4.3: Pasch Geometries

produced by Mark Barsamian, 2021.03.11

for Ohio University MATH 3110/5110 College Geometry

Topics

- **Is the Plane Separation Axiom (PSA) always true?**
- **Pasch's Postulate**
- **Two Equivalent Statements in a Metric Geometry**
- **The Missing Strip Plane**
- **Definition of Pasch Geometry**
- **Peano's Axiom**

Reading: Section 4.3 Pasch Geometries, p 75 - 80 in *Geometry: A Metric Approach with Models, Second Edition* by Millman & Parker

Homework: Section 4.3 # 1, 2, 3, 4, 7

Recall from Section 2.1: Theorem about Intersecting Lines in *Incidence Geometry*

Theorem 2.1.6 Given two lines l_1 and l_2 in an *incidence geometry*,

If $l_1 \cap l_2$ has two or more distinct points,

then l_1 and l_2 are the same line. That is, $l_1 = l_2$.

The *contrapositive* of the statement of Theorem 2.1.6 can be stated as a *corollary*.

Corollary 2.1.7 (contrapositive of Theorem 2.1.6)

Given two lines l_1 and l_2 in an *incidence geometry*,

If lines l_1 and l_2 are known to be distinct lines (that is, $l_1 \neq l_2$),

then either lines l_1 and l_2 do not intersect or they intersect in exactly one point.

Recall Definitions of Partition of a Set and Convex Set from Section 4.1

Definition of Partition of a Set

Words: $\{A_1, A_2, A_3, \dots\}$ *is a partition of set* A.

Meaning: The following three requirements are all satisfied.

- Each of the A_i is a non-empty subset of A.
- A is the union of all the A_i. That is,

$$A = \bigcup_i A_i$$

- The sets $A_1, A_2, A_3, \dots$ are mutually disjoint. That is, if $i \neq j$ then $A_i \cap A_j = \phi$

Definition of Convex

- **Words:** S is convex
- **Usage:** A metric geometry $(\mathcal{P}, \mathcal{L}, d)$ is given, and $S \subset \mathcal{P}$ is a set of points.
- **Meaning:** for every two distinct points $A, B \in S$, the segment $\overline{AB} \subset S$.
- **Quantified version:** $\forall A, B \in S, A \neq B (\overline{AB} \subset S)$.
- **Universal Conditional Version:** $\forall A, B \in \mathcal{P}, A \neq B$(If $A, B \in S$ then $\overline{AB} \subset S$)

Also recall from Section 4.1 the Definition of the Plane Separation Axiom

Definition: The Plane Separation Axiom (PSA) (**My version of the definition**)

- **Words:** A metric Geometry $(\mathcal{P}, \mathcal{L}, d)$ satisfies the **plane separation axiom** (PSA)
- **Meaning:** For every line $l \in \mathcal{L}$, there are two associated sets of points called *half planes*, denoted H_1 and H_2, with the following properties:
  (i) The three sets l, H_1, H_2 form a partition of the set $\mathcal{P}$ of all points.
  (ii) Each of the *half planes* is convex.
  (iii) If $A \in H_1$ and $B \in H_2$, then $\overline{AB}$ *intersects* line l.
- **Additional Terminology:**
  - Line l is called the ***edge*** of *half planes* H_1 and H_2.
  - **Words:** *Points A, B lie on the **same side** of line l.*
    - **Meaning:** Points A, B are elements of the same half plane associated to l.
  - **Words:** *Points A, B lie on **opposite sides** of line l.*
    - **Meaning:** Points A, B are elements of different half planes associated to l.

And recall from Section 4.1 that it is useful to know the Contrapositives of PSA (ii) and (iii)

PSA (ii) and (iii) and their Contrapositives

PSA (ii): If distinct points P, Q are in the same *half plane*,

then $\overline{PQ}$ does not intersect line l.

PSA (ii) (contrapositive): If $\overline{PQ}$ does intersect line l,

then P, Q are *not* in the same *half plane*.

PSA (iii) If P, Q are not in the same *half plane*,

then $\overline{PQ}$ intersects line l.

PSA (iii) (contrapositive) If $\overline{PQ}$ does not intersect line l,

then P, Q are distinct points in the same *half plane*.

From Section Section 4.2: PSA for the Euclidean and Poincaré planes

Definition of Cartesian half planes

For real numbers a, m, b, define the define the *Cartesian half planes* as follows

$$H_a^+ = \{(x, y) \in \mathbb{R}^2 | x > a\}$$

$$H_a^- = \{(x, y) \in \mathbb{R}^2 | x < a\}$$

$$H_{m,b}^+ = \{(x, y) \in \mathbb{R}^2 | y > mx + b\}$$

$$H_{m,b}^- = \{(x, y) \in \mathbb{R}^2 | y < mx + b\}$$

Definition of Poincaré half planes

For real numbers a, c, r with $r > 0$, define the define the *Poincaré half planes* as follows

$${}_aH^+ = \{(x, y) \in \mathbb{H} | x > a\}$$

$${}_aH^- = \{(x, y) \in \mathbb{H} | x < a\}$$

$${}_cH_r^+ = \{(x, y) \in \mathbb{H} | (x - c)^2 + y^2 > r^2\}$$

$${}_cH_r^- = \{(x, y) \in \mathbb{H} | (x - c)^2 + y^2 < r^2\}$$

Proposition 4.2.4 The Euclidean plane $\left(\mathbb{R}^2, \boldsymbol{\mathcal{L}_E}, \boldsymbol{d_E}\right)$satisfies the *PSA*

With *Cartesian half planes* H_a^+ and H_a^- (or $H_{m,b}^+$ and $H_{m,b}^-$) playing the role of half planes for the *vertical line* L_a (or for the *non-vertical line* $L_{m,b}$), the *Euclidean plane* satisfies the *Plane Separation Axiom (PSA)*.

Result of Exercise 4.2#5 The Taxicab plane $\left(\mathbb{R}^2, \boldsymbol{\mathcal{L}_E}, \boldsymbol{d_T}\right)$ satisfies the *PSA*

With *Cartesian half planes* H_a^+ and H_a^- (or $H_{m,b}^+$ and $H_{m,b}^-$) playing the role of half planes for the *vertical line* L_a (or for the *non-vertical line* $L_{m,b}$), the *Taxicab plane* satisfies the *Plane Separation Axiom (PSA)*.

Proposition 4.2.5 The Poincaré plane satisfies the *PSA*

With *Poincaré half planes* ${}_aH^+$ and ${}_aH^-$ (or ${}_cH_r^+$ and ${}_cH_r^+$) playing the role of half planes for the *type I line* ${}_aL$ (or for the *type II line* ${}_cL_r$), the *Poincaré plane* satisfies the *Plane Separation Axiom (PSA)*.

Section 4.3 Pasch Geometries

In the previous section, we saw that our three most familiar metric geometries—the Euclidean, Taxicab, and Poincaré planes—all satisfy the *Plane Separation Axiom (PSA)*. A reasonable question is

Question: Is the **Plane Separation Axiom (PSA)** true for *all* metric geometries?

The object of Section 4.3 is to answer that question.

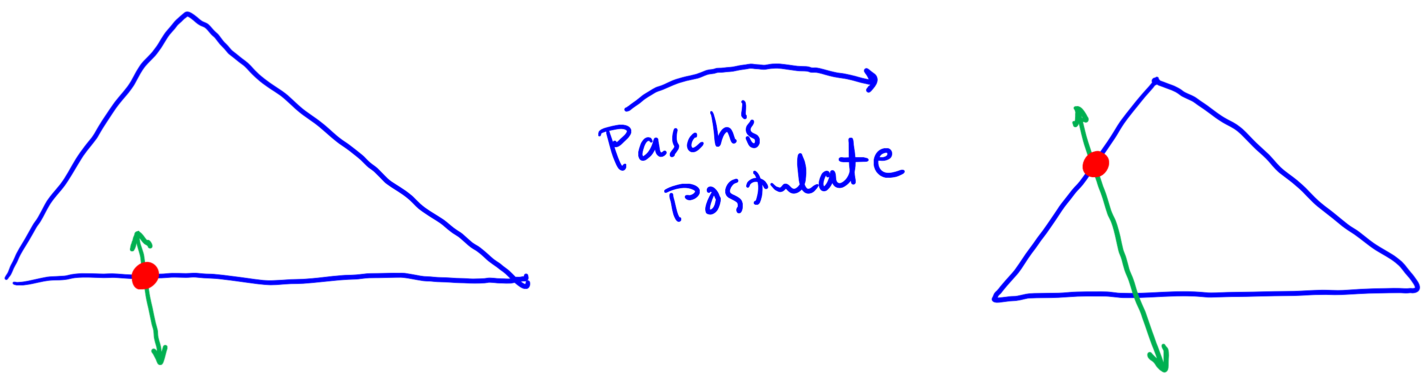
In order to answer the question, we will

- Learn about *Pasch's Postulate*
- Determine the relationship between the *Plane Separation Axiom* and *Pasch's Postulate*
- Study an example of a metric geometry called the *Missing Strip Plane*

Pasch's Postulate

Definition: Pasch's Postulate (PP)

- **Words:** A metric Geometry $(\mathcal{P}, \mathcal{L}, d)$ satisfies **Pasch's Postulate (PP)**
- **Meaning:** For every line and for every triangle, if the line intersects a side of the triangle at a point that is not a vertex, then the line intersects at least one of the opposite sides.



Two Equivalent Statements in a Metric Geometry

Theorem About Two Equivalent Statements in a Metric Geometry

Given: Metric Geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$

Claim: The following statements are equivalent (TFAE)

(1) The metric geometry satisfies the *Plane Separation Axiom (PSA)*.

(2) The metric geometry satisfies *Pasch's Postulate (PP)*.

Proof Strategy:

Proof Part 1: Prove that (1) → (2)

This is the subject of the book's Theorem 4.3.1.

I will also do a proof in this video.

Proof Part 2: Prove that (2) → (1)

This direction of the proof is very difficult. It is done in the book on pages 76,77, in Theorems 4.3.2 and 4.3.3.

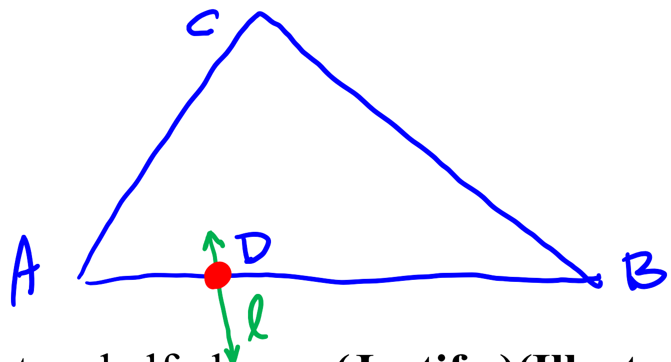
I won't do a proof in this video.

Proof Part 1: Prove that (1) → (2)

(1) Suppose that Metric Geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$ satisfies the *Plane Separation Axiom (PSA)*.

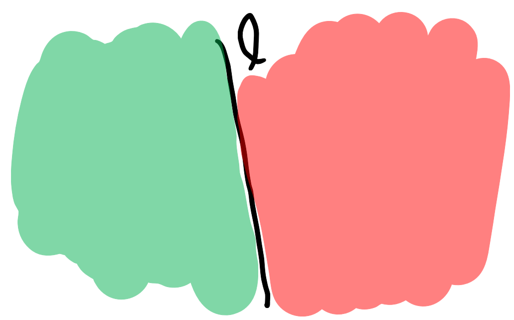
(2) Suppose that a line l intersects side $\overline{AB}$ of triangle ΔABC at a point D that is not a vertex.

(Illustrate.)



(3) Associated to line l are two half planes. **(Justify.)(Illustrate.)**

The two half planes exist by the Plane Separation Axiom.

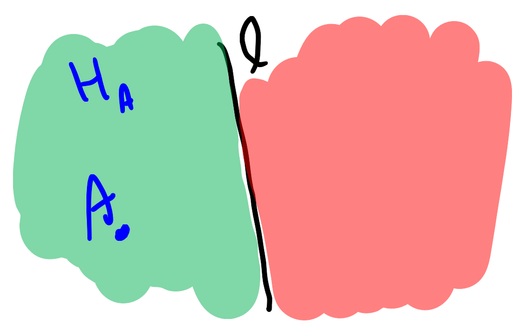


(4) Point A is in one of those half planes. **(Justify.)** Let H_A be the half plane that contains point A.

(Illustrate.)

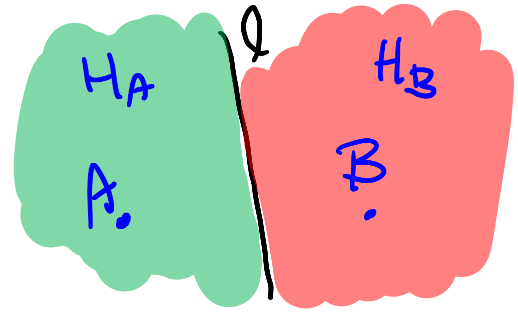
By PSA (i), we know that $\{l, H_1, H_2\}$ is a partion of the set of all points.

And we know that A is not in l by ~~(1)~~ (2).



(5) Point B is not in H_A. **(Justify.)** Let H_B be the half plane that contains point B. **(Illustrate.)**

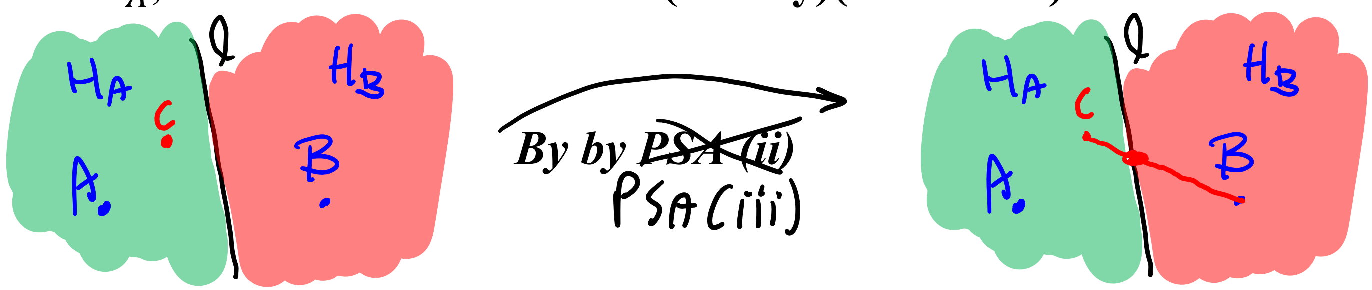
By ~~(1)~~ (2) and PSA (ii) (contrapositive), we know that A, B are not in the same half plane.



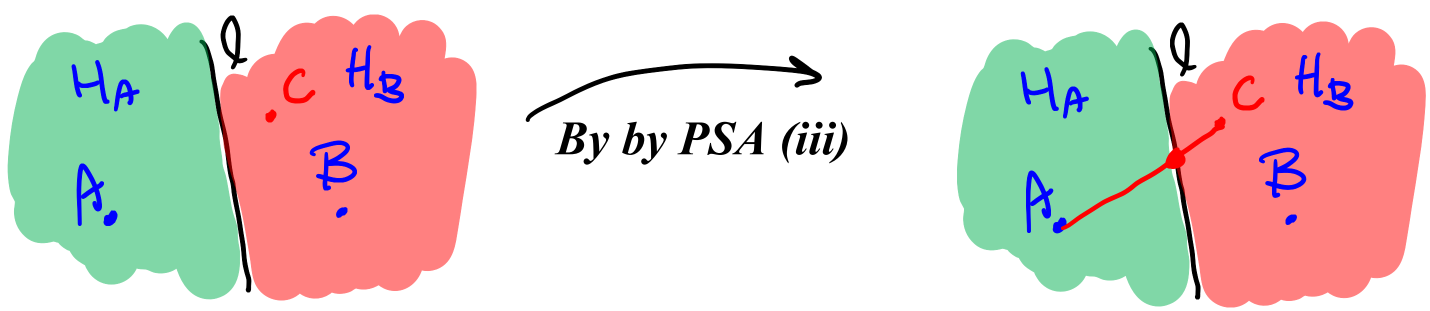
(6) There are three possibilities for point C: Either $C \in H_A$ or $C \in H_B$ or $C \in l$. **(Justify.)**

Because ~~$\{l, H_1, H_2\}$~~ $\{l, H_A, H_B\}$ is a ~~partion~~ Partition of the set of all points, by PSA (i)

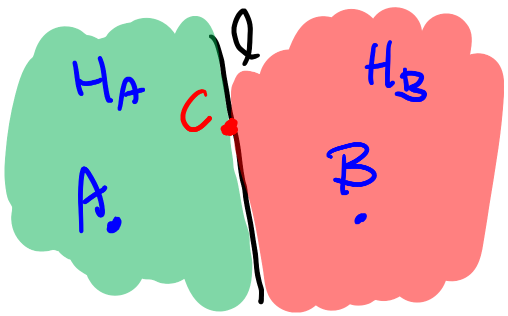
(7) **(Case (i))** If $C \in H_A$, then $\overline{BC}$ intersects line l. **(Justify)(Illustrate.)**



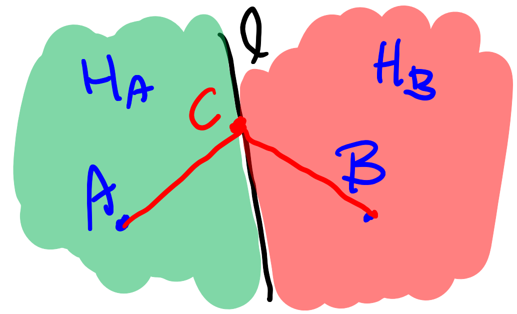
(8) **(Case (ii))** If $C \in H_B$, then $\overline{AC}$ intersects line l. **(Justify)(Illustrate.)**



(9) **(Case (iii))** If $C \in l$, then both $\overline{AC}$ and $\overline{BC}$ intersect line l. **(Illustrate.)**



Because C is an element of both segments



(10) **(Conclusion of cases)** Conclude that line l intersects $\overline{AC}$ or $\overline{BC}$. **(Justify)**

Because it is true in every case.

(11) Therefore, Metric Geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$ also satisfies *Pasch's Postulate (PP)* **(Justify)**

By (2), (10), and definition of Pasch's Postulate

End of Proof Part 1

Proof Part 2: Prove that (2) → (1): See the Book

The Missing Strip Plane

Recall the question posed at the start of this video:

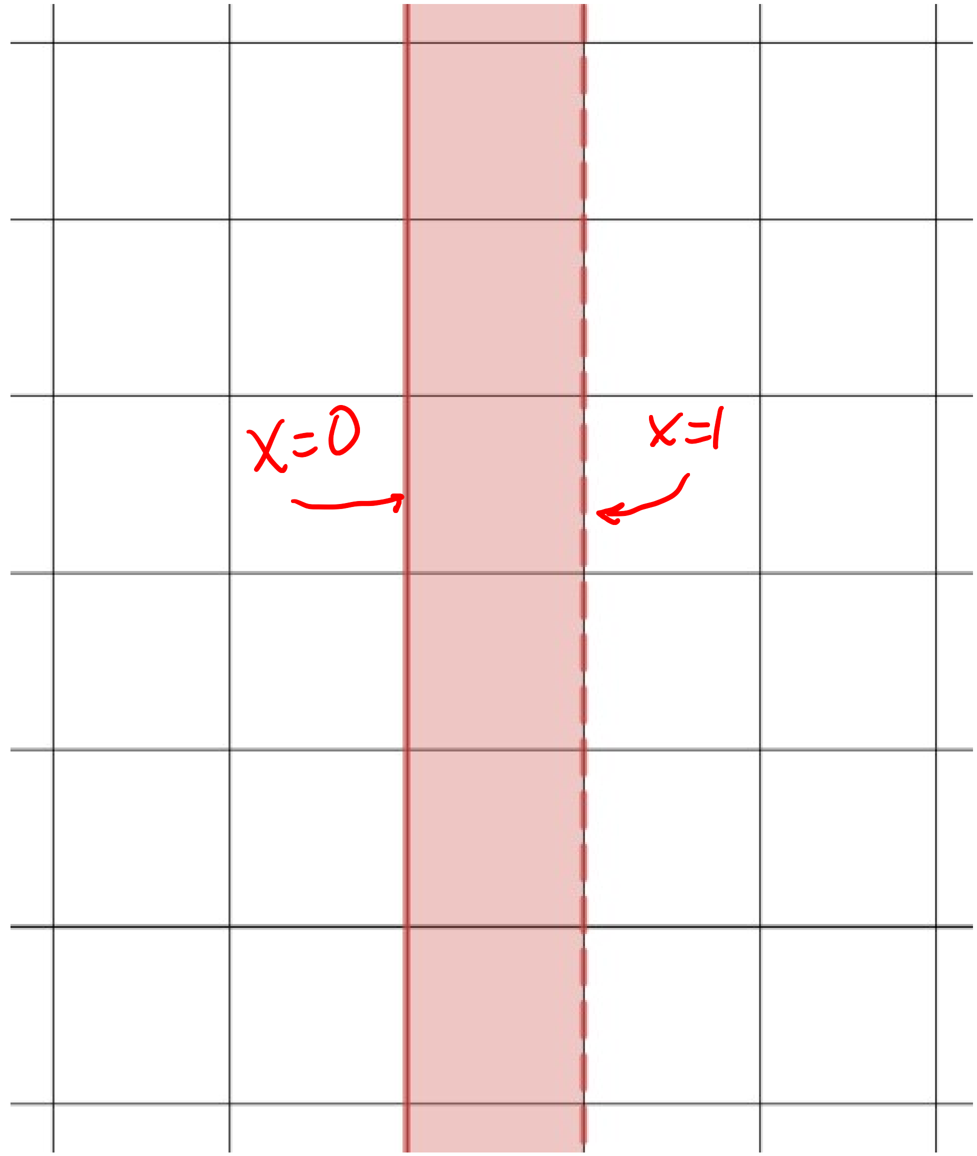
Question: Is the **Plane Separation Axiom (PSA)** true for *all* metric geometries?

The answer to that question will be found in a new geometry called the *Missing Strip Plane*.

We start by defining a peculiar set of points in $\mathbb{R}^2$

$$strip = \{(x, y) \in \mathbb{R}^2 | 0 \leq x < 1\}$$

The set *strip* is shown at right.

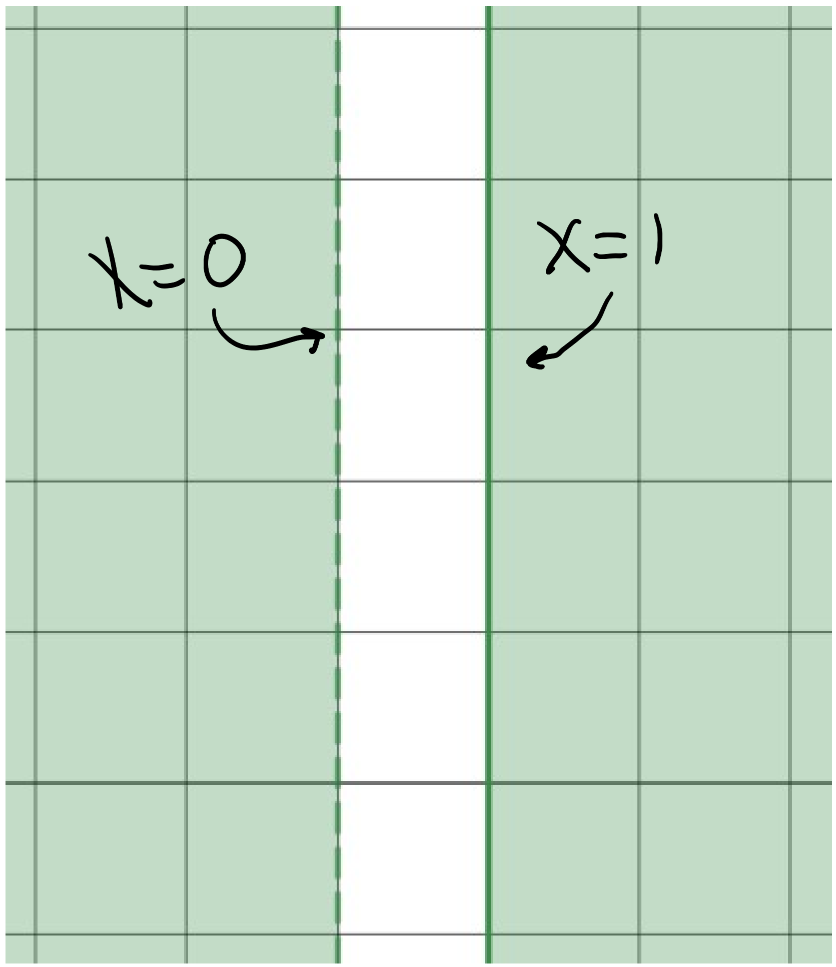


Use that set called $strip$ to define the *Missing Strip Plane*.

Definition: The **Missing Strip Plane** is the pair $(\mathcal{P}, \mathcal{L})$ where

$$\mathcal{P}_{ms} = \mathbb{R}^2 - strip = \{(x, y) \in \mathbb{R}^2 | x < 0 \text{ or } 1 \leq x\}$$

$$\mathcal{L}_{ms} = \{l - strip | l \text{ is a cartesian line}\}$$



[Example 1] Lines in the Missing Strip Plane.

Let

$$A = (2,5)$$

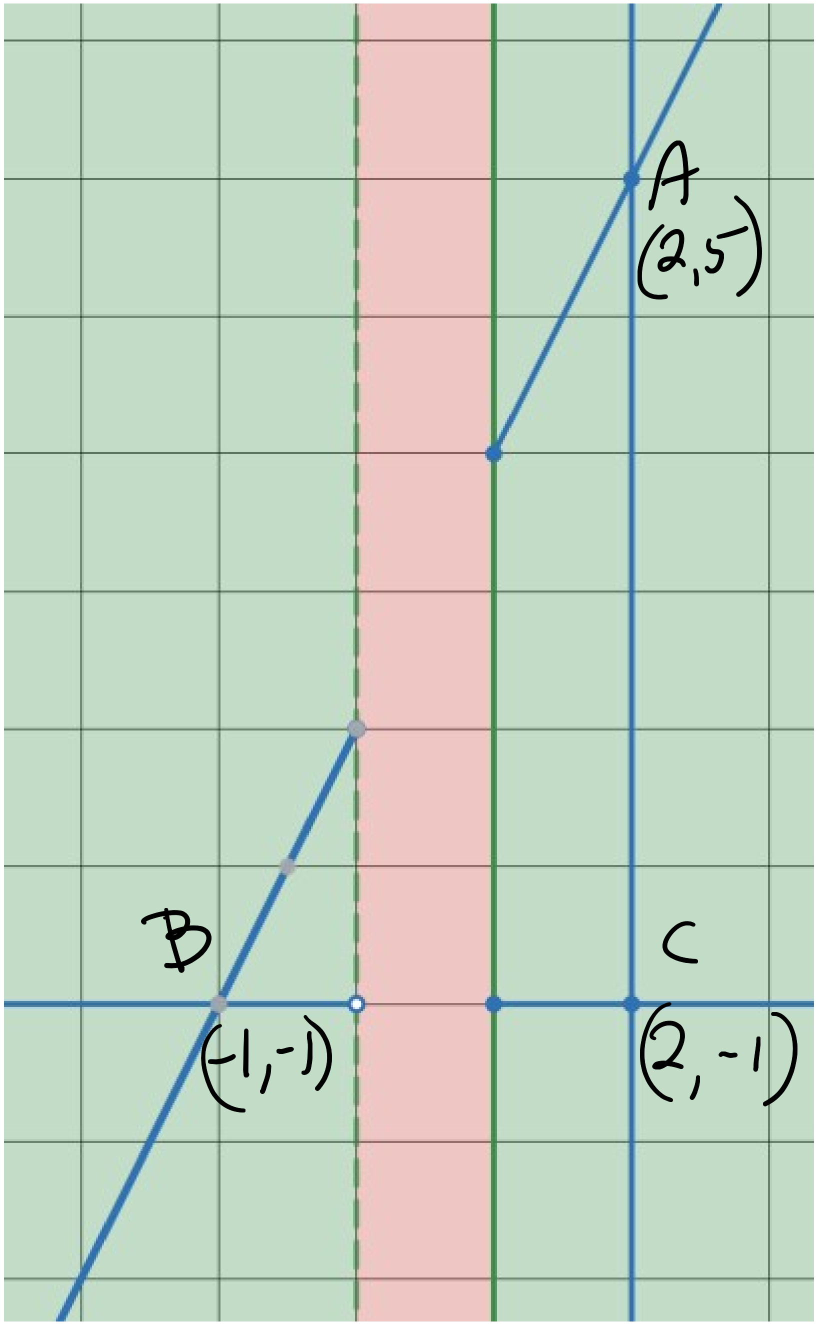
$$B = (-1,-1)$$

$$C = (2,-1)$$

Missing Strip lines

$$\overleftrightarrow{AB}, \overleftrightarrow{BC}, \overleftrightarrow{CA}$$

are shown in the drawing at right.



End of [Example 1]

In Homework Exercise 4.3#4, you will prove that the *Missing Strip Plane* is an *incidence geometry*. (Note that in order to do this, you will need to first prove that the *Missing Strip Plane* is an *abstract geometry*.)

In the book on page 79, the authors mention that you will be proving that the *Missing Strip Plane* is an *incidence geometry*. Then, they go on to prove that the *Missing Strip Plane* is a *metric geometry*. Their proof of this fact is very interesting. The proof uses Theorem 2.2.8, which we did not discuss in our course.

Since the *Missing Strip Plane* is a *metric geometry*, it will have *segments* and *triangles*.

[Example 2] *Segments* and *Triangles* in the *Missing Strip Plane.*

Let

$$A = (2,5)$$

$$B = (-1,-1)$$

$$C = (2,-1)$$

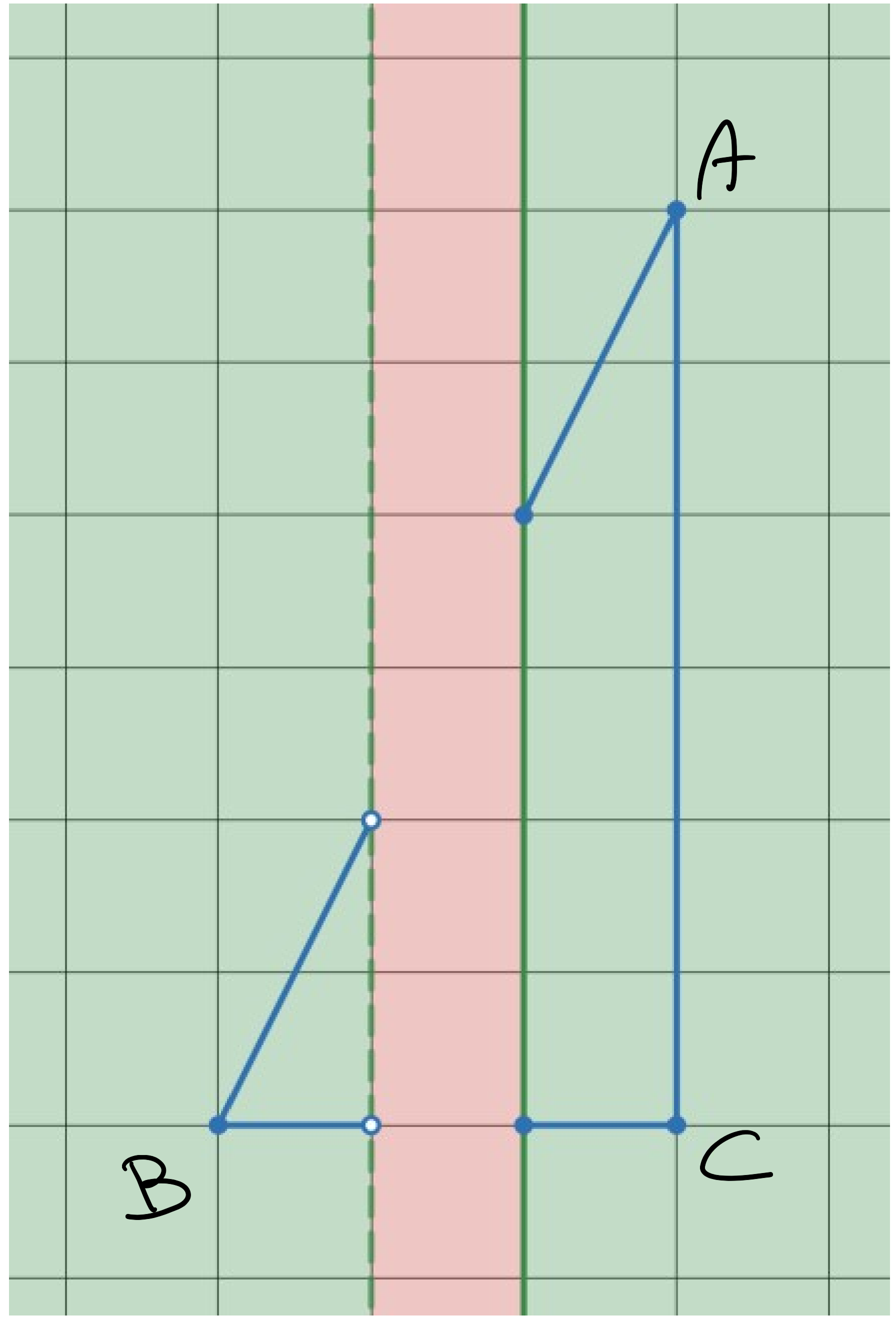
Missing Strip segments

$$\overline{AB}, \overline{BC}, \overline{CA}$$

and *Missing Strip triangle*

$$\Delta ABC$$

are shown in the drawing at right.



End of [Example 2]

[Example 3] The *Missing Strip Plane* does not satisfy *Pasch's Postulate*.

Let

$$A = (2,5)$$

$$B = (-1,-1)$$

$$C = (2,-1)$$

$$D = (-2,2)$$

$$E = (3,2)$$

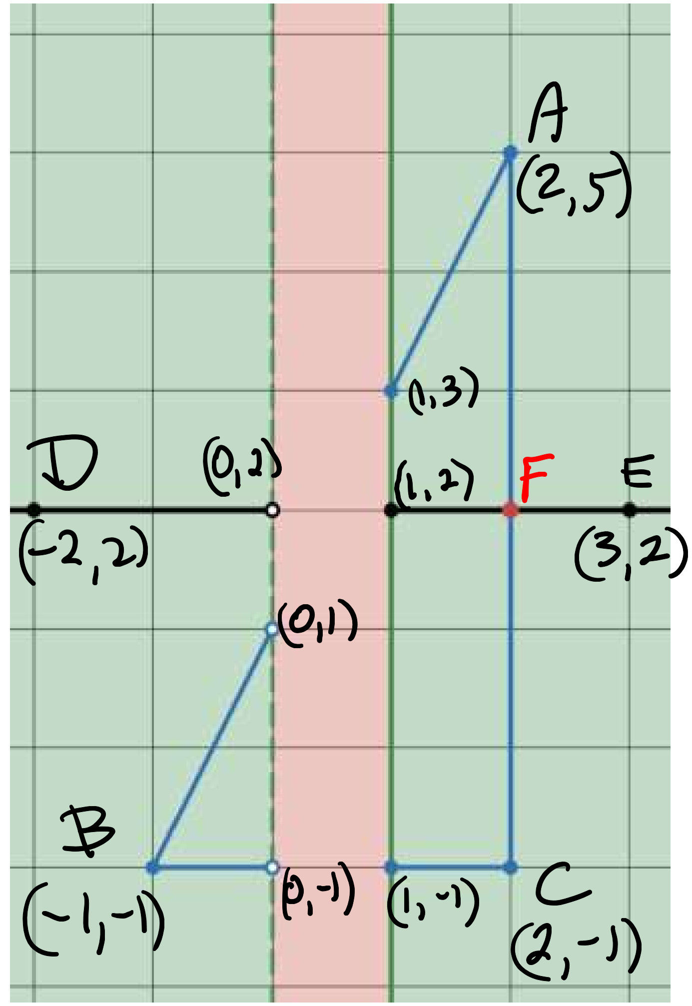
Missing Strip triangle

$$\Delta ABC$$

and *Missing Strip line*

$$\overleftrightarrow{DE}$$

are shown in the drawing at right.



Observe that line $\overleftrightarrow{DE}$ intersects side ~~$\overline{AB}$~~ $\overline{AC}$ of ΔABC at point ~~E~~ F such that ~~$A - E - B$~~ A - F - C. But line $\overleftrightarrow{DE}$ does not intersect either of the other two sides of the triangle. This shows that the *Missing Strip Plane* does not satisfy *Pasch's Postulate*.

End of [Example 3]

The above example shows that the *Missing Strip Plane* does not satisfy *Pasch's Postulate (PP)*.

The *Theorem about Two Equivalent Statements in a Metric Geometry* (from ten pages ago) tells us that it does not satisfy the *Plane Separation Axiom (PSA)*, either.

That provides the answer to our question from a few pages ago:

Question: Is the ***Plane Separation Axiom (PSA)*** true for *all* metric geometries?

Answer: No, there are metric geometries that do not satisfy *PSA*.

Pasch Geometries

We have seen that not all metric geometries satisfy the *Plane Separation Axiom (PSA)*.

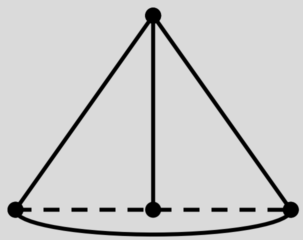
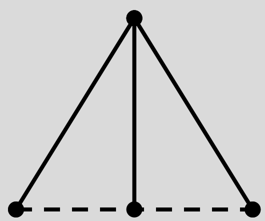
Therefore, if we want to be sure that a geometry does satisfy the *PSA*, we need to make it a requirement.

Definition of Pasch Geometry

A ***Pasch Geometry*** is a *metric geometry* that satisfies the *Plane Separation Axiom (PSA)*.

Remark: By the *Theorem About Two Equivalent Statements in a Metric Geometry*, we see that Pasch Geometries are also the metric geometries that satisfy *Pasch's Postulate (PP)*.

Some of the Geometries that We have Studied so Far

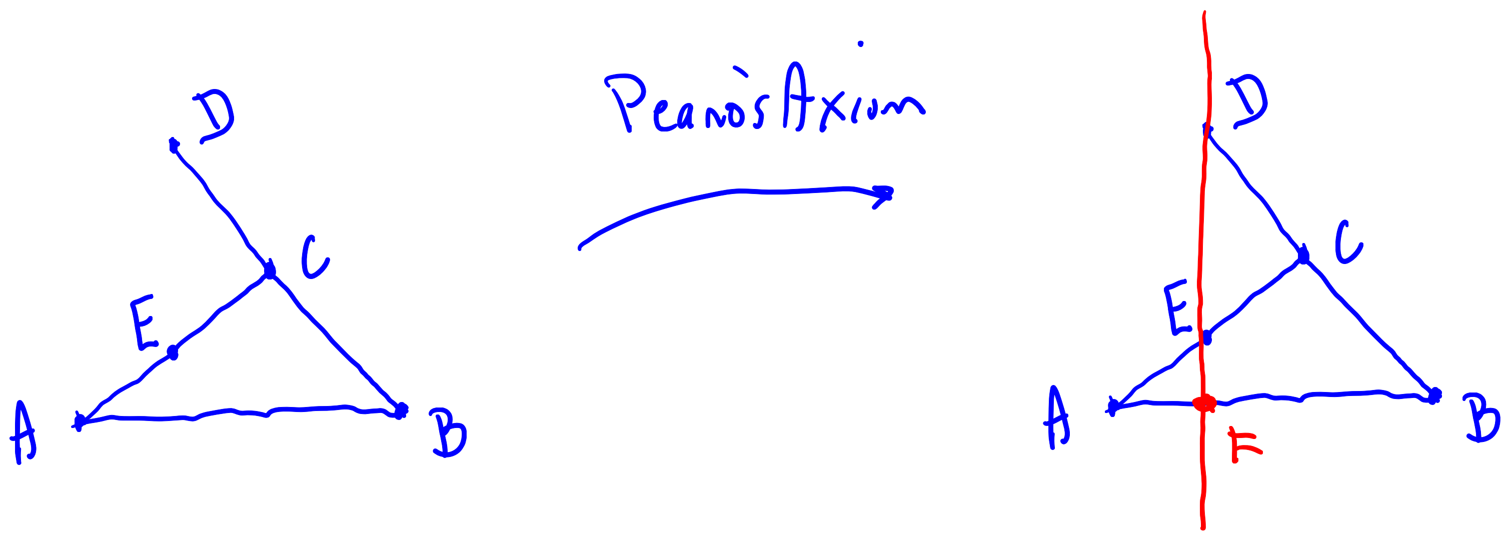
Geometry	Abstract?	Incidence?	Metric?	Pasch?
	yes	no	no	no
Riemann Sphere	yes	no	no	no
	yes	yes	no	no
Missing Strip plane	yes	yes	yes	no
Euclidean plane	yes	yes	yes	yes
Taxicab plane	yes	yes	yes	yes
Max plane	yes	yes	yes	yes
Poincaré plane	yes	yes	yes	yes

Peano’s Axiom

Definition: Peano’s Axiom (PA)

- **Words:** A metric Geometry $(\mathcal{P}, \mathcal{L}, d)$ satisfies ***Peano’s Axiom (PA)***
- **Meaning:** Given triangle ΔABC and points D, E such that $B - C - D$ and $A - E - C$, there exists a point $F \in \overleftrightarrow{DE}$ such that $A - F - B$ and $D - E - F$.

Illustration of *Peano’s Axiom*:



Is Peano's Axiom Always True?

Earlier in this video, we posed the following question and eventually found an answer.

Question: Is the **Plane Separation Axiom (PSA)** true for *all* metric geometries?

Answer: No, we saw an example in the *Missing Strip Plane* metric geometry that illustrated that the *Missing Strip Plane* did not satisfy *Pasch's Postulate*. A theorem about the equivalence of Pasch's Postulate and the Plane Separation Axiom then told us that the *Missing Strip Plane* also does not satisfy the *Plane Separation Axiom*.

An obvious question now is,

Question: Is the **Peano's Axiom (PA)** true for *all* metric geometries?

Answer: No. (In one of your homework exercises, you will be asked to provide an example in the *Missing Strip Plane* metric geometry that illustrates that the *Missing Strip Plane* does not satisfy Peano's Axiom.)

The following Theorem tells us about one situation where we know that *Peano's Axiom will* be satisfied.

Theorem: ***PSA → PA***

Given: Metric Geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$

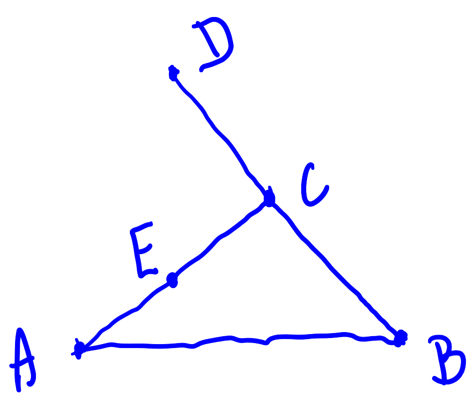
Claim: If the metric geometry satisfies the *Plane Separation Axiom (PSA)*, then the metric geometry satisfies *Peano's Axiom (PA)*.

Justify and Illustrate the Statements in the Following Proof

Proof

(1) Suppose Metric Geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$ satisfies the Plane Separation Axiom

(2) Suppose that triangle ΔABC is given, along with points D, E such that $B - C - D$ and $A - E - C$. **(Illustrate.)**



Part 1: Show that there exists a point $F \in \overleftrightarrow{DE}$ such that $A - F - B$

(3) Pasch's Postulate is also satisfied. **(Justify.)**

By (1) and Theorem 4.3.1 that says if a metric geometry satisfies PSA then it satisfies PP.

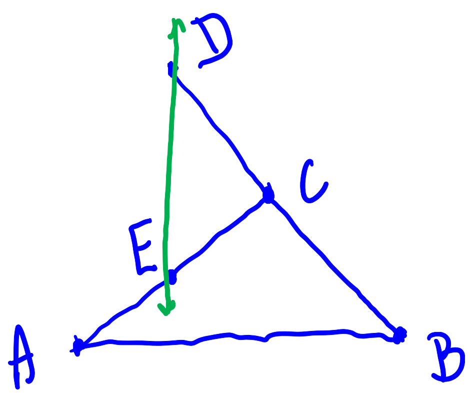
Remark: Theorem 4.3.3 that says if a metric geometry satisfies PP then it satisfies PSA.

I incorporated Theorems 4.3.1 and 4.3.3 into the theorem that I called

"Theorem About Two Equivalent Statements in a Metric Geometry"

(4) Line $\overleftrightarrow{DE}$ intersects segment $\overline{BA}$ or segment $\overline{BC}$. **(Justify.) ~~(Illustrate.)~~**

By (2) and Pasch's Postulate applied to line $\overleftrightarrow{\mathbf{DE}}$ that intersects ΔABC at point E that lies on side $\overline{\mathbf{AC}}$ and that is not an endpoint.



Pasch's Postulate

(5) Line $\overleftrightarrow{DE}$ can't intersect segment $\overline{BC}$. **(Justify.)**

Lines $\overleftrightarrow{\boldsymbol{DE}}$ ***and*** $\overleftrightarrow{\boldsymbol{BC}}$ ***intersect at point*** $\boldsymbol{D}$**.**

They are not allowed to have any other intersections, by Corollary 2.1.7.

Point D is not in $\overline{\boldsymbol{BC}}$ ***because*** $\boldsymbol{B - C - D}$ ***by (2).***

(6) Therefore line $\overleftrightarrow{DE}$ intersects segment $\overline{BA}$. **(Justify.)**

By (4) and (5).

(7) The point of intersection can't be point B. **(Justify.)**

We already know that B is not on line $\overleftrightarrow{\boldsymbol{DE}}$ ***because lines*** $\overleftrightarrow{\boldsymbol{DE}}$ ***and*** $\overleftrightarrow{\boldsymbol{BC}}$ ***intersect at point D.***

(8) The point of intersection can't be point A. **(Justify.)**

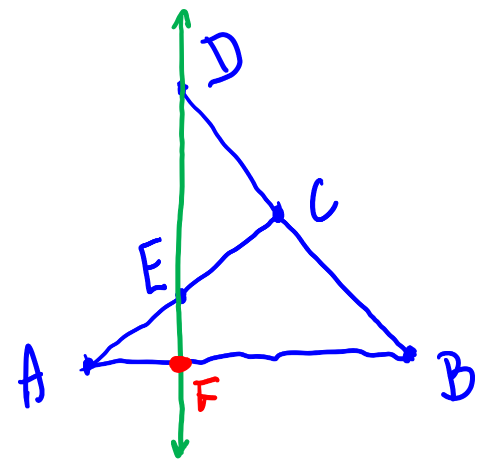
We know that lines $\overleftrightarrow{\boldsymbol{DE}}$ ***and*** $\overleftrightarrow{\boldsymbol{AC}}$ ***intersect at point E.***

They are not allowed to have any other intersections, by Corollary 2.1.7.

Therefore, the point of intersection cannot be A.

(9) Therefore, the point of intersection must be a point F such that $A - F - B$. **(Justify.) (Illustrate.)**

Because line $\overleftrightarrow{DE}$ intersects segment $\overline{AB}$ (by (6)) and not at one of the endpoints (by (7),(8)), the only remaining option is a point F such that $A - F - B$ (by definition of line segment).



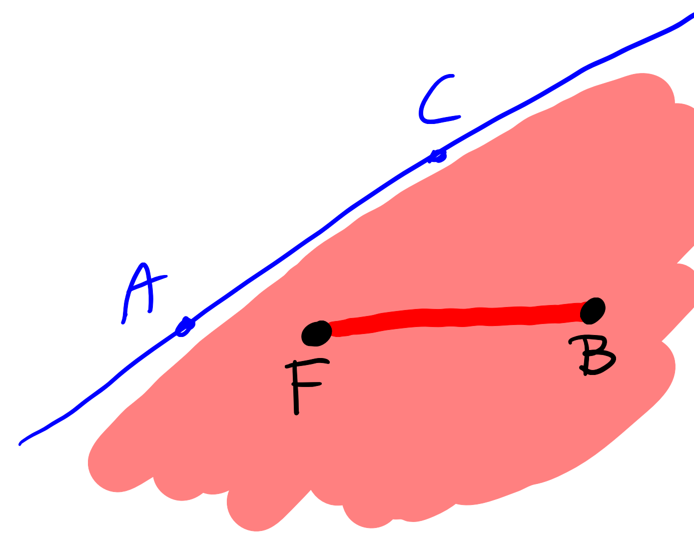
Part 2: Show that $D - E - F$ is also true

(10) Points F, B are in the same half plane of line $\overleftrightarrow{AC}$. **(Justify.) (Illustrate.)**

We know that $\overleftrightarrow{AC}$ does not intersect $\overline{FB}$ because

- ***lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{AB}$ only intersect at point A***
- ***and A is not on $\overline{FB}$ because $A - F - B$.***

Therefore, by PSA (iii) (contrapositive), points F, B are in the same half plane.



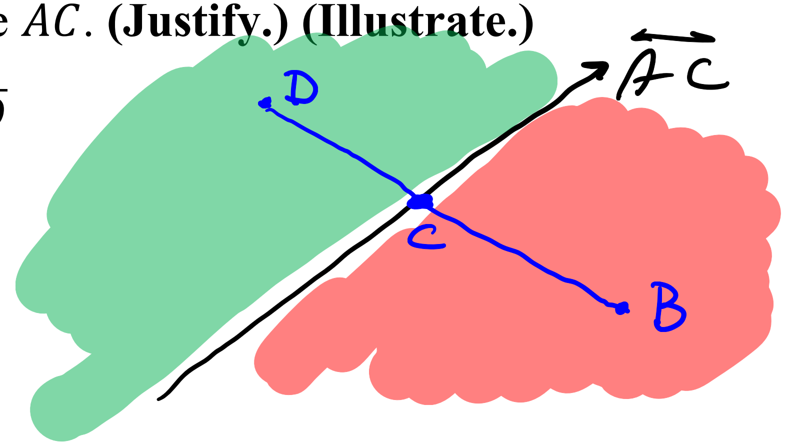
(11) Points B, D are in different half planes of line $\overleftrightarrow{AC}$. **(Justify.) (Illustrate.)**

Observe that line $\overleftrightarrow{AC}$ intersects segment $\overline{BD}$

at point C such that $B - C - D$ (by (2)),

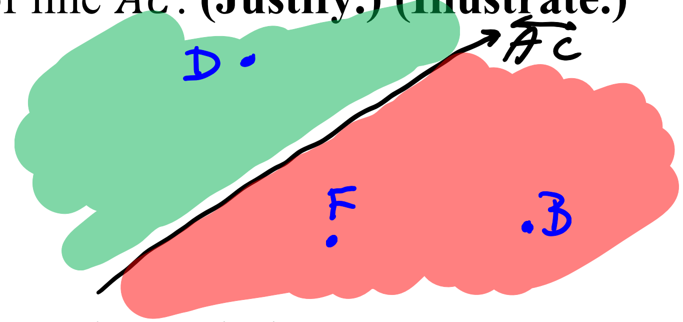
so by PSA (ii) (contrapositive),

points B, D are in different half planes.



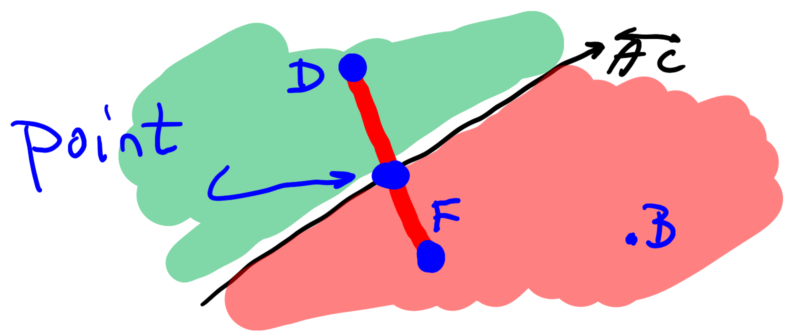
(12) Therefore, F, D are in different half planes of line $\overleftrightarrow{AC}$. **(Justify.) (Illustrate.)**

By (10),(11)



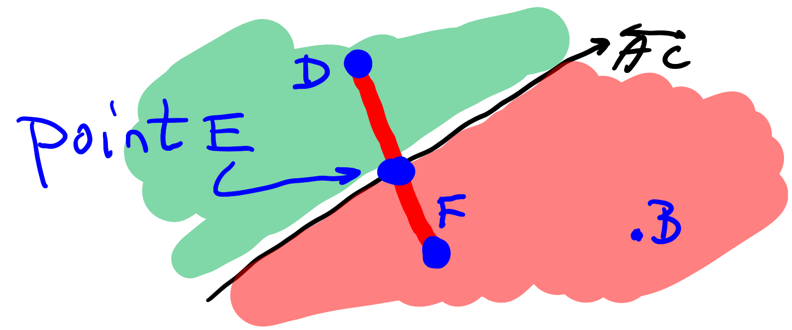
(13) Therefore, line $\overleftrightarrow{AC}$ intersects segment $\overline{FD}$ at a point such that $F - point - D$. **(Justify.) (Illustrate.)**

By (12) and PSA (iii)



(14) But line $\overleftrightarrow{AC}$ is known to intersect line $\overleftrightarrow{DF}$ at point E. The two lines can only intersect at one point. **(Justify.)** Thefore, the point of intersection in statement (13) must be point E.

That is, $D - E - F$ is true. **(Illustrate.)**



By Corollary 2.1.7

(15) We have proven that there exists a point F on line $\overleftrightarrow{DE}$ such that $A - F - B$ and $D - E - F$. **(Justify.)**

By (9),(14)

(16) Therefore, Metric Geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$ also satisfies *Peano's Axiom (PA)* **(Justify)**

By (2),(16), and definition of Peano's Axiom

End of Proof

End of Video