4.4: Interiors and the Crossbar Theorem

produced by Mark Barsamian, 2021.03.11

for Ohio University MATH 3110/5110 College Geometry

Topics

- **Plane Separation Properties of Interiors of Rays and Segments**
- **The Z Theorem**
- **Interiors of Angles and Triangles**
- **The Crossbar Theorem**
- **The Converse of the Crossbar Theorem**

Reading: Section 4.4: Interiors and the Crossbar Theorem, p 81 - 85 in *Geometry: A Metric Approach with Models, Second Edition* by Millman & Parker

Homework: Section 4.4 #2, 4, 5, 6, 9, 10, 11, 12, 15

Recall from Section 2.1: Theorem about Intersecting Lines in *Incidence Geometry*

Theorem 2.1.6 Given two lines l_1 and l_2 in an *incidence geometry*,

If $l_1 \cap l_2$ has two or more distinct points,

then l_1 and l_2 are the same line. That is, $l_1 = l_2$.

The *contrapositive* of the statement of Theorem 2.1.6 can be stated as a *corollary*.

Corollary 2.1.7 (contrapositive of Theorem 2.1.6)

Given two lines l_1 and l_2 in an *incidence geometry*,

If lines l_1 and l_2 are known to be distinct lines (that is, $l_1 \neq l_2$),

then either lines l_1 and l_2 do not intersect or they intersect in exactly one point.

Recall Some Properties of Betweenness from Section 3.2

Corollary 3.2.4 Fact about Three Distinct Collinear Points in a Metric Geometry

Given: Three distinct collinear points P, Q, R in a metric geometry

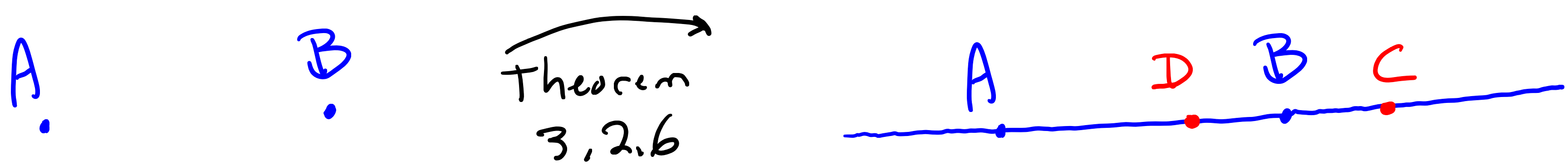
Claim: Exactly one of the points is between the other two.

Theorem 3.2.6 Existence of Points with Certain Betweenness Relationships

Given: Distinct points A, B in a *metric geometry*

Claim: (i) There exists a point C with $A - B - C$

(ii) There exists a point D with $A - D - B$



Recall Definitions of Segment and Ray and their Interiors

Definition of Segment

Symbol: $\overline{AB}$

Spoken: *segment* A B.

Usage: A, B are distinct points in a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$.

Meaning: the set

$$\overline{AB} = \{C \in \mathcal{P} | C = A \text{ or } A - C - B \text{ or } C = B\}$$

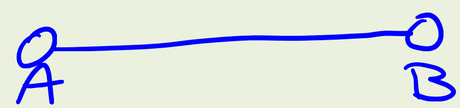
Additional Terminology

The **end points** (or **vertices**) of $\overline{AB}$ are the points A and B.

The **interior of the segment** is the set of all points of the segment that are *not* endpoints:

$$\text{int}(\overline{AB}) = \overline{AB} - \{A, B\} = \{C \in \mathcal{P} | A - C - B\}$$

Symbol: length$(\overline{AB})$



Spoken: the **length** of segment $\overline{AB}$

Meaning: the number AB. That is, the length is the number $d(A, B)$.

Definition of Ray

Symbol: $\overrightarrow{AB}$

Spoken: *ray A B*.

Usage: A, B are distinct points in a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$.

Meaning: the set

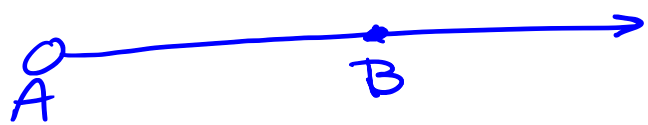
$$\overrightarrow{AB} = \{C \in \mathcal{P} | C = A \text{ or } A - C - B \text{ or } C = B \text{ or } A - B - C\}$$
$$= \overline{AB} \cup \{C \in \mathcal{P} | A - B - C\}$$

Additional Terminology

The **initial point** (or **vertex**) of $\overrightarrow{AB}$ is the point A.

The **interior of the ray** is the set of all points of the ray except the initial point:

$$\operatorname{int}(\overrightarrow{AB}) = \overrightarrow{AB} - \{A\} = \{C \in \mathcal{P} | A - C - B \text{ or } C = B \text{ or } A - B - C\}$$



Recall the Plane Separation Axiom

Definition: The Plane Separation Axiom (PSA) (**My version of the definition**)

- **Words:** A metric Geometry $(\mathcal{P}, \mathcal{L}, d)$ satisfies the **plane separation axiom** (PSA)
- **Meaning:** For every line $l \in \mathcal{L}$, there are two associated sets of points called *half planes*, denoted H_1 and H_2, with the following properties:

  (i) The three sets l, H_1, H_2 form a partition of the set $\mathcal{P}$ of all points.

  (ii) Each of the *half planes* is convex.

  (iii) If $A \in H_1$ and $B \in H_2$, then $\overline{AB}$ *intersects* line l.
- **Additional Terminology:**
  - Line l is called the ***edge*** of *half planes* H_1 and H_2.
  - **Words:** *Points* A, B *lie on the* ***same side*** *of line* l.
  - **Meaning:** Points A, B are elements of the same half plane associated to l.
  - **Words:** *Points* A, B *lie on* ***opposite sides*** *of line* l.
  - **Meaning:** Points A, B are elements of different half planes associated to l.

PSA (ii) and (iii) and their Contrapositives

PSA (ii): If distinct points P, Q are in the same *half plane*, then $\overline{PQ}$ does not intersect line l.

PSA (ii) (contrapositive): If $\overline{PQ}$ does intersect line l, then P, Q are *not* in the same *half plane*.

PSA (iii) If P, Q are not in the same *half plane*, then $\overline{PQ}$ intersects line l.

PSA (ii) (contrapositive) If $\overline{PQ}$ does not intersect line l, then P, Q are distinct points in the same *half plane*.

Definition: Pasch's Postulate (PP)

- **Words:** A metric Geometry $(\mathcal{P}, \mathcal{L}, d)$ satisfies **Pasch's Postulate (PP)**
- **Meaning:** For every line and for every triangle, if the line intersects a side of the triangle at a point that is not a vertex, then the line intersects at least one of the opposite sides.

Theorem About Two Equivalent Statements in a Metric Geometry

Given: Metric Geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$

Claim: The following statements are equivalent (TFAE)

(1) The metric geometry satisfies the *Plane Separation Axiom (PSA)*.

(2) The metric geometry satisfies *Pasch's Postulate (PP)*.

Definition of Pasch Geometry

A ***Pasch Geometry*** is a *metric geometry* that satisfies the *Plane Separation Axiom (PSA)*.

Remark: By the *Theorem About Two Equivalent Statements in a Metric Geometry*, we see that Pasch Geometries are also the metric geometries that satisfy *Pasch's Postulate (PP)*.

Section 4.4 Interiors and the Crossbar Theorem

Plane Separation Properties of Interiors of Rays and Segments

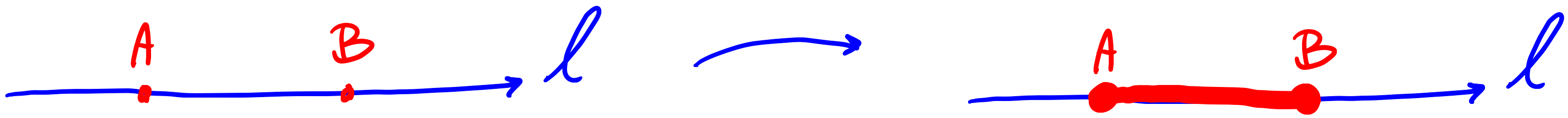
The Plane Separation Axiom (PSA) discusses some of the ways in which points and line segments intersect a line and its half planes.

It will be useful to articulate some more ways in which line segments intersect a line and its half planes and to also consider ways in rays intersect a line and its half planes.

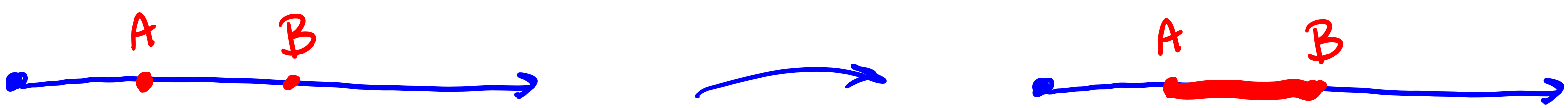
For that, we will start by considering convexivity properties.

I will point out, but will not prove, the following convexivity properties of lines, rays, segments, and their interiors:

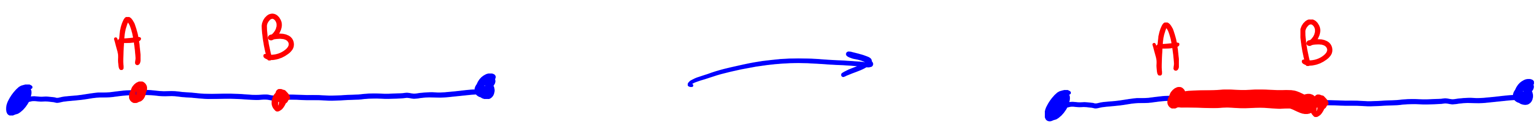
In a metric geometry, a *line* is a convex set.



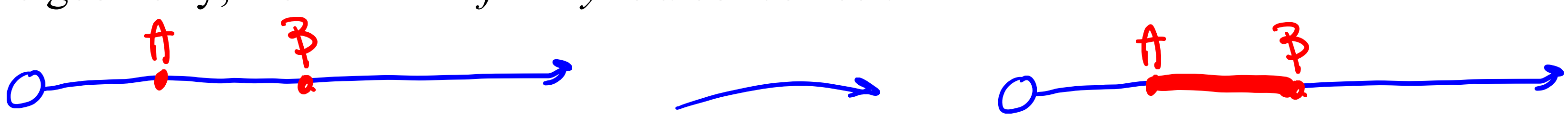
In a metric geometry, a *ray* is a convex set.



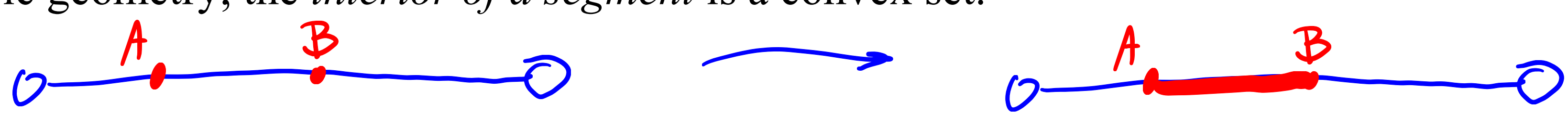
In a metric geometry, a *segment* is a convex set.



In a metric geometry, the *interior of a ray* is a convex set.



In a metric geometry, the *interior of a segment* is a convex set.

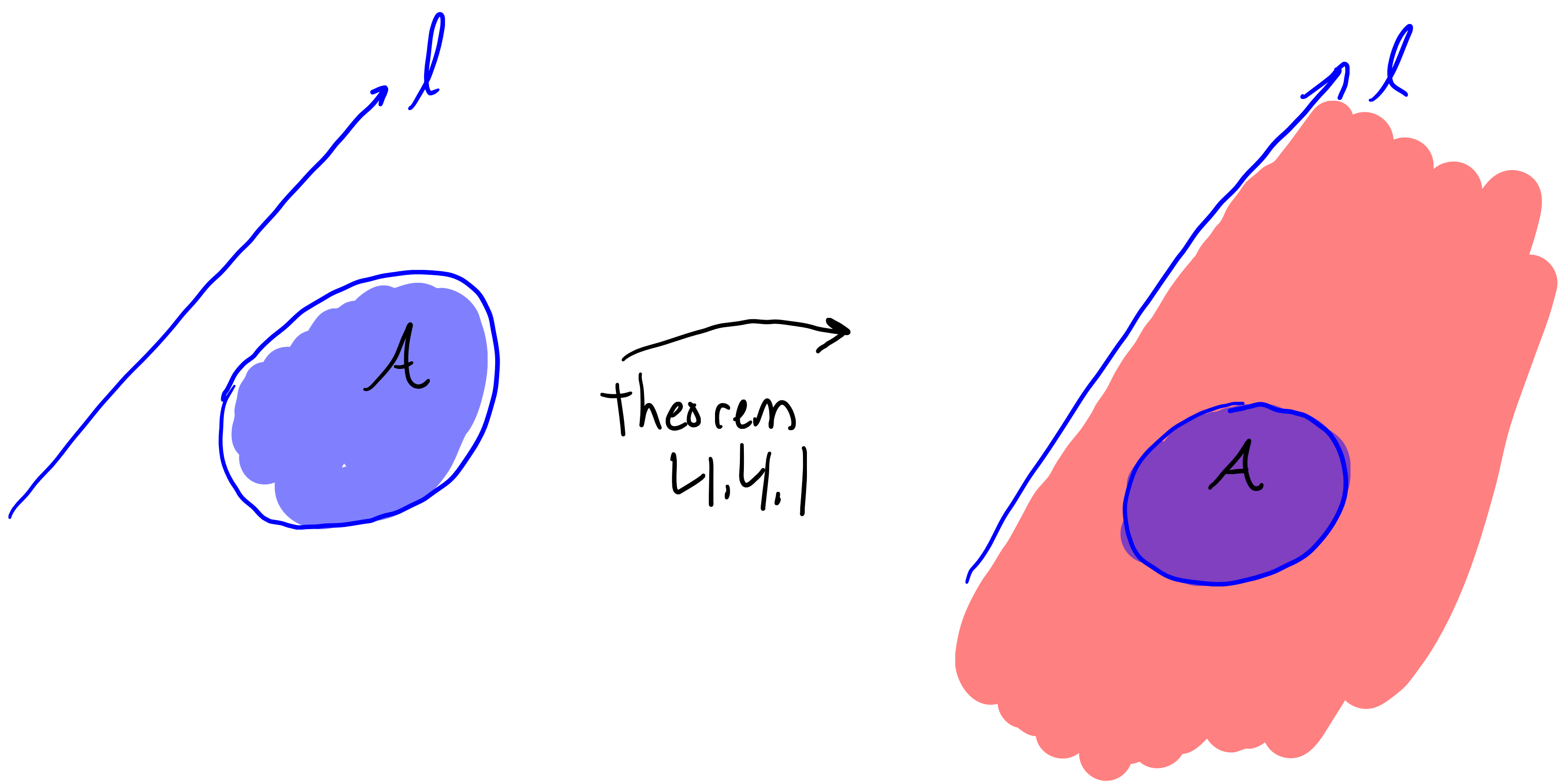


The following theorem is proven in the book. The proof is not difficult, so I won't discuss it here.

Theorem 4.4.1 In a Pasch Geometry,

if $\mathcal{A}$ is a nonempty convex set that does not intersect line l,

then all points of $\mathcal{A}$ lie on the same side of l.



The proof of the following theorem is straightforward, and so I won't discuss the proof here. (You'll prove it in a homework exercise. Your proof should use Theorem 4.4.1)

Theorem 4.4.2 In a Pasch Geometry,

let $\mathcal{A}$ be a line, ray, segment, interior of a ray, or interior of a segment.

(i) If l is a line with $\mathcal{A} \cap l = \phi$, then all of $\mathcal{A}$ lies on one side of l.

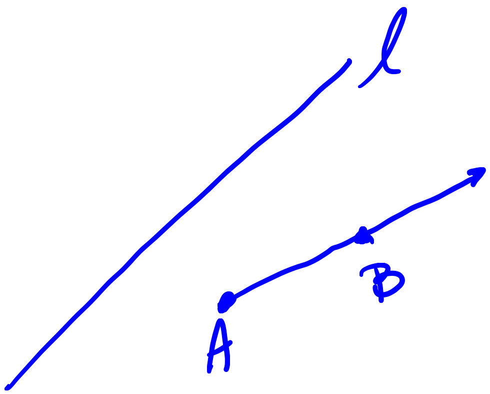
(ii) If $A - B - C$ and $\overleftrightarrow{AC} \cap l = \{B\}$

then $\text{int}\left(\overrightarrow{BA}\right)$ and $\text{int}(\overline{BA})$ both lie on the same side of l,

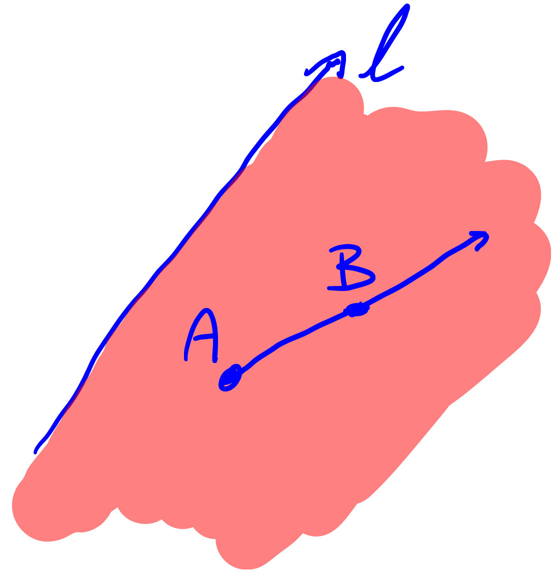
while $\text{int}\left(\overrightarrow{BC}\right)$ and $\text{int}(\overline{BC})$ both lie on the other side of l.

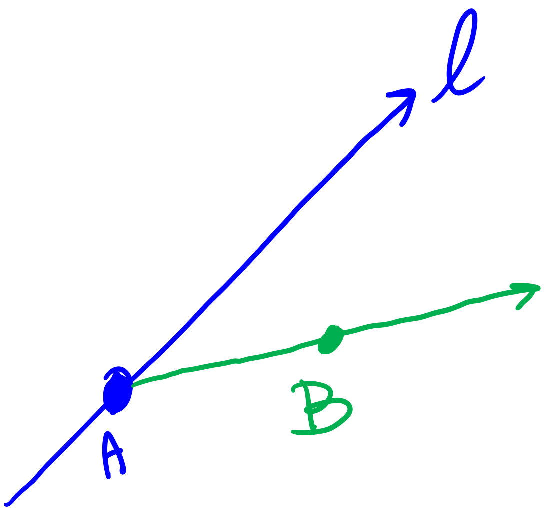
It is worthwhile making a variety of drawings that illustrate the statement of Theorem 4.4.2. This will not only help you understand the statement of the theorem, but also help train your eyes so that in future proofs, you will recognize situations where you need to use the theorem.

Illustrations of Theorem 4.4.2 (i)



th 4.4.2





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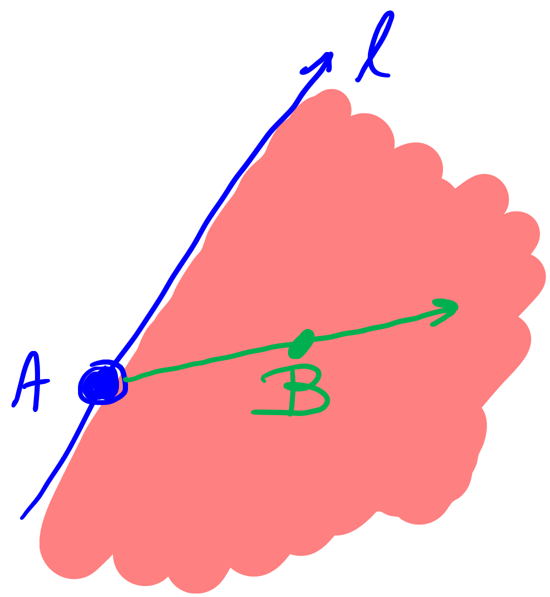
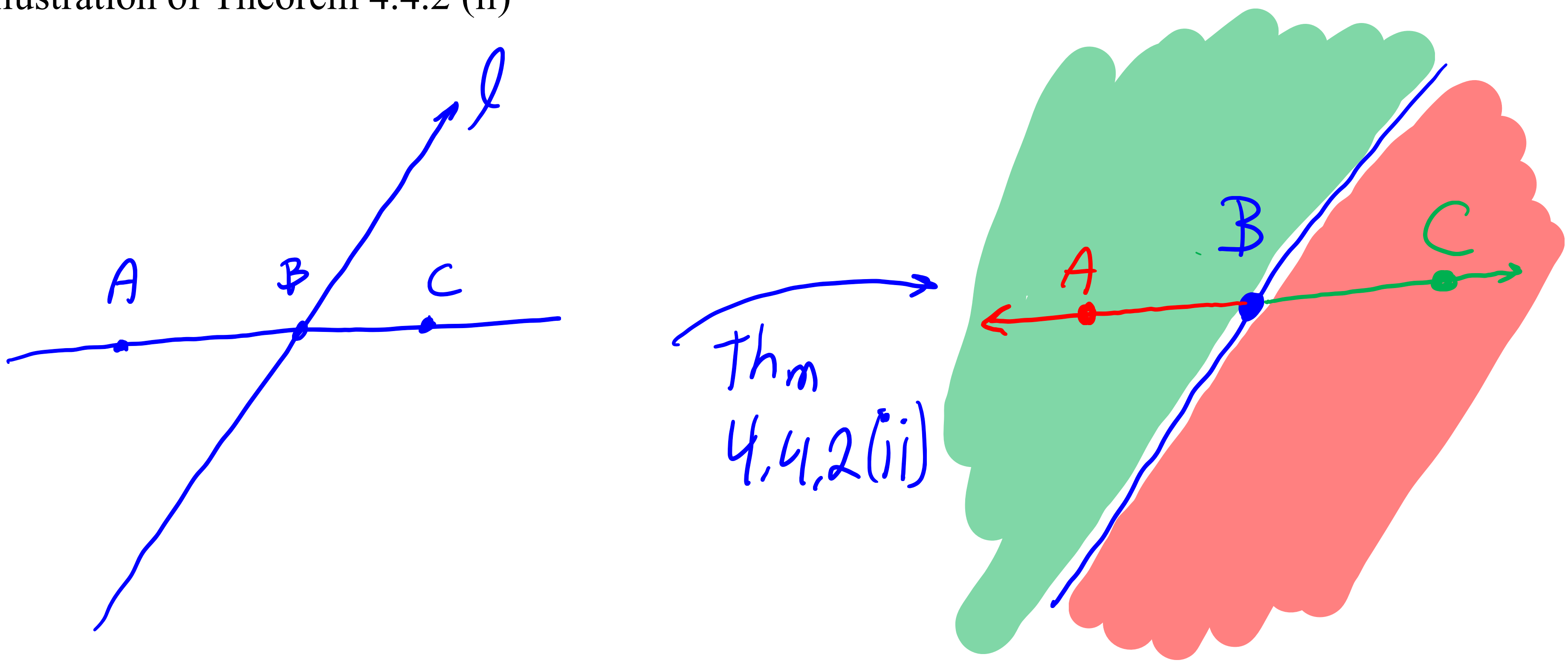


Illustration of Theorem 4.4.2 (ii)

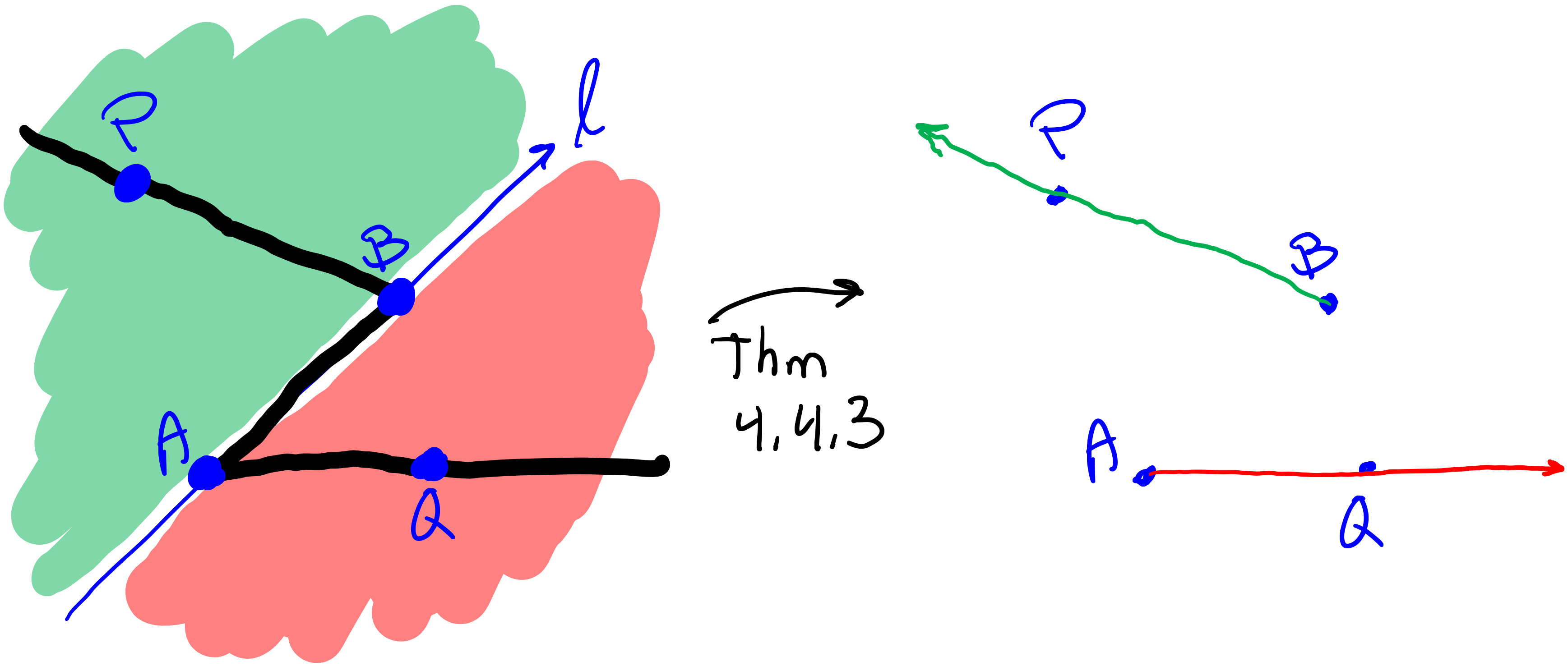


The Z Theorem

The following theorem is easily proven using Theorem 4.4.2. See the book for a proof.

Theorem 4.4.3 (*The Z Theorem*) In a Pasch geometry, if P and Q are on opposite sides of $\overleftrightarrow{AB}$, then $\overrightarrow{BP} \cap \overrightarrow{AQ} = \phi$. In particular, $\overline{BP} \cap \overline{AQ} = \phi$.

Illustration of the Statement of Theorem 4.4.3



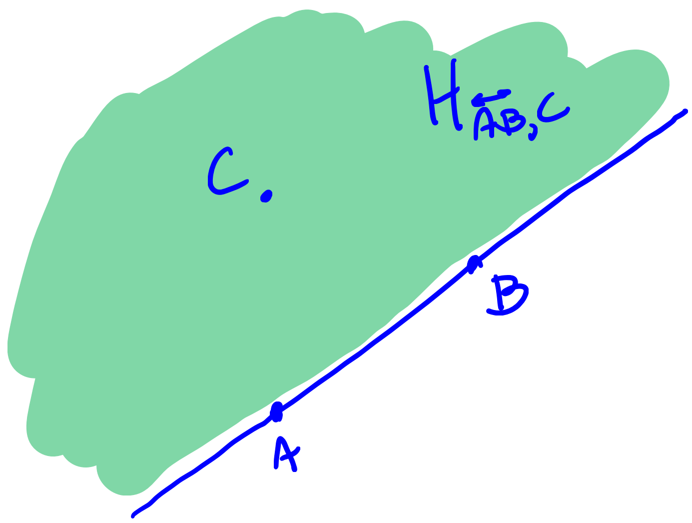
Interiors of Angles and Triangles

Definition of More Descriptive Half Plane Notation

Symbol: $H_{\overleftrightarrow{AB},C}$

Usage: A, B, C are non-collinear points in a Pasch geometry.

Meaning: The half plane of line A, B that contains C.



Definition of Angle and Triangle Interiors

Symbol: $\mathrm{int}(\angle ABC)$

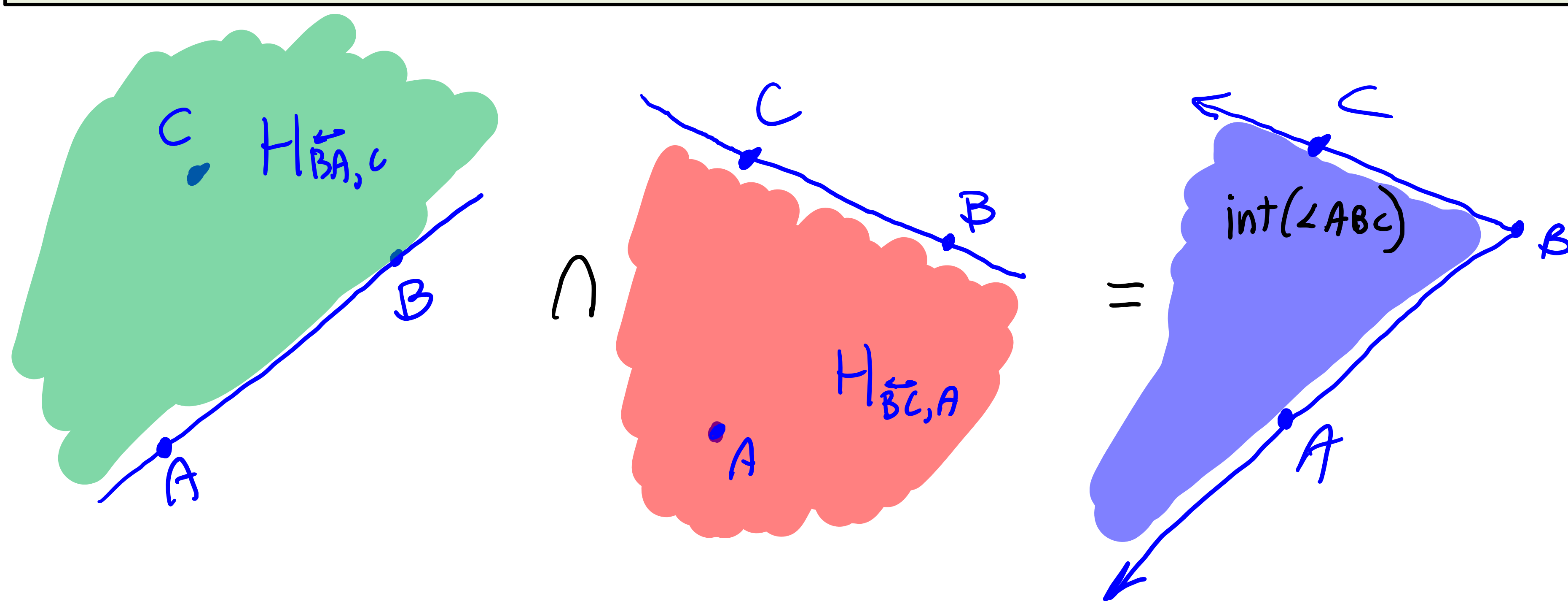
Spoken: *the interior of angle* A, B, C

Meaning: $H_{\overleftrightarrow{BA},C} \cap H_{\overleftrightarrow{BC},A}$

Symbol: $\mathrm{int}(\Delta ABC)$

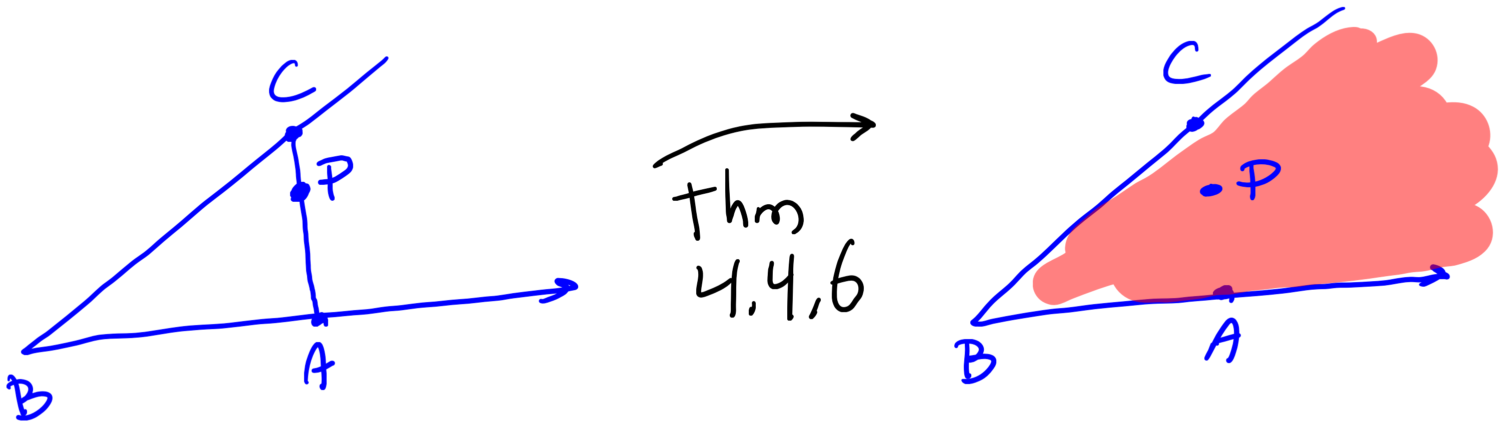
Spoken: *the interior of triangle* A, B, C

Meaning: $H_{\overleftrightarrow{AB},C} \cap H_{\overleftrightarrow{BC},A} \cap H_{\overleftrightarrow{CA},B}$



Theorem 4.4.6 Given $\angle ABC$ in a Pasch geometry, if $A - P - C$, then $P \in \text{int}(\angle ABC)$.

Illustration of the Statement of the Theorem

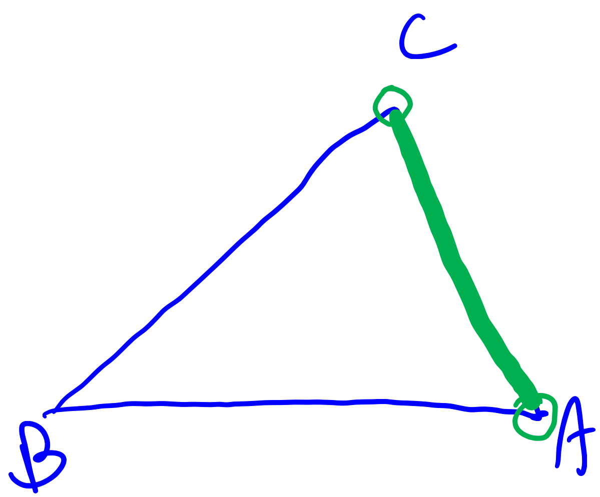


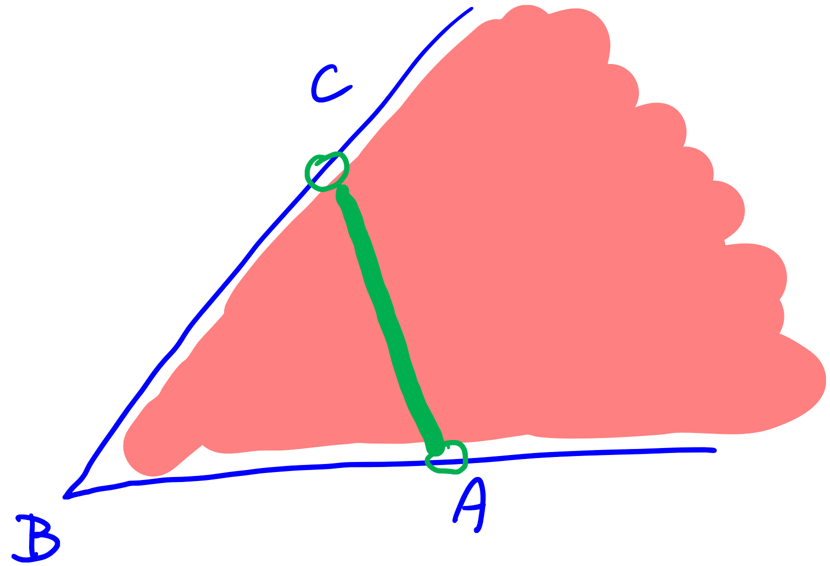
You'll justify and illustrate the steps in a given proof of Theorem 4.4.6 in your homework.

Observe this immediate consequence (corollary) of Theorem 4.4.6

In a Pasch geometry, all points in the interior of one side of a triangle are in the interior of the opposite angle. That is, in any triangle ΔABC the following subset relationship is true

$$\mathrm{int}(\overline{AC}) \subset \mathrm{int}(\angle ABC)$$



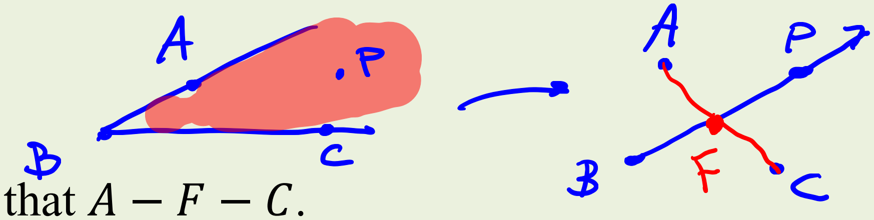


The Crossbar Theorem

Theorem 4.4.7 (The Crossbar Theorem)

In a Pasch geometry, if $P \in \text{int}(\angle ABC)$,

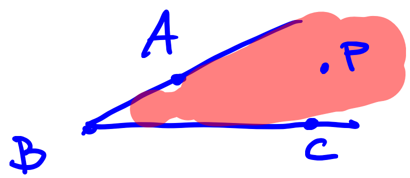
then $\overrightarrow{BP}$ intersects $\overline{AC}$ at a unique point F such that $A - F - C$.



Proof

Part 1: Introduce point *E* and use Pasch's Postulate

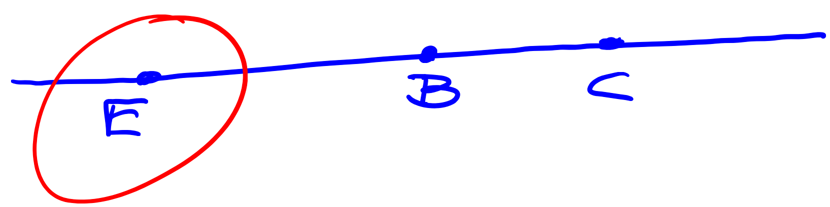
(1) In a Pasch geometry, suppose that $P \in \text{int}(\angle ABC)$. **(Illustrate)**



(2) There exists a point E such that $E - B - C$. **(Justify) (Illustrate)**

By Theorem 3.2.6(i) applied

to given points C, B.



(3) Pasch's Postulate is satisfied. **(Justify)**

By (1) and Theorem 4.3.1 that says if a metric geometry satisfies PSA then it satisfies PP.

Remark: Theorem 4.3.3 that says if a metric geometry satisfies PP then it satisfies PSA.

I incorporated Theorems 4.3.1 and 4.3.3 into the theorem that I called

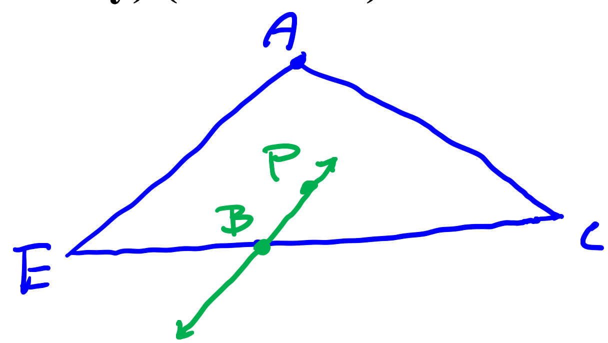
"Theorem About Two Equivalent Statements in a Metric Geometry"

(4) line $\overleftrightarrow{BP}$ intersects segment $\overline{AE}$ or segment $\overline{AC}$. **(Justify) (Illustrate)**

By Pasch's Postulate applied to

line $\overleftrightarrow{BP}$ that intersects ΔAEC

at point B such that $E - B - C$.

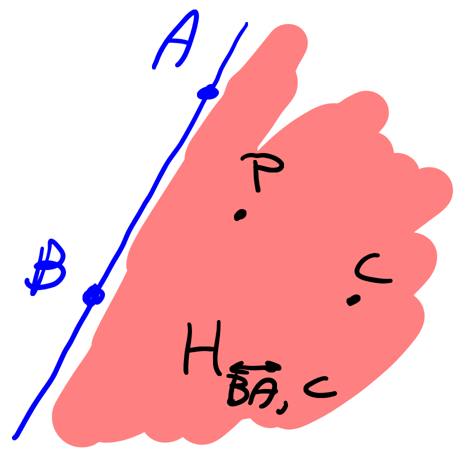


Part 2: Show that line $\overleftrightarrow{BP}$ does not intersect segment $\overline{AE}$.

(5) P and C are on the same side of $\overleftrightarrow{AB}$. **(Justify) (Illustrate)**

Because $P \underset{by\,(1)}{\in} \text{int}(\angle ABC) \underset{definition}{=} H_{\overleftrightarrow{BA},C} \cap H_{\overleftrightarrow{BC},A}$

so $P \in H_{\overleftrightarrow{BA},C}$



(6) C and E are on opposite sides of $\overleftrightarrow{AB}$. **(Justify) (Illustrate)**

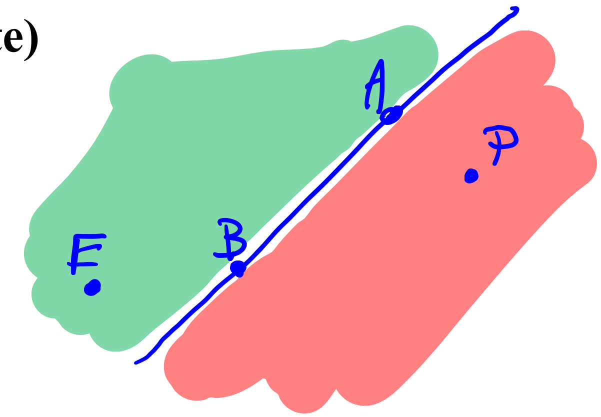
By (1),(4), we know $\overleftrightarrow{AB}$ intersects $\overline{EC}$

at point B such that $E - B - C$.

PSA (ii) (contrapositive) tells us that

C and E are not in the same half plane of $\overleftrightarrow{AB}$.

(7) Therefore, P and E are on opposite sides of $\overleftrightarrow{AB}$. **(Justify) (Illustrate)**



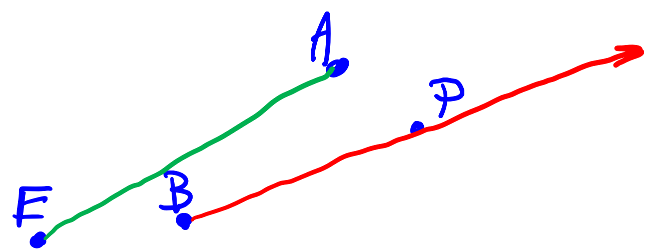
By (5),(6)

(8) $\overrightarrow{BP} \cap \overline{AE} = \phi$. **(Justify) (Illustrate)**

By Theorem 4.4.3 (the Z Theorem)

applied to points P, E on opposite

sides of $\overleftrightarrow{AB}$.

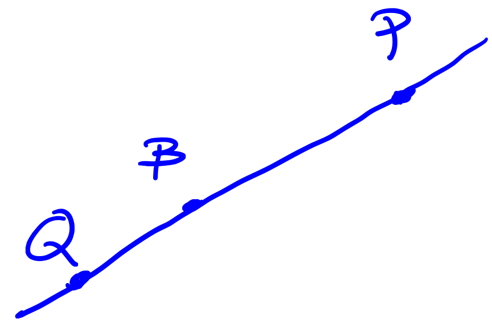


(9) There exists a point Q such that $P - B - Q$.

(justify) (illustrate)

By Theorem 3.2.6(i) applied

to given points P, B.



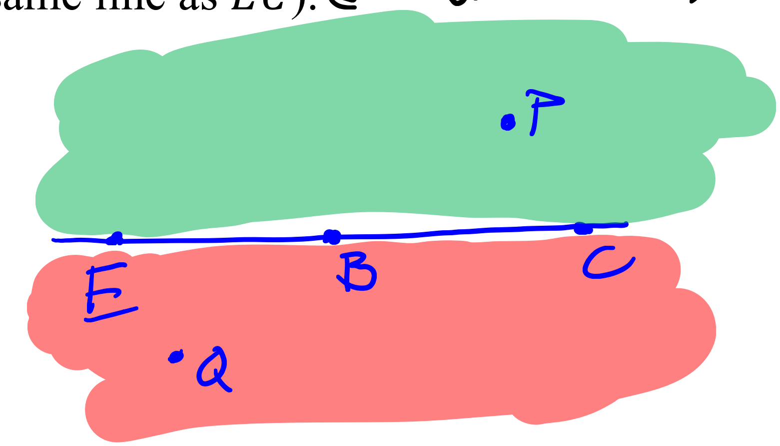
(10) Q and P are on opposite sides of $\overleftrightarrow{BC}$ (which is the same line as $\overleftrightarrow{EC}$). (justify)(illustrate)

By (1),(9), we know $\overleftrightarrow{BC}$ intersects $\overline{QP}$

at point B such that $P - B - Q$.

PSA (ii) (contrapositive) tells us that

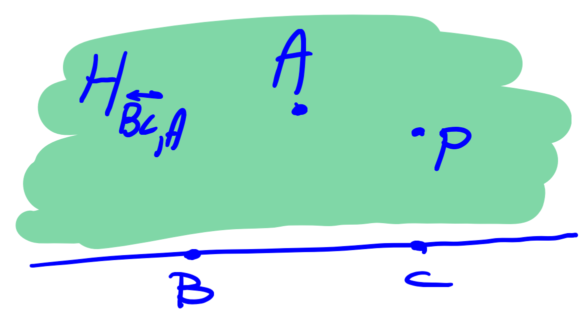
Q and P are not in the same half plane of $\overleftrightarrow{BC}$.



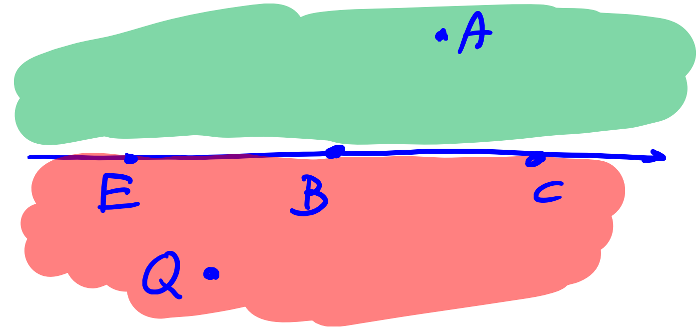
(11) P and A are on the same side of $\overleftrightarrow{BC}$. **(Justify) (Illustrate)**

Because $P \underset{by\,(1)}{\in} \text{int}(\angle ABC) \underset{definition}{=} H_{\overleftrightarrow{BA},C} \cap H_{\overleftrightarrow{BC},A}$

so $P \in H_{\overleftrightarrow{BC},A}$



(12) Q and A are on opposite sides of $\overleftrightarrow{BC}$ (which is the same line as $\overleftrightarrow{EC}$). **(Justify) (Illustrate)**



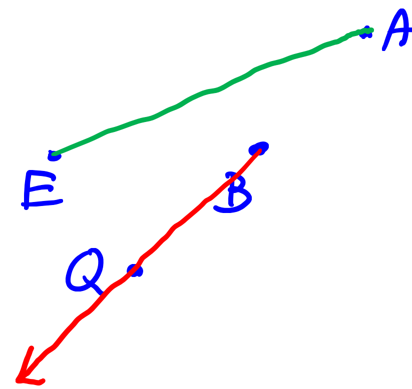
By (10),(11)

(13) $\overrightarrow{BQ} \cap \overline{AE} = \phi$. **(Justify) (Illustrate)**

By (12) and Theorem 4.4.3 (the Z Theorem)

applied to points $\boldsymbol{Q}, \boldsymbol{A}$ on opposite

sides of $\overleftrightarrow{\boldsymbol{BC}}$.

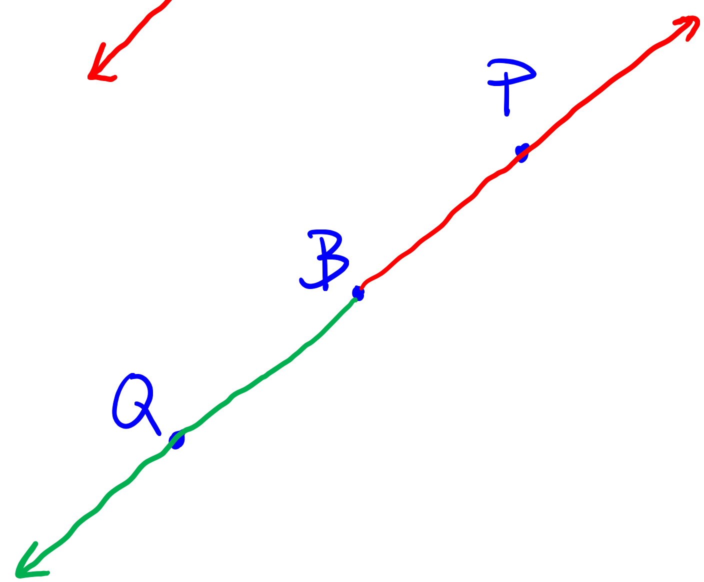


(14) $\overleftrightarrow{BP} = \overrightarrow{BP} \cap \overrightarrow{BQ}$ **(Justify) (Illustrate)**

In Exercise 3.3#14, you showed that

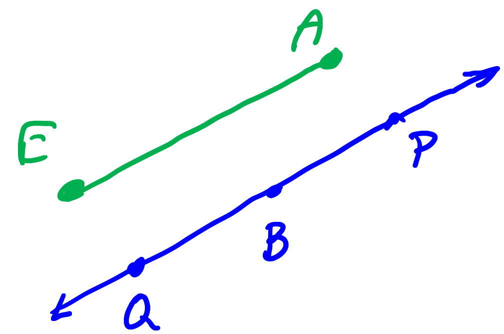
If $\boldsymbol{P} \in \overleftrightarrow{\boldsymbol{BQ}} - \overrightarrow{\boldsymbol{BQ}}$, then $\overleftrightarrow{\boldsymbol{BP}} = \overrightarrow{\boldsymbol{BP}} \cap \overrightarrow{\boldsymbol{BQ}}$

That is, if $\boldsymbol{P} - \boldsymbol{B} - \boldsymbol{Q}$, then $\overleftrightarrow{\boldsymbol{BP}} = \overrightarrow{\boldsymbol{BP}} \cap \overrightarrow{\boldsymbol{BQ}}$



(15) $\overleftrightarrow{BP} \cap \overline{AE} = \phi$. That is, line $\overleftrightarrow{BP}$ does not intersect segment $\overline{AE}$. **(Justify) (Illustrate)**

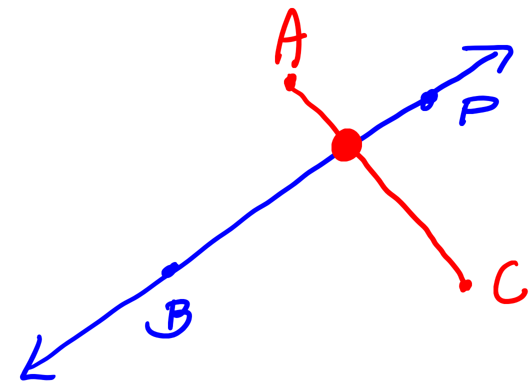
By (8),(13),(14)



Part 3: Prove that $\overleftrightarrow{BP}$ must intersect segment $\overline{AC}$.

(16) Line $\overleftrightarrow{BP}$ must intersect segment $\overline{AC}$. **(Justify)**

By (4),(15)

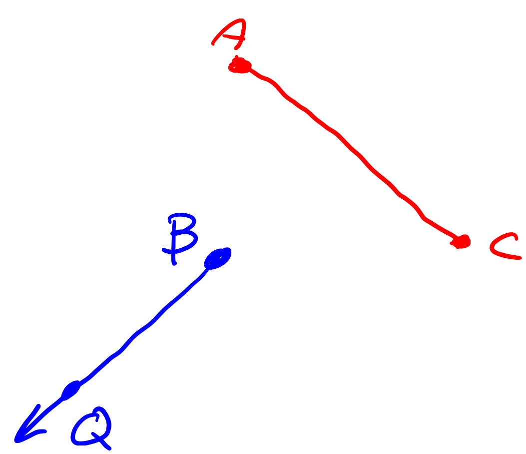


(17) $\overrightarrow{BQ} \cap \overline{AC} = \phi$. **(Justify) (Illustrate)**

By (12) and Theorem 4.4.3 (the Z Theorem)

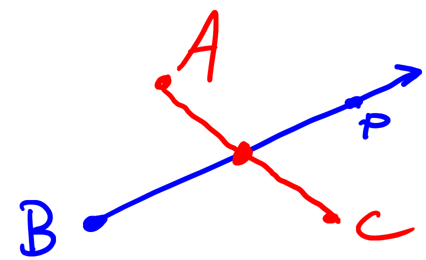
applied to points Q, A on opposite

sides of $\overleftrightarrow{BC}$.



(18) Therefore, $\overrightarrow{BP}$ must intersect segment $\overline{AC}$. **(Justify) (Illustrate)**

By (16),(14),(17)



Part 3: Prove property of the intersection of $\overrightarrow{BP}$ and $\overline{AC}$.

(19) The intersection of line $\overleftrightarrow{BP}$ and segment $\overline{AC}$ must just be a single point. Call the point F. **(Justify)**

By Corollary 2.1.7, two distinct lines cannot intersect in more than one point.

(20) $F \neq A$. **(Justify)**

We know that P is not on line $\overleftrightarrow{BA}$, because P $\in int(\angle ABC)$.

So line $\overleftrightarrow{BP}$ is not the same line as $\overleftrightarrow{BA}$. That is, they are distinct lines.

Lines $\overleftrightarrow{BP}$ and $\overleftrightarrow{BA}$ intersect at B.

By Corollary 2.1.7, two distinct lines cannot intersect in more than one point.

In other words, $\overleftrightarrow{BP}$ and $\overleftrightarrow{BA}$ cannot also intersect at A.

(21) $F \neq C$. **(Justify)**

By reasoning similar to the justification for (20).

(22) Therefore, $A - F - C$. **(Justify)**

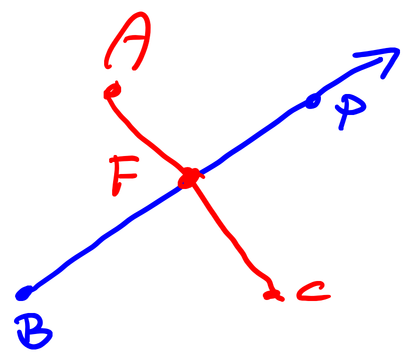
Because ray $\overrightarrow{BP}$ intersects segment $\overline{AC}$ at a single point F (by (19))

and F is not one of the endpoints (by (20),(21)),

the only remaining option is that $A - F - C$ (by definition of line segment).

Conclusion

(23) We have proven that $\overrightarrow{BP}$ intersects $\overline{AC}$ at a unique point F such that $A - F - C$. **(Illustrate)**



End of Proof

The Converse of the Crossbar Theorem

Recall that a *conditional statement*: *If A then B*

is logically equivalent to its *contrapositive: If NOT(B) then NOT(A).*

As a result of this, any time one proves a theorem that has the form of a conditional statement, one knows that the contrapositive version of the same statement is automatically true. The contrapositive statement is not another theorem: it is just a different way of saying the theorem that has already been proven.

But the original statement is *not* logically equivalent to its *converse*: *If B then A*.

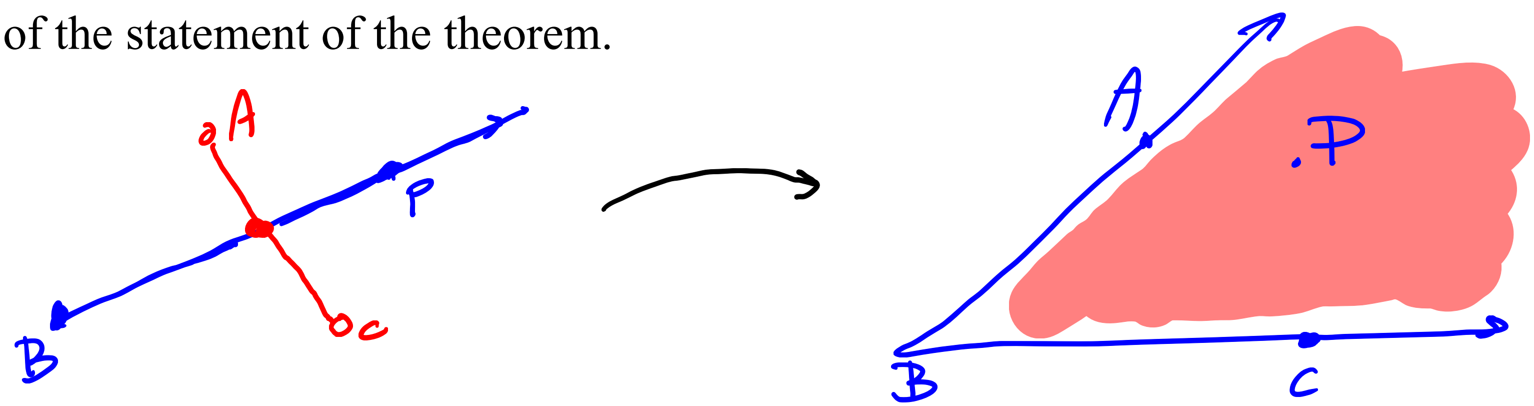
As a result of this, when a known theorem has the form of a conditional statement, the converse statement is *not* automatically true. (The converse statement is *not* just a different way of saying the theorem that has already been proven.) If the converse statement is true, then it constitutes another theorem, and it will have to be proven with a new proof.

That is the situation with the Crossbar Theorem. The Converse of the Statement of the Crossbar Theorem is a new theorem that has to be proven with a new proof.

Theorem (Converse of the Statement of the Crossbar Theorem)

Given $\angle ABC$ and point P in a Pasch Geometry, if $\overrightarrow{BP}$ intersects $\text{int}(\overline{AC})$, then $P \in \text{int}(\angle ABC)$.

Illustration of the statement of the theorem.



You will prove the theorem in *suggested exercise* 4.4#12.

Hint: In *assigned homework exercises* ~~H06 [3],[4]~~, you study and write proofs that are about proving that a point is in the interior of some angle, or that some set of points is a subset of the interior of some angle. The same kind of techniques used in those *two assigned homework exercises* will be useful for *suggested exercise* 4.4#12.

End of Video