.2 PSA for the Euclidean and Poincaré Planes, p 70 - 74 in *Geometry: A Metric Approach with Models, Second Edition* by Millman & Parker

Homework: None

Recall Definitions of Partition of a Set and Convex Set from Section 4.1

Definition of Partition of a Set

Words: $\{A_1, A_2, A_3, \ldots\}$ *is a partition of set* A.

Meaning: The following three requirements are all satisfied.

- Each of the A_i is a non-empty subset of A.
- A is the union of all the A_i. That is,

$$A = \bigcup_i A_i$$

- The sets $A_1, A_2, A_3, \ldots$ are mutually disjoint. That is, if $i \neq j$ then $A_i \cap A_j = \phi$

Definition of Convex

- **Words:** S is convex
- **Usage:** A metric geometry $(\mathcal{P}, \mathcal{L}, d)$ is given, and $S \subset \mathcal{P}$ is a set of points.
- **Meaning:** for every two distinct points $A, B \in S$, the segment $\overline{AB} \subset S$.
- **Quantified version:** $\forall A, B \in S, A \neq B(\overline{AB} \subset S)$.
- **Universal Conditional Version:** $\forall A, B \in \mathcal{P}, A \neq B(\text{If } A, B \in S \text{ then } \overline{AB} \subset S)$

Also recall from Section 4.1 the Definition of the Plane Separation Axiom

Definition: The Plane Separation Axiom (PSA) (**My version of the definition**)

- **Words:** A metric Geometry $(\mathcal{P}, \mathcal{L}, d)$ satisfies the **plane separation axiom** (PSA)
- **Meaning:** For every line $l \in \mathcal{L}$, there are two associated sets of points called *half planes*, denoted H_1 and H_2, with the following properties:
  (i) The three sets l, H_1, H_2 form a partition of the set $\mathcal{P}$ of all points.
  (ii) Each of the *half planes* is convex.
  (iii) If $A \in H_1$ and $B \in H_2$, then $\overline{AB}$ *intersects* line l.
- **Additional Terminology:**
  - Line l is called the ***edge*** of *half planes* H_1 and H_2.
  - **Words:** *Points A, B lie on the* ***same side*** *of line l.*
    - **Meaning:** Points A, B are elements of the same half plane associated to l.
  - **Words:** *Points A, B lie on* ***opposite sides*** *of line l.*
    - **Meaning:** Points A, B are elements of different half planes associated to l.

Section 4.2: PSA for the Euclidean and Poincaré planes

The goal of Section 4.2 of the book is to prove that both the Euclidean plane and the Poincaré plane satisfy the *Plane Separation Axiom (PSA)*. The approach taken by the authors is as follows:

- Define the sets that are the half planes in the geometry.
- Prove that the geometry, with the half planes as defined, does satisfy the *PSA*.

The authors do this for the Euclidean plane, and then they do it for the Poincaré plane.

The material is dense. It is also difficult, for a few reasons.

- The definition of the half planes for the Euclidean plane uses some new vector notation.
- The proof that the Euclidean plane sastisfies the *PSA* uses a lot of moderate to hard calculations involving vectors.
- The proof that the Poincaré plane satisfies the *PSA* involves a lot of hard calculations involving the hyperbolic trig functions.

In this video, and in our course, we will not cover the proofs. There are two reasons for this.

- The proof style is not one that we will encounter again in the course.
- The authors' definition of the half planes for the Euclidean plane is flawed.

What I will do in this video is focus instead on two things:

(1) I will present clear definitions of *Cartesian half planes* and *Poincaré half planes*.

(2) I will present the results that the *Euclidean*, *Taxicab*, and *Poincaré planes* satisfy the *PSA*.

(3) Then I will make informal drawings that illustrate examples showing how *Poincaré half planes* exhibit behavior consistent with the *PSA*. (I don't think that drawings of the behavior of the *Cartesian half planes* are necessary.)

Definition of Cartesian half planes

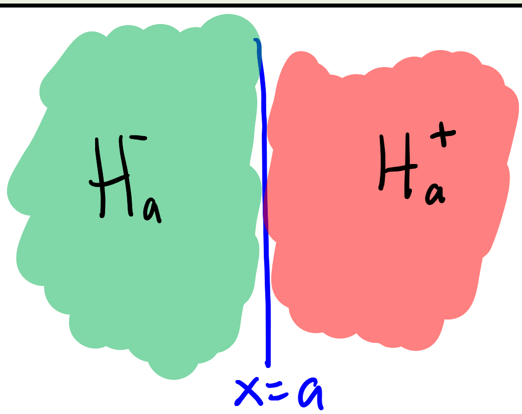
For real numbers a, m, b, define the define the *Cartesian half planes* as follows

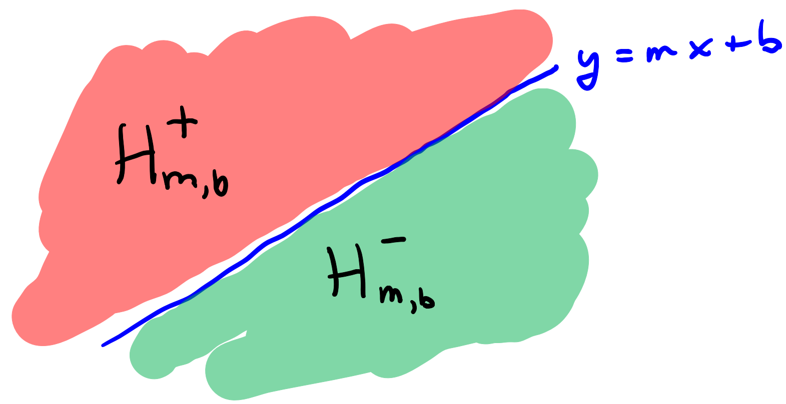
$$H_a^+ = \{(x,y) \in \mathbb{R}^2 | x > a\}$$

$$H_a^- = \{(x,y) \in \mathbb{R}^2 | x < a\}$$

$$H_{m,b}^+ = \{(x,y) \in \mathbb{R}^2 | y > mx + b\}$$

$$H_{m,b}^- = \{(x,y) \in \mathbb{R}^2 | y < mx + b\}$$





Note that the book defines *Euclidean half planes*, rather than *Cartesian half planes*. This is a stylistic choice: the resulting planes are the same. But I think it is important to call the half planes *Cartesian* because their definition only uses calculations involving *Cartesian points and lines* and does not in any way use the *Euclidean distance function*. But also note that the book's definition of their Euclidean half planes uses vector notation, including some symbols that we have not used before. A careful reading of the book's definition will show that there is a significant flaw in it.

Textbook's Definition of Euclidean half planes

Let $l = \overleftrightarrow{PQ}$ be a Euclidean line. The Euclidean half planes determined by l are

$$H^+ = \{A \in \mathbb{R}^2 | \langle (A-P), (Q-P)^\perp \rangle > 0\}$$

$$H^- = \{A \in \mathbb{R}^2 | \langle (A-P), (Q-P)^\perp \rangle < 0\}$$

The problem with this definition is that different choices of points on line l can result in different sets being called H^+ and H^-.

For example, consider the non-vertical line l defined by the equation $y = 0$. The points $(0,0)$ and $(1,0)$ lie on line l.

If we choose $P = (0,0)$ and $Q = (1,0)$, then the Textbook's definition results in the sets

$$H^+ = \{(x,y) \in \mathbb{R}^2 | y > 0\}$$

$$H^- = \{(x,y) \in \mathbb{R}^2 | y < 0\}$$

If we choose $P = (1,0)$ and $Q = (0,0)$, then the Textbook's definition results in the sets

$$H^+ = \{(x,y) \in \mathbb{R}^2 | y < 0\}$$

$$H^- = \{(x,y) \in \mathbb{R}^2 | y > 0\}$$

We see that different choices of P, Q on l resulted in different sets being called H^+ and H^-. That is, half planes are *not well-defined* by the textbook's definition. Therefore, we won't use it.

Definition of Poincaré half planes

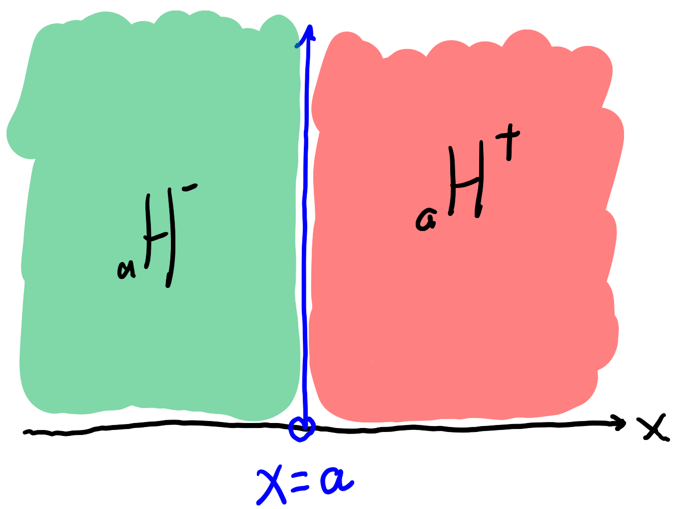
For real numbers a, c, r with $r > 0$, define the define the *Poincaré half planes* as follows

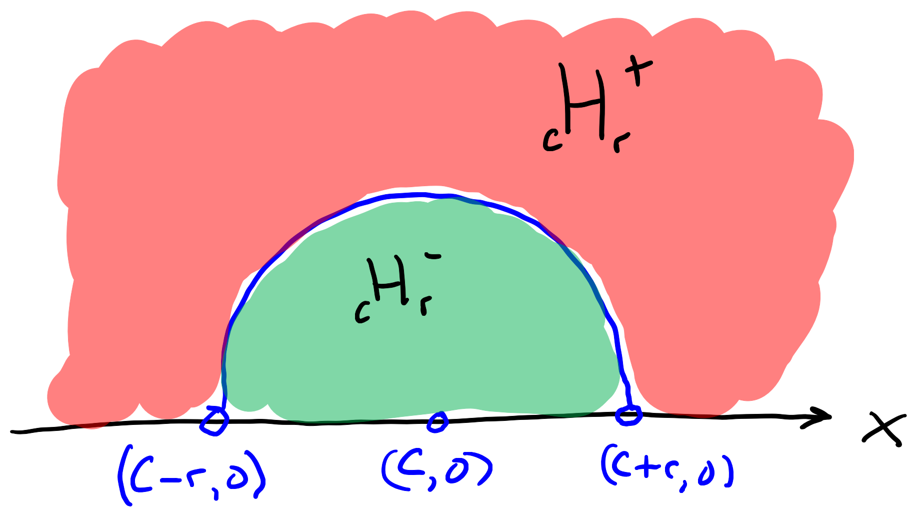
$${}_aH^+ = \{(x,y) \in \mathbb{H} | x > a\}$$

$${}_aH^- = \{(x,y) \in \mathbb{H} | x < a\}$$

$${}_cH_r^+ = \{(x,y) \in \mathbb{H} | (x-c)^2 + y^2 > r^2\}$$

$${}_cH_r^- = \{(x,y) \in \mathbb{H} | (x-c)^2 + y^2 < r^2\}$$





Proposition 4.2.4 The Euclidean plane $\left(\mathbb{R}^2, \mathcal{L}_E, \boldsymbol{d_E}\right)$satisfies the *PSA*

With *Cartesian half planes* H_a^+ and H_a^- (or $H_{m,b}^+$ and $H_{m,b}^-$) playing the role of half planes for the *vertical line* L_a (or for the *non-vertical line* $L_{m,b}$), the *Euclidean plane* satisfies the *Plane Separation Axiom (PSA)*.

The proof of this proposition is provided in the book on pages 72-73. The proof shows that the Euclidean plane does indeed have properties (i),(ii),(iii) of the *PSA*. The book uses a different definition of half-planes than I am using in these notes, so the book's proof looks a little different than the proof would look in these notes. But here is an outline of the proof that would be needed.

(i) Prove that

$\{L_a, H_a^+, H_a^-\}$ is a partition of $\mathbb{R}^2$

$\left\{L_{m,b}, H_{m,b}^+, H_{m,b}^-\right\}$ is a partition of $\mathbb{R}^2$.

(ii) Prove that the *Cartesian half planes* $H_a^+, H_a^-, H_{m,b}^+, H_{m,b}^-$ are convex sets.

(iii) Prove that

If $A \in H_a^+$ and $B \in, H_a^-$, then *Euclidean segment* $\overline{AB}$ *intersects* line L_a.

If $A \in H_{m,b}^+$ and $B \in H_{m,b}^-$, then *Euclidean segment* $\overline{AB}$ *intersects* line $L_{m,b}$.

Result of Exercise 4.2#5 The Taxicab plane $\left(\mathbb{R}^2, \mathcal{L}_E, d_T\right)$ satisfies the *PSA*

With *Cartesian half planes* H_a^+ and H_a^- (or $H_{m,b}^+$ and $H_{m,b}^-$) playing the role of half planes for the *vertical line* L_a (or for the *non-vertical line* $L_{m,b}$), the *Taxicab plane* satisfies the *Plane Separation Axiom (PSA)*.

When discussing *segments*, *rays*, *angles*, and *triangles* in Chapter 3, we did not discuss how those objects look in the *Taxicab plane*. All of those objects look just like their Euclidean counterparts. Because of that, I will not make any drawings to illustrate how the *Taxicab plane* satisfies the *PSA*. The proof that the *Taxicab plane* satisfies the *PSA* would be very similar to the proof that the *Euclidean plane* satisfies the *PSA*.

Proposition 4.2.5 The Poincaré plane satisfies the *PSA*

With *Poincaré half planes* ${}_aH^+$ and ${}_aH^-$ (or ${}_cH_r^+$ and ${}_cH_r^+$) playing the role of half planes for the *type I line* ${}_aL$ (or for the *type II line* ${}_cL_r$), the *Poincaré plane* satisfies the *Plane Separation Axiom (PSA)*.

There is a proof of this proposition on pages 73-74 of the book. The proof is a very interesting application of the hyperbolic trig functions $\operatorname{sech}(t)$ and $\tanh(t)$, and also uses the Intermediate Value Theorem. I won't discuss it, but here is an outline of the the proof:

(i) Prove that

$\{{}_aL, {}_aH^+, {}_aH^-\}$ is a partition of $\mathbb{H}$

$\{{}_cL_r, {}_cH_r^+, {}_cH_r^+\}$ is a partition of $\mathbb{H}$.

(ii) Prove that the *Poincaré half planes* ${}_aH^+, {}_aH^-, {}_cH_r^+, {}_cH_r^-$ are convex sets.

(iii) Prove that

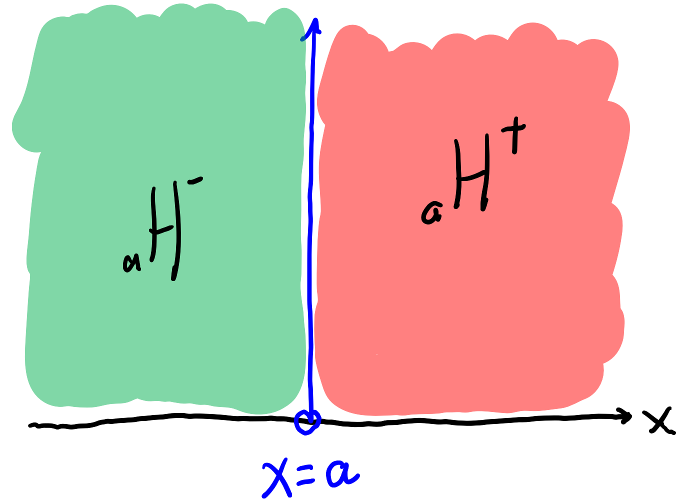
If $A \in {}_aH^+$ and $B \in {}_aH^-$, then *Poincaré segment* $\overline{AB}$ *intersects* line ${}_aL$.

If $A \in {}_cH_r^+$ and $B \in {}_cH_r^-$, then *Poincaré segment* $\overline{AB}$ *intersects* line ${}_cL_r$.

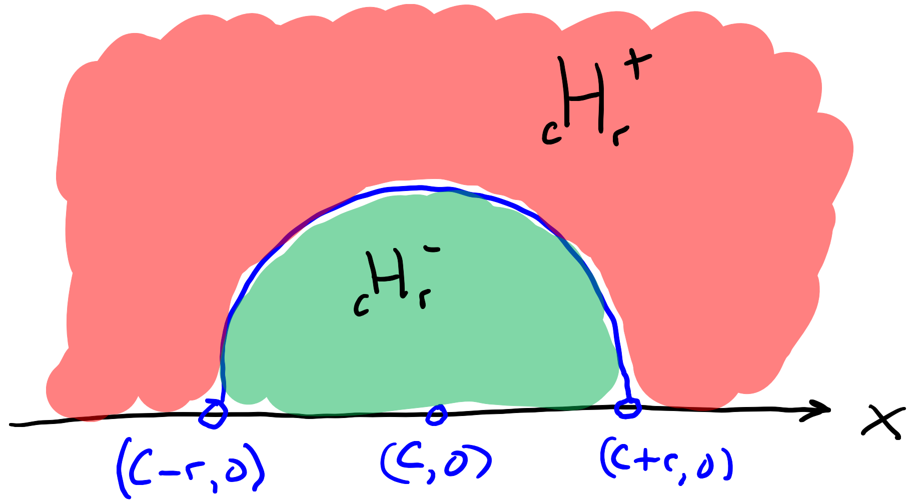
Although I won't discuss the proof of Proposition 4.2.5, I will make some drawings to illustrate how *Poincaré planes* exhibits behavior consistent with the *PSA*.

Drawings showing how the *Poincaré plane* satisfies PSA (i)

$\{ {}_aL,\ {}_aH^+,\ {}_aH^- \}$ is a partition of $\mathbb{H}$

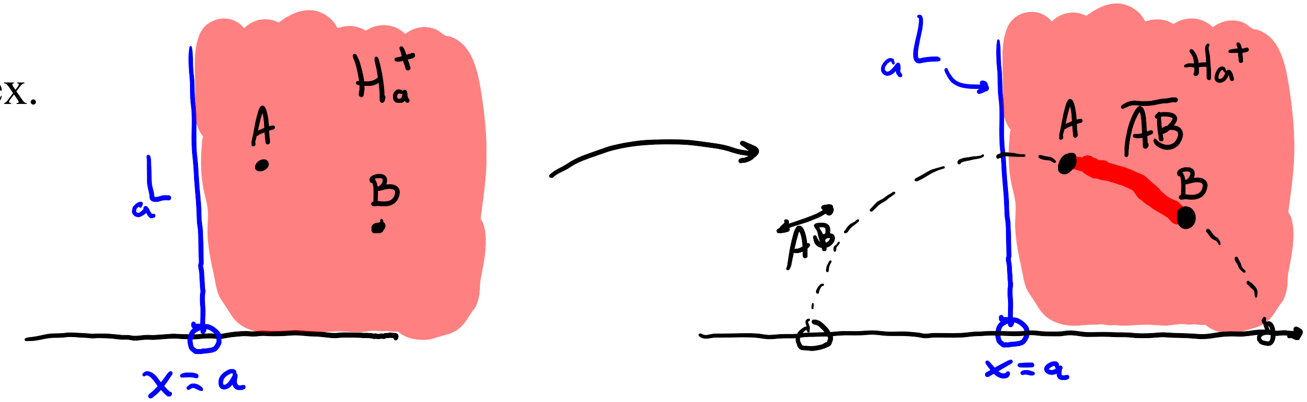


$\{ {}_cL_r,\ {}_cH_r^+,\ {}_cH_r^+ \}$ is a partition of $\mathbb{H}$.

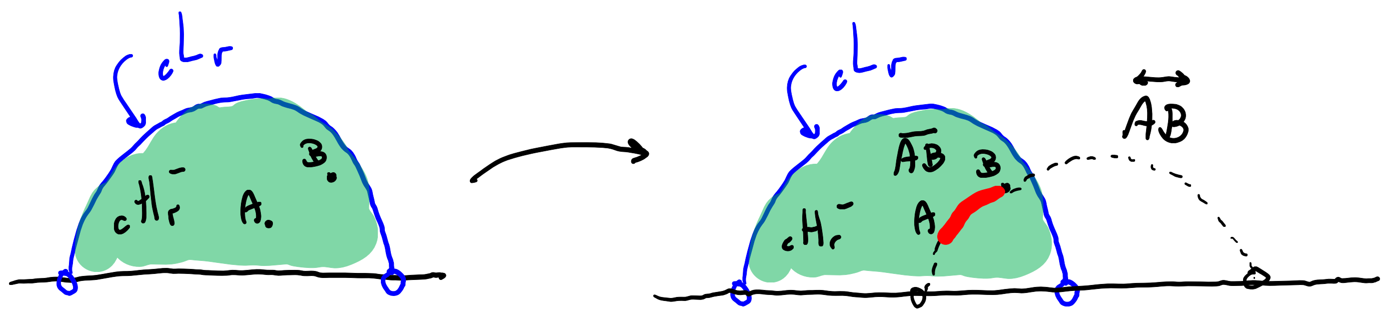


Drawings showing how the *Poincaré plane* satisfies PSA (ii)

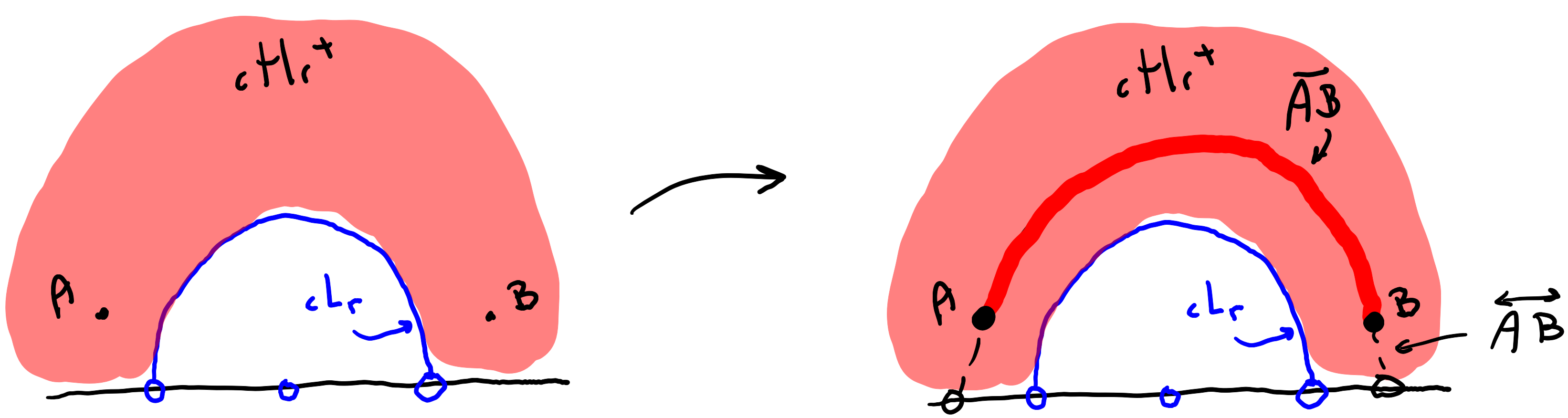
Poincaré half plane $_{a}H^{+}$ is convex.



Poincaré half plane $_{c}H_{r}^{-}$ is convex.

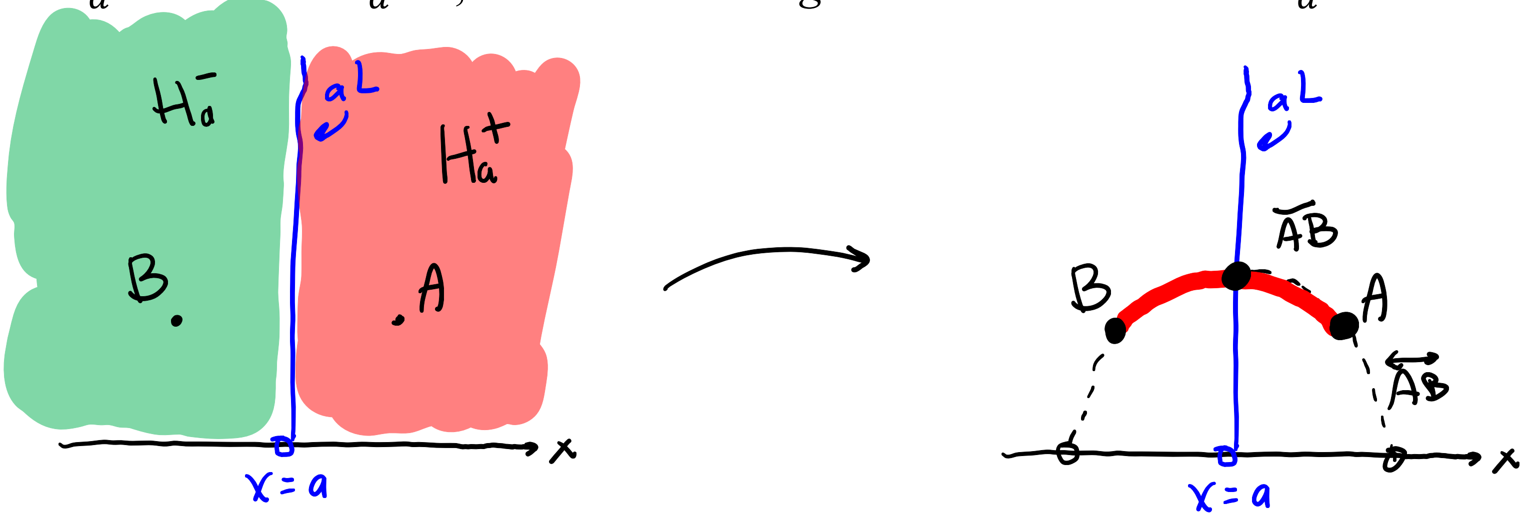


Poincaré half plane $_{c}H_{r}^{+}$ is convex.

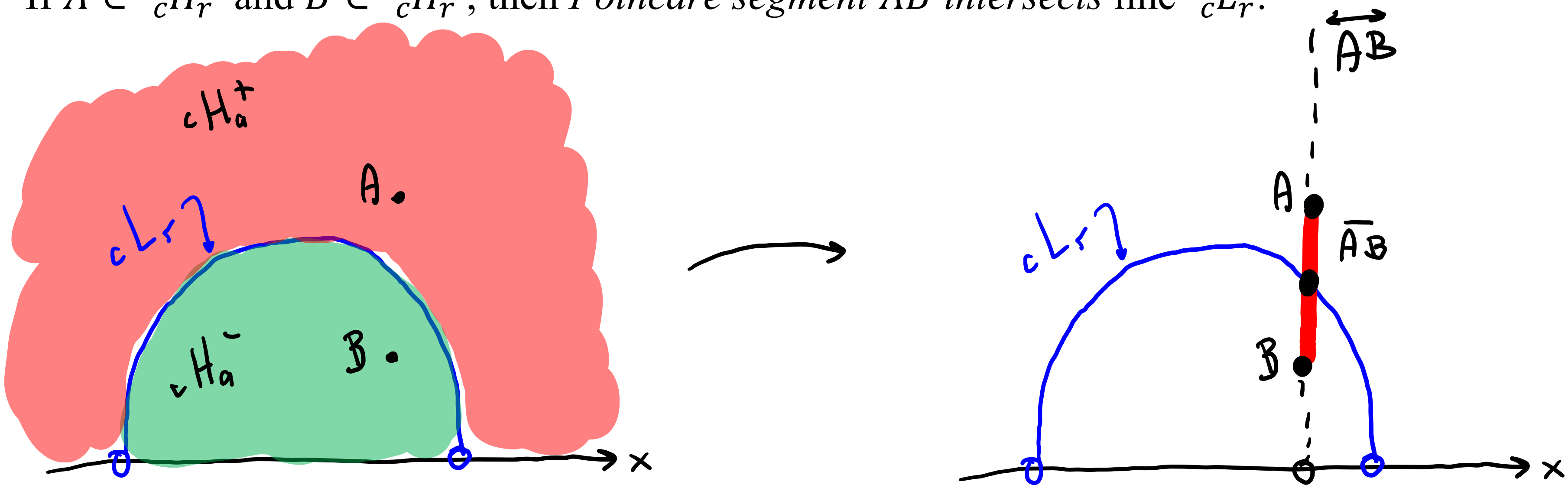


Drawings showing how the *Poincaré plane* satisfies PSA (iii)

If $A \in {}_aH^+$ and $B \in {}_aH^-$, then *Poincaré segment* $\overline{AB}$ *intersects* line ${}_aL$.



If $A \in {}_cH_r^+$ and $B \in {}_cH_r^-$, then *Poincaré segment* $\overline{AB}$ *intersects* line ${}_cL_r$.



End of Video