Video 5.1: The Measure of an Angle

produced by Mark Barsamian, 2021.03.18

for Ohio University MATH 3110/5110 College Geometry

Topics

- **Angle Measure**
- **Protractor Geometry**
- **Euclidean Angle Measure**
- **Measure of a Poincaré Angle**

Reading: Section 4.5: The Measure of an Angle, p 90 – 96

in *Geometry: A Metric Approach with Models, Second Edition* by Millman & Parker

Homework: Section 5.1 # 1, 2, 3, 7

Angle Measure

Definition: The symbol $\mathcal{A}$ denotes the set of all angles in a Pasch Geometry

Definition of Angle Measure

Words: Angle measure (or protractor) based on r_0

Usage: There is a Pasch geometry in the discussion, and r_0 is a fixed positive real number

Meaning: a function $m: \mathcal{A} \to \mathbb{R}$ that has these three properties (the *Axioms of Angle Measure*)

(i) $0 < m(\angle ABC) < r_0$

(ii) (This statement is called the *Angle Constuction Axiom*)

Given

- a half plane H
- a ray $\overrightarrow{BC}$ on the edge of that half plane
- a number θ such that $0 < \theta < r_0$

There exists a unique ray $\overrightarrow{BA}$ with $A \in H$ such that $m(\angle ABC) = \theta$.

(iii) (This statement is called the *Angle Addition Axiom*)

Angle measure is “additive” in the following sense:

If $D \in int(\angle ABC)$, then $m(\angle ABD) + m(\angle DBC) = m(\angle ABC)$

The three *Axioms of Angle Measure* are a "wish list" of properties that a function must have in order to be qualified to be called an "angle measure".

Illustration of the *Angle Construction Axiom*

Given

- a half plane H
- a ray $\overrightarrow{BC}$ on the edge of that half plane
- a number θ such that $0 < \theta < r_0$

There exists a unique ray $\overrightarrow{BA}$ with $A \in H$ such that $m(\angle ABC) = \theta$.

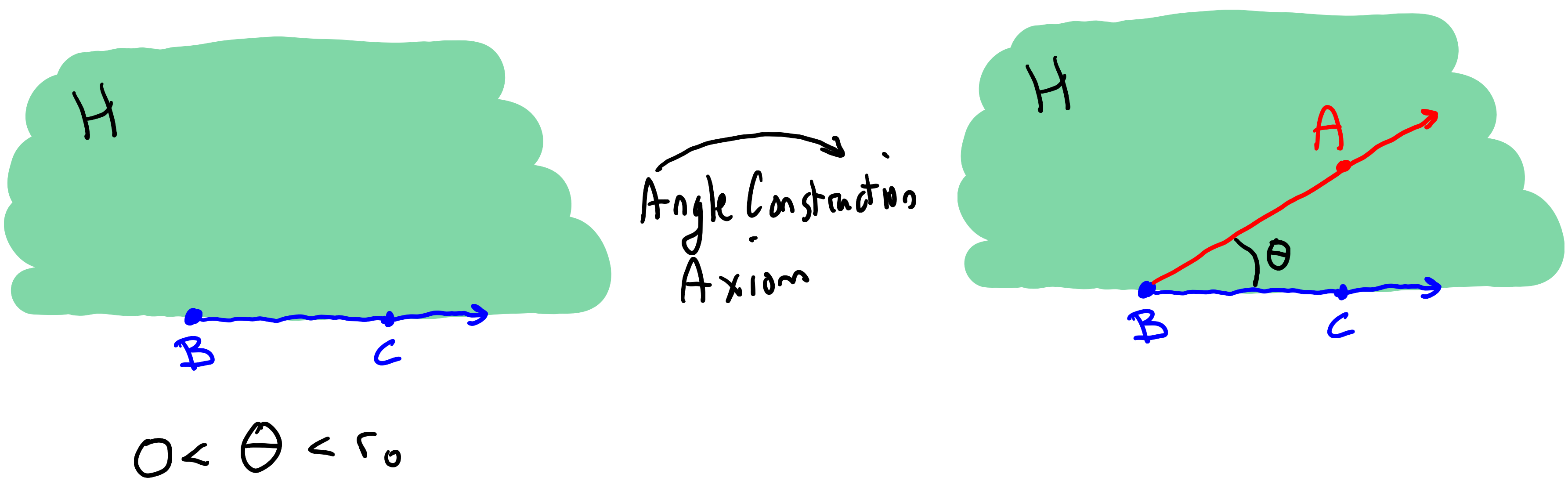
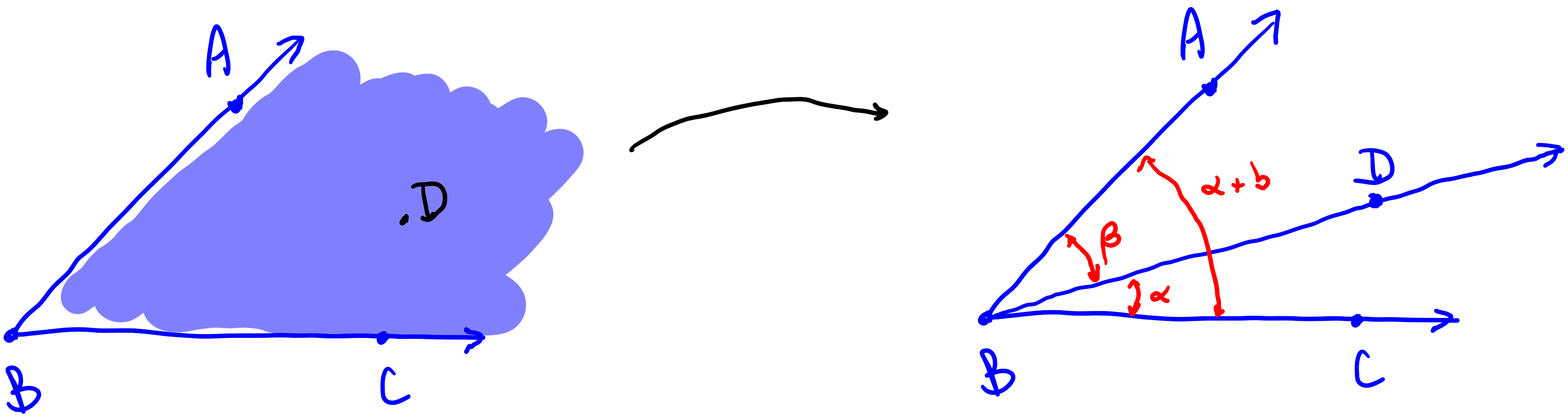


Illustration of the *Angle Addition Axiom*

If $D \in int(\angle ABC)$, then $m(\angle ABD) + m(\angle DBC) = m(\angle ABC)$



What's up with r_0?

Most commonly we use $r_0 = 180$. In that case, $0 < m(\angle ABC) < 180$

(note: no straight angles!)

When $r_0 = 180$, the angle measures are sometimes labeled as *degrees*.

So if $m(\angle ABC) = 2$, we could say $m(\angle ABC) = 2\ degrees$ or $m(\angle ABC) = 2°$

Also common to use $r_0 = \pi$. In that case, $0 < m(\angle ABC) < \pi$

When $r_0 = \pi$, the angle measures are sometimes labeled as *radians*.

So if $m(\angle ABC) = 2$, we could say $m(\angle ABC) = 2\ radians$.

A more obscure choice is to use $r_0 = 200$. In that case, $0 < m(\angle ABC) < 200$

When $r_0 = 200$, the angle measures are sometimes labeled as *gradians*.

So if $m(\angle ABC) = 2$, we could say $m(\angle ABC) = 2\ gradians$.

Remark: Calculators offer *Degrees*, *Radians*, *Gradians* with a *DRG* key

In our book, from now on, we will only use $r_0 = 180$. Since that is the only value of r_0 that we will ever use, we won't bother saying "degrees". We'll just say $m(\angle ABC) = 2$, for example, not $m(\angle ABC) = 2^\circ$.

Protractor Geometry

Definition: A **Protractor Geometry** is an ordered quadruple $(\mathcal{P}, \mathcal{L}, d, m)$ such that the ordered triple $(\mathcal{P}, \mathcal{L}, d)$ is a *Pasch Geometry* and m is an *angle measure* for $(\mathcal{P}, \mathcal{L}, d)$.

In today's video, we will be discussing two examples of Protractor Geometry:

the *Euclidean plane* the *Poincaré plane*.

Euclidean Angle Measure

Definition of Euclidean Angle Measure

Words: Euclidean angle measure

Meaning: The function $m_E\colon \mathcal{A} \to \mathbb{R}$ (where $\mathcal{A}$ is the set of angles in $\mathbb{R}^2$) defined by

$$m_E(\angle ABC) = \cos^{-1}\left(\frac{\langle A-B, C-B\rangle}{\|A-B\|\|C-B\|}\right)$$

Remarks:

- Using *Degrees* for the $\cos^{-1}$, because we are always using $r_0 = 180$ in our course.
- The expression $A - B$ is the *Euclidean Vector from B to A*.
- Fact (Prop 5.1.2): This function does have properties i,ii,iii set forth in the *Definition of Angle Measure*, so it is qualified to be called an *angle measure* for the *Euclidean plane* $(\mathbb{R}^2, \mathcal{L}_E, d_E)$. In other words, $(\mathbb{R}^2, \mathcal{L}_E, d_E, m_E)$ is a *protractor geometry*. It is also called the *Euclidean plane*.
- Fact (Exercise 5.1#5) The function m_E is also qualified to be called an *angle measure* for the *Taxicab plane* $(\mathbb{R}^2, \mathcal{L}_E, d_T)$. In other words, $(\mathbb{R}^2, \mathcal{L}_E, d_T, m_E)$ is a *protractor geometry*. It is also called the *Taxicab plane*.

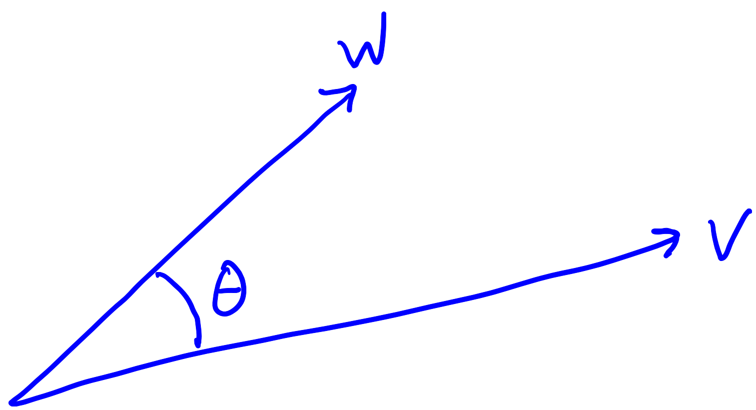
Remark on connection to dot product

$$V \cdot W = \|V\|\|W\| \cos(\theta) \quad \text{using dot product notation}$$

$$\frac{V \cdot W}{\|V\|\|W\|} = \cos(\theta) \quad \text{here, we assume that both vectors are non-zero}$$

$$\frac{\langle V, W \rangle}{\|V\|\|W\|} = \cos(\theta) \quad \text{using angle bracket notation instead of dot product}$$

$$\cos^{-1}\left(\frac{\langle V, W \rangle}{\|V\|\|W\|}\right) = \theta$$



[Example 1] Let $A = (0,1), B = (0,5), C = (3,4)$

(a) Compute the *Euclidean angle measure of* $\angle ABC$.

Give an *exact, simplified answer* and a *decimal approximation*, rounded to 2 places.

Show all steps clearly.

Solution:

Compute Euclidean Vector from B to A:

$$A - B = (0,1) - (0,5) = (0,-4)$$

Compute Norm of the Euclidean Vector from B to A:

$$\|A - B\| = \|(0,-4)\| = \sqrt{0^2 + (-4)^2} = 4$$

Compute Euclidean Vector from B to C:

$$C - B = (3,4) - (0,5) = (3,-1)$$

Compute Norm of the Euclidean Vector from B to C:

$$\|C - B\| = \|(3,-1)\| = \sqrt{3^2 + (-1)^2} = \sqrt{10}$$

Compute Inner Product of the Two Euclidean Vectors at B:

$$\langle A - B, C - B \rangle = \langle (0,-4), (3,-1) \rangle = (0)(3) + (-4)(-1) = 4$$

Divide the Inner Product by the Product of the Two Norms:

$$\frac{\langle A-B, C-B\rangle}{\|A-B\|\|C-B\|} = \frac{4}{(4)(\sqrt{10})} = \frac{1}{\sqrt{10}}$$

Compute Euclidean Angle Measure of $\angle ABC$:

$$m_E(\angle ABC) = \cos^{-1}\left(\frac{\langle A-B, C-B\rangle}{\|A-B\|\|C-B\|}\right) = \underbrace{\cos^{-1}\left(\frac{1}{\sqrt{10}}\right)}_{exact} \approx \underbrace{71.57}_{approx}$$

(b) Compute the *Euclidean angle measure of* $\angle BCA$.

Give an *exact, simplified answer* and a *decimal approximation*, rounded to 2 places.

Show all steps clearly.

Solution:

Compute Euclidean Vector from C to B:

$$B - C = (0,5) - (3,4) = (-3,1)$$

Compute Norm of the Euclidean Vector from C to B:

$$\|B - C\| = \|(-3,1)\| = \sqrt{(-3)^2 + 1^2} = \sqrt{10}$$

Compute Euclidean Vector from C to A:

$$A - C = (0,1) - (3,4) = (-3,-3)$$

Compute Norm of the Euclidean Vector from C to A:

$$\|A - C\| = \|(-3,-3)\| = \sqrt{(-3)^2 + (-3)^2} = 3\sqrt{2}$$

Compute Inner Product of the Two Euclidean Vectors at C:

$$\langle B - C, A - C \rangle = \langle (-3,1), (-3,-3) \rangle = (-3)(-3) + (1)(-3) = 6$$

Divide the Inner Product by the Product of the Two Norms:

$$\frac{\langle B - C, A - C \rangle}{\|B - C\|\|A - C\|} = \frac{6}{(\sqrt{10})(3\sqrt{2})} = \frac{2}{\sqrt{20}} = \frac{1}{\sqrt{5}}$$

Compute Euclidean Angle Measure of $\angle BCA$:

$$m_E(\angle BCA) = \cos^{-1}\left(\frac{\langle B - C, A - C \rangle}{\|B - C\|\|A - C\|}\right) = \underbrace{\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)}_{exact} \approx \underbrace{63.43}_{approx}$$

(C) Compute the *Euclidean angle measure of* $\angle CAB$.

Give an *exact, simplified answer* and a *decimal approximation*, rounded to 2 places.

Show all steps clearly.

Solution:

Compute Euclidean Vector from A to C:

$$C - A = (3,4) - (0,1) = (3,3)$$

Compute Norm of the Euclidean Vector from A to C:

$$\|C-A\| = \|(3,3)\| = \sqrt{3^2+3^2} = 3\sqrt{2}$$

Compute Euclidean Vector from A to B:

$$B - A = (0,5) - (0,1) = (0,4)$$

Compute Norm of the Euclidean Vector from A to B:

$$\|B-A\| = \|(0,4)\| = \sqrt{0^2+4^2} = 4$$

Compute Inner Product of the Two Euclidean Vectors at A:

$$\langle C-A, B-A\rangle = \langle (3,3),(0,4)\rangle = (3)(3)+(3)(4) = 12$$

Divide the Inner Product by the Product of the Two Norms:

$$\frac{\langle C-A, B-A\rangle}{\|C-A\|\|B-A\|} = \frac{12}{(3\sqrt{2})(4)} = \frac{1}{\sqrt{2}}$$

Compute Euclidean Angle Measure of $\angle ABC$:

$$m_E(\angle CAB) = \cos^{-1}\left(\frac{\langle C-A, B-A\rangle}{\|C-A\|\|B-A\|}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \underbrace{45}_{exact}$$

(D) Compute the *Approximate Angle Sum* of *Euclidean triangle* ΔABC.

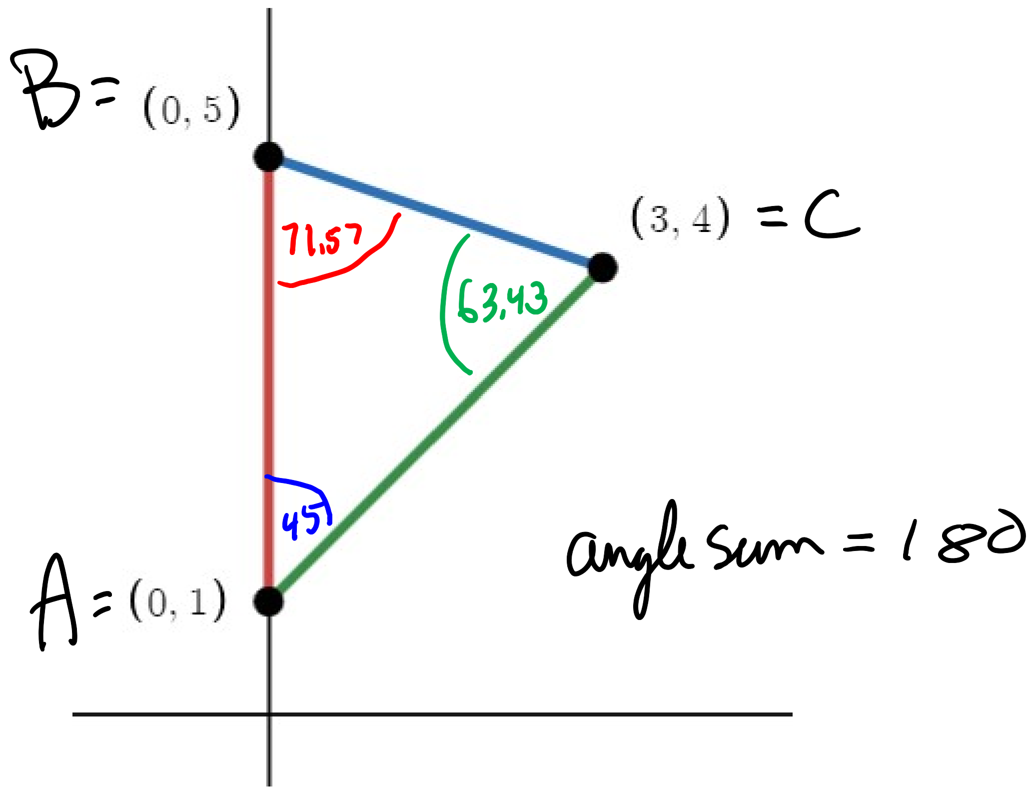
Give a *decimal approximation*, rounded to 2 places.

Solution:

$$\begin{aligned} \text{Sum}_E(\Delta ABC) &= m_E(\angle ABC) + m_E(\angle BCA) + m_E(\angle CAB) \\ &\approx 71.57 + 63.43 + 45 \\ &= 180.00 \end{aligned}$$

(E) Illustrate your results from the previous parts.

Solution:

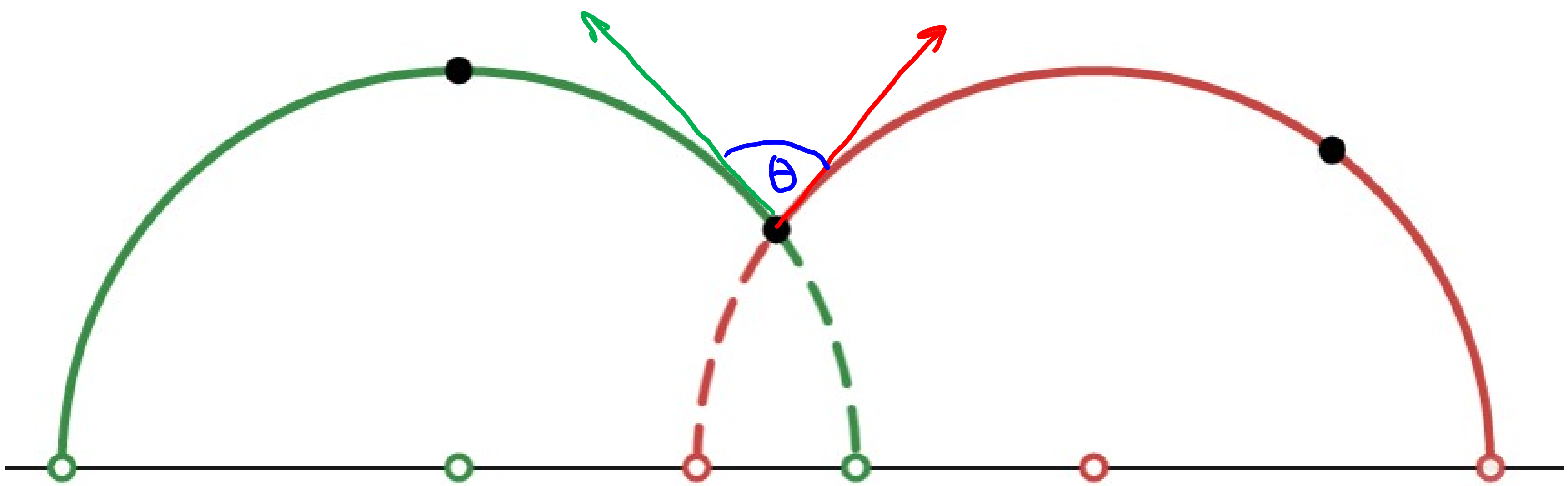


End of [Example 1]

Measure of a Poincaré Angle

We have a way to measure angles in the *Euclidean plane*. What do we do in the *Poincaré plane*?

The method is to find the *Euclidean Tangent Vector* to the *circular arcs* at the point where they intersect, then measure the angle between those *tangent vectors* using *Euclidean angle measure*.



The book presents this definition of the Euclidean Tangent to a Poincaré Ray

Definition of the Euclidean Tangent to a Poincaré Ray

If $P = (x_P, y_P) \in \mathbb{H}$ and $Q = (x_Q, y_Q) \in \mathbb{H}$

then the **Euclidean Tangent to Poincaré Ray $\overrightarrow{PQ}$** is defined to be the vector

$$T_{PQ} = \begin{cases} (0, y_Q - y_P) & if\ \overleftrightarrow{PQ}\ is\ a\ type\ I\ line \\ (y_P, c - x_P) & if\ \overleftrightarrow{PQ}\ is\ a\ type\ II\ line\ \ {}_cL_r\ and\ x_P < x_Q \\ -(y_P, c - x_P) & if\ \overleftrightarrow{PQ}\ is\ a\ type\ II\ line\ \ {}_cL_r\ and\ x_P > x_Q \end{cases}$$

The presentation of this definition is very concise. It will be helpful to have a clear procedure for computing T_{PQ} in practice

Procedure for Computing T_{PQ}, the Euclidean Tangent to a Poincaré Ray $\overrightarrow{PQ}$

If $x_P = x_Q$, then $\overleftrightarrow{PQ}$ is a *type I line* ${}_{x_P}L$ and $T_{PQ} = (0, y_Q - y_P)$

If $x_P < x_Q$, then $\overleftrightarrow{PQ}$ is a *type II line* ${}_cL_r$. Find c, r and then build $T_{PQ} = (y_P, c - x_P)$

If $x_P > x_Q$, then $\overleftrightarrow{PQ}$ is a *type II line* ${}_cL_r$. Find c, r and then build $T_{PQ} = -(y_P, c - x_P)$

Observe that in the case that $\overleftrightarrow{PQ}$ is a *type II line* ${}_cL_r$, the computation of T_{PQ} only needs the value of c for the line. The value of r is not used. But I feel that it is important to find both c and r for two reasons:

- When making an illustration of $\overrightarrow{PQ}$, it is important to label the line ${}_cL_r$.
- To double-check my results, I plot P and Q and line ${}_cL_r$ in Desmos.

Recall this Procedure from Section 2.1 for finding Poincaré lines:

Procedure for Finding the *Poincaré Line* Passing Through Two Distinct Points in $\mathbb{H}$

Suppose $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are any two distinct points of $\mathbb{H}$.

If $x_1 = x_2$ then let $a = x_1 = x_2$. In this case, ${}_aL \in \mathcal{L}_H$ and $P, Q \in {}_aL$.

If $x_1 \neq x_2$ then define constants c, r by the following formulas:

$$c = \frac{x_2^2 - x_1^2 + y_2^2 - y_1^2}{2(x_2 - x_1)}$$

$$r = \sqrt{(x_1 - c)^2 + y_1^2}$$

In this case, ${}_cL_r \in \mathcal{L}_H$ and $P, Q \in {}_cL_r$.

Definition of Measure of a Poincaré Angle

Words: the Poincaré angle measure

Meaning: The function ~~m_E~~ m_H: $\mathcal{A} \to \mathbb{R}$ (where $\mathcal{A}$ denotes the set of angles in $\mathbb{H}$)defined by

$$m_H(\angle ABC) = \cos^{-1}\left(\frac{\langle T_{BA}, T_{BC}\rangle}{\|T_{BA}\|\|T_{BC}\|}\right)$$

Remarks:

- Observe that the expression on the right side of the equal sign is the Euclidean angle measure of the Euclidean angle formed by the Euclidean tangent Vectors T_{BA} and T_{BC}.
- Using *Degrees* for the $\cos^{-1}$, because we are always using $r_0 = 180$ in our course.
- The expression T_{BA} is the *Euclidean Tangent to Poincaré ray* $\overrightarrow{BA}$ *at* B.
- Fact (Prop 5.1.4): This function does have properties i,ii,iii set forth in the *Definition of Angle Measure*, so it is qualified to be called an *angle measure* the *Poincaré plane* $(\mathbb{H}, \mathcal{L}_H, d_H)$. In other words, $(\mathbb{H}, \mathcal{L}_H, d_H, m_H)$ is a *protractor geometry*. It is also called the *Poincaré plane*.

[Example 2] Let $A = (0,1), B = (0,5), C = (3,4)$

(a) Compute the *Measure of Poincaré angle* $\angle ABC$.

Give an *exact, simplified answer* and a *decimal approximation*, rounded to 2 places.

Show all steps clearly.

Solution:

Compute Poincaré Line $\overleftrightarrow{BA}$

Since $x_A = x_B = 0$, *Poincaré Line* $\overleftrightarrow{BA}$ will be the *type I line* ${}_0L$

Compute Euclidean Vector Tangent to Poincaré Ray $\overrightarrow{BA}$: at point B

Since $x_A = x_B = 0$, the Euclidean Tangent Vector will be

$$T_{BA} = (0, y_A - y_B) = (0,1 - 5) = (0, -4)$$

Compute Norm of the Euclidean Vector Tangent to Poincaré Ray $\overrightarrow{BA}$:

$$\|T_{BA}\| = \|(0, -4)\| = \sqrt{0^2 + (-4)^2} = 4$$

Compute Poincaré Line $\overleftrightarrow{BC}$

Since $x_B \neq x_C$ *Poincaré Line* $\overleftrightarrow{BC}$ will be a *type II line*

$$c = \frac{x_C^2 - x_B^2 + y_C^2 - y_B^2}{2(x_C - x_B)} = \frac{3^2 - 0^2 + 4^2 - 5^2}{2(3 - 0)} = 0$$

$$r = \sqrt{(x_B - c)^2 + y_B^2} = \sqrt{(0-0)^2 + 5^2} = 5$$

So Poincaré line $\overleftrightarrow{BC}$ is the *type II line* ${}_0L_5$.

Compute Euclidean Vector Tangent to Poincaré Ray $\overrightarrow{BC}$: at point B

Since $x_B < x_C$, the Euclidean Tangent will be

$$T_{BC} = (y_B, c - x_B) = (5, 0 - 0) = (5,0)$$

Compute Norm of the Euclidean Vector Tangent to Poincaré Ray $\overrightarrow{BC}$:

$$\|T_{BC}\| = \|(5,0)\| = \sqrt{5^2 + 0^2} = 5$$

Compute Inner Product of the Two Euclidean Tangent Vectors at B:

$$\langle T_{BA}, T_{BC} \rangle = \langle (0,-4), (5,0) \rangle = (0)(5) + (-4)(0) = 0$$

Divide the Inner Product by the Product of the Two Norms

$$\frac{\langle T_{BA}, T_{BC} \rangle}{\|T_{BA}\| \|T_{BC}\|} = \frac{0}{(4)(5)} = 0$$

Compute Measure of Poincaré Angle $\angle ABC$:

$$m_H(\angle ABC) = \cos^{-1}\left(\frac{\langle T_{BA}, T_{BC} \rangle}{\|T_{BA}\| \|T_{BC}\|}\right) = \cos^{-1}(0) = 90$$

exact answer

(b) Compute the *Measure of Poincaré angle* $\angle BCA$.

Give an *exact, simplified answer* and a *decimal approximation*, rounded to 2 places.

Show all steps clearly.

Solution:

Compute Poincaré Line $\overleftrightarrow{CB}$

In part (a), we found that Poincaré line $\overleftrightarrow{BC}$ is the *type II line* ${}_0L_5$.

Compute Euclidean Vector Tangent to Poincaré Ray $\overrightarrow{CB}$: at point C

Since $x_C > x_B$, the Euclidean Tangent will be

$$T_{CB} = -(y_C, c - x_C) = -(4, 0 - 3) = (-4, 3)$$

Compute Norm of the Euclidean Vector Tangent to Poincaré Ray $\overrightarrow{CB}$:

$$\|T_{CB}\| = \|(-4, 2)\| = \sqrt{(-4)^2 + 3^2} = 5$$

Compute Poincaré Line $\overleftrightarrow{CA}$

Since $x_C \neq x_A$ *Poincaré Line* $\overleftrightarrow{A}$ ($\overleftrightarrow{CA}$) will be a *type II line*

$$c = \frac{x_A^2 - x_C^2 + y_A^2 - y_C^2}{2(x_A - x_C)} = \frac{0^2 - 3^2 + 1^2 - 4^2}{2(0 - 3)} = 4$$

$$r = \sqrt{(x_C - c)^2 + y_C^2} = \sqrt{(3 - 4)^2 + 4^2} = \sqrt{17}$$

So Poincaré line $\overleftrightarrow{CA}$ is the *type II line* ${}_4L_{\sqrt{17}}$.

Compute Euclidean Vector Tangent to Poincaré Ray $\overrightarrow{CA}$:

Since $x_C > x_A$, the Euclidean Tangent will be

$$T_{CA} = -(y_C, c - x_C) = -(4, 4 - 3) = (-4, -1)$$

Compute Norm of the Euclidean Vector Tangent to Poincaré Ray $\overrightarrow{CA}$:

$$\|T_{CA}\| = \|(-4, -1)\| = \sqrt{(-4)^2 + (-1)^2} = \sqrt{17}$$

Compute Inner Product of the Two Euclidean Tangent Vectors at C:

$$\langle T_{CB}, T_{CA} \rangle = \langle (-4, 3), (-4, -1) \rangle = (-4)(-4) + (3)(-1) = 13$$

Divide the Inner Product by the Product of the Two Norms

$$\frac{\langle T_{CB}, T_{CA} \rangle}{\|T_{CB}\| \|T_{CA}\|} = \frac{13}{(5)\left(\sqrt{17}\right)} = \frac{13}{5\sqrt{17}}$$

Compute Measure of Poincaré Angle $\angle BCA$:

$$m_H(\angle BCA) = \cos^{-1}\left(\frac{\langle T_{CB}, T_{CA} \rangle}{\|T_{CB}\| \|T_{CA}\|}\right) = \underbrace{\cos^{-1}\left(\frac{13}{5\sqrt{17}}\right)}_{exact} \approx \underbrace{50.91}_{approx}$$

(c) Compute the *Measure of Poincaré angle* $\angle CAB$.

Give an *exact, simplified answer* and a *decimal approximation*, rounded to 2 places.

Show all steps clearly.

Solution:

Compute Poincaré Line $\overleftrightarrow{AC}$

In part (b), we found that Poincaré line $\overleftrightarrow{AC}$ is the *type II line* ${}_4L_{\sqrt{17}}$.

Compute Euclidean Vector Tangent to Poincaré Ray $\overrightarrow{AC}$:

Since $x_A < x_C$, the Euclidean Tangent will be

$$T_{AC} = (y_A, c - x_A) = (1, 4 - 0) = (1,4)$$

Compute Norm of the Euclidean Vector Tangent to Poincaré Ray $\overrightarrow{AC}$:

$$\|T_{AC}\| = \|(1,4)\| = \sqrt{1^2 + 4^2} = \sqrt{17}$$

Compute Poincaré Line $\overleftrightarrow{AB}$

In part (a), we found that Poincaré line $\overleftrightarrow{AB}$ is the *type I line* ${}_0L$.

Compute Euclidean Vector Tangent to Poincaré Ray $\overrightarrow{AB}$:

Since $x_A = x_B = 0$, the Euclidean Tangent Vector will be

$$T_{AB} = (0, y_B - y_A) = (0, 5 - 1) = (0,4)$$

Compute Norm of the Euclidean Vector Tangent to Poincaré Ray $\overrightarrow{AB}$:

$$\|T_{AB}\| = \|(0,4)\| = \sqrt{0^2 + 4^2} = 4$$

Compute Inner Product of the Two Euclidean Tangent Vectors at A:

$$\langle T_{AC}, T_{AB} \rangle = \langle (1,4), (0,4) \rangle = (1)(0) + (4)(4) = 16$$

Divide the Inner Product by the Product of the Two Norms

$$\frac{\langle T_{AC}, T_{AB} \rangle}{\|T_{AC}\| \|T_{AB}\|} = \frac{16}{(\sqrt{17})(4)} = \frac{4}{\sqrt{17}}$$

Compute Measure of Poincaré Angle $\angle CAB$:

$$m_H(\angle CAB) = \cos^{-1}\left(\frac{\langle T_{AC}, T_{AB} \rangle}{\|T_{AC}\| \|T_{AB}\|}\right) = \underbrace{\cos^{-1}\left(\frac{4}{\sqrt{17}}\right)}_{exact} \approx \underbrace{14.04}_{approx}$$

(d) Compute the *Approximate Angle Sum* of *Poincaré triangle* ΔABC.

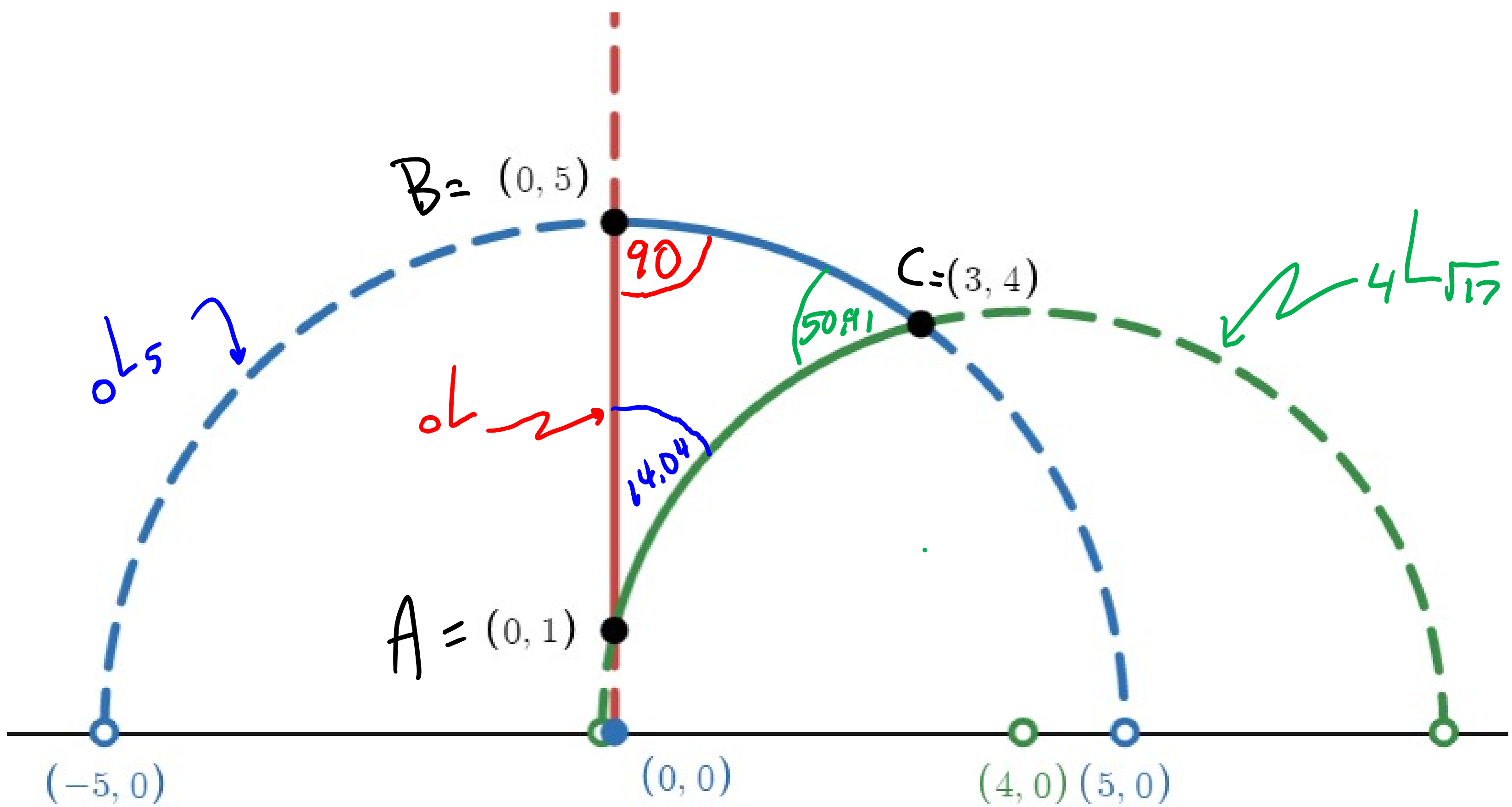
Give a *decimal approximation*, rounded to 2 places.

Solution:

$$\begin{aligned} \text{Sum}_H(\Delta ABC) &= m_H(\angle ABC) + m_H(\angle BCA) + m_H(\angle CAB) \\ &\approx 90 + 50.91 + 14.04 \\ &= 154.95 \end{aligned}$$

(e) Illustrate your results from the previous parts.

Solution:



End of [Example 2]

angle sum ≈ 154,95

It is interesting to compare the results of **[Examples 1,2]** since they used the same points.

	[Example 1] (Euclidean)	[Example 2] (Poincaré)
$m(\angle ABC)$	71.57	90
$m(\angle BCA)$	63.43	50.91
$m(\angle BCA)$	45	14.04
Triangle angle sum	180	154.95

Observe three things:

- Sometimes the Euclidean angle has a larger measure than the Poincaré angle. But sometimes it is the other way around.
- The Euclidean triangle angle sum is 180. This is a familiar result.
- The Poincaré triangle has angle sum less than 180. This is surprising.

End of Video