Video 5.3a: The Angle Addition Axiom and the Linear Pair Theorem

produced by Mark Barsamian, 2021.03.20

for Ohio University MATH 3110/5110 College Geometry

Topics

- **Statements about Points in the Interior of Angles and Big & Small Angle Measures**
- **The Linear Pair Theorem**
- **Converse Statements that are Also Theorems**

Reading: Section 5.3

in *Geometry: A Metric Approach with Models, Second Edition* by Millman & Parker

Homework: See Course Web Page

Recall Useful Theorems from Section 4.4

Theorem 4.4.8

In a Pasch Geometry, if C, P are on the same side of line $\overleftrightarrow{AB}$,

then $P \in \text{int}(\angle ABC)$ if and only if A, C are on opposite sides of $\overleftrightarrow{BP}$.

Alternate wording:

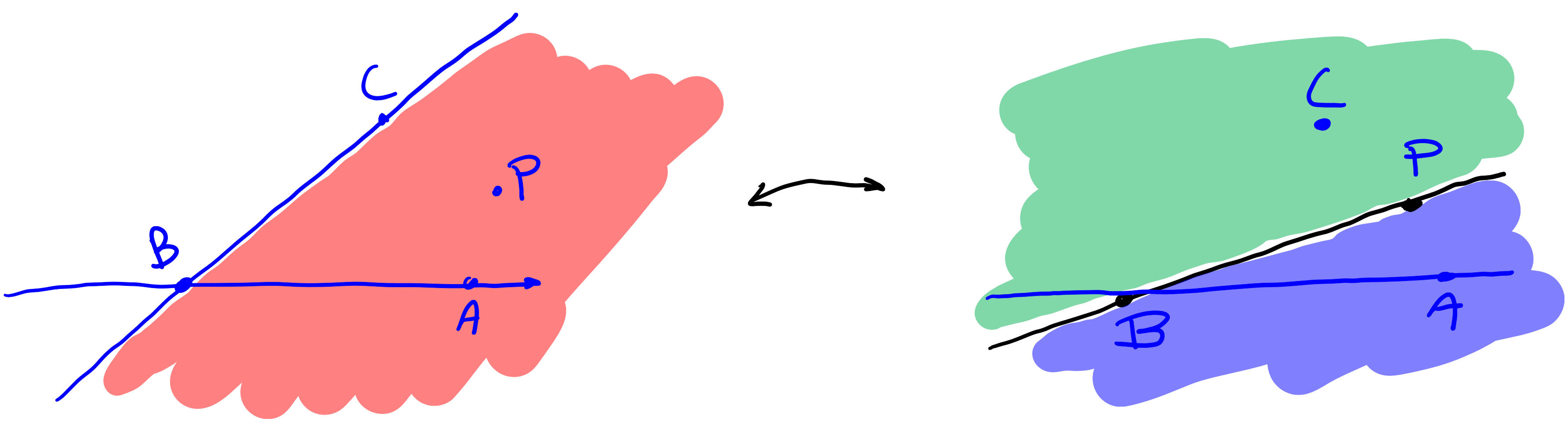
Given points C, P on the same side of a line $\overleftrightarrow{AB}$ in a Pasch geometry,

the following are equivalent (TFAE)

(i) P, A are on the same side of line $\overleftrightarrow{BC}$

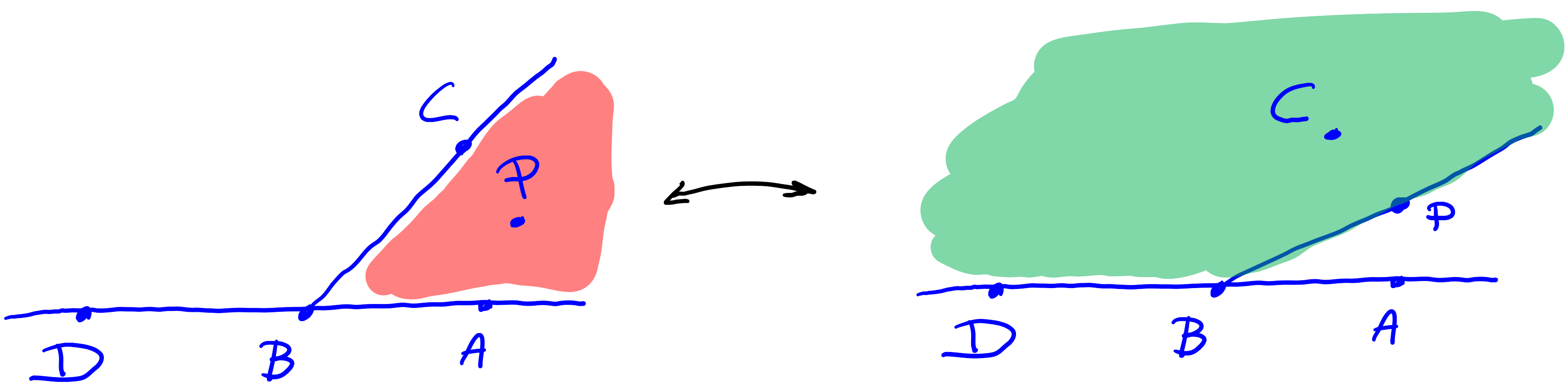
(ii) A, C are on opposite sides of $\overleftrightarrow{BP}$

Illustration



> **Theorem 4.4.9**
>
> In a Pasch Geometry, if $A - B - D$, then $P \in \text{int}(\angle ABC)$ if and only if $C \in \text{int}(\angle DBP)$
>
> **Alternate wording:**
>
> In a Pasch geometry, given points A, B, D such that $A - B - D$,
>
> the following are equivalent (TFAE)
>
> (i) $P \in \text{int}(\angle ABC)$
>
> (ii) $C \in \text{int}(\angle DBP)$

Illustration



Recall Definition of Angle Measure from Section 5.1

Definition: The symbol $\mathcal{A}$ denotes the set of all angles in a Pasch Geometry

Definition of Angle Measure

Words: Angle measure (or protractor) based on r_0

Usage: There is a Pasch geometry in the discussion, and r_0 is a fixed positive real number

Meaning: a function $m: \mathcal{A} \to \mathbb{R}$ that has these three properties (the *Axioms of Angle Measure*)

(i) $0 < m(\angle ABC) < r_0$

(ii) (This statement is called the *Angle Constuction Axiom*)

Given

- a half plane H
- a ray $\overrightarrow{BC}$ on the edge of that half plane
- a number θ such that $0 < \theta < r_0$

There exists a unique ray $\overrightarrow{BA}$ with $A \in H$ such that $m(\angle ABC) = \theta$.

(iii) (This statement is called the *Angle Addition Axiom*)

Angle measure is "additive" in the following sense:

If $D \in int(\angle ABC)$, then $m(\angle ABD) + m(\angle DBC) = m(\angle ABC)$

And Recall the Definition Protractor Geometry from Section 5.1

Definition: A **Protractor Geometry** is an ordered quadruple $(\mathcal{P}, \mathcal{L}, d, m)$ such that the ordered triple $(\mathcal{P}, \mathcal{L}, d)$ is a *Pasch Geometry* and m is an *angle measure* for $(\mathcal{P}, \mathcal{L}, d)$.

Section 5.3

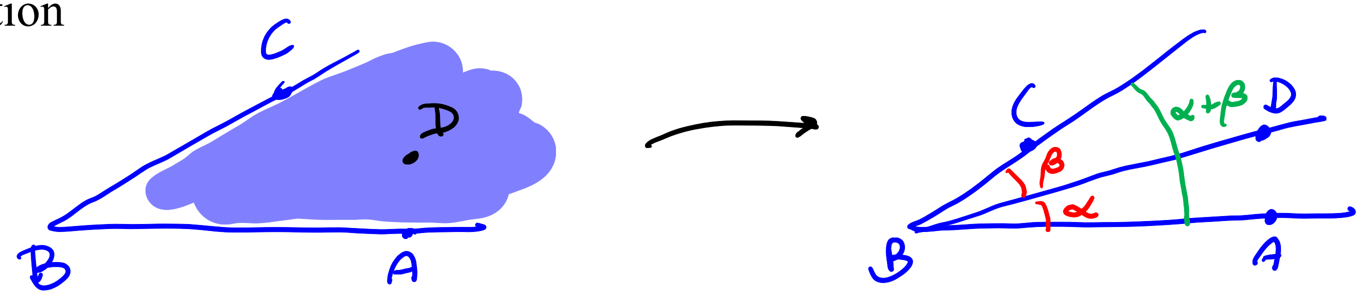
We will discuss Section 5.3 in two videos. The section is titled *Perpendicularity and Angle Congruence*, but those topics will be taken up in the second of the two videos. In this video, we will discuss some important consequences of Angle Measure Axiom (iii), the *Angle Addition Axiom*.

Statements about Points in the Interior of Angles and Big & Small Angle Measures

Recall that Angle Measure Axiom (iii) (the Angle Addition Axiom) says that

(iii) If $D \in int(\angle ABC)$, then $m(\angle ABD) + m(\angle DBC) = m(\angle ABC)$

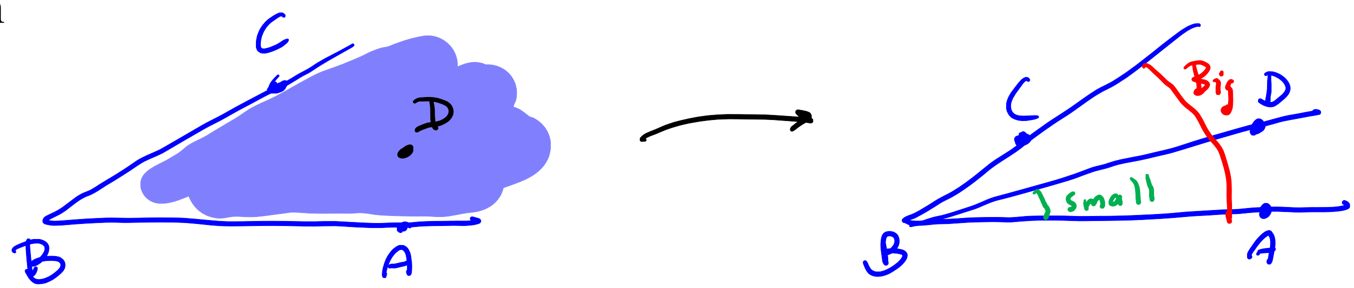
Illustration



Because we know that $m(\angle DBC)$ is positive (by Angle Measure Axiom (i)), this automatically gives us the following weaker statement. (The statement is such a simple corollary that the book does not even state it as a theorem or a corollary.)

> **Unstated Corollary:** If $D \in int(\angle ABC)$ then $m(\angle ABD) < m(\angle ABC)$.

Illustration



Remember that when a theorem has the form of a conditional statement, the contrapositive statement is automatically true, and is not a separate theorem.

Contrapositive of Unstated Corollary: If $m(\angle ABD) \nless m(\angle ABC)$, then $D \notin int(\angle ABC)$.

But remember that when a theorem has the form of a conditional statement, the converse statement is *not* automatically true. If the converse statement is true, then it constitutes a separate theorem, and it has to be proved.

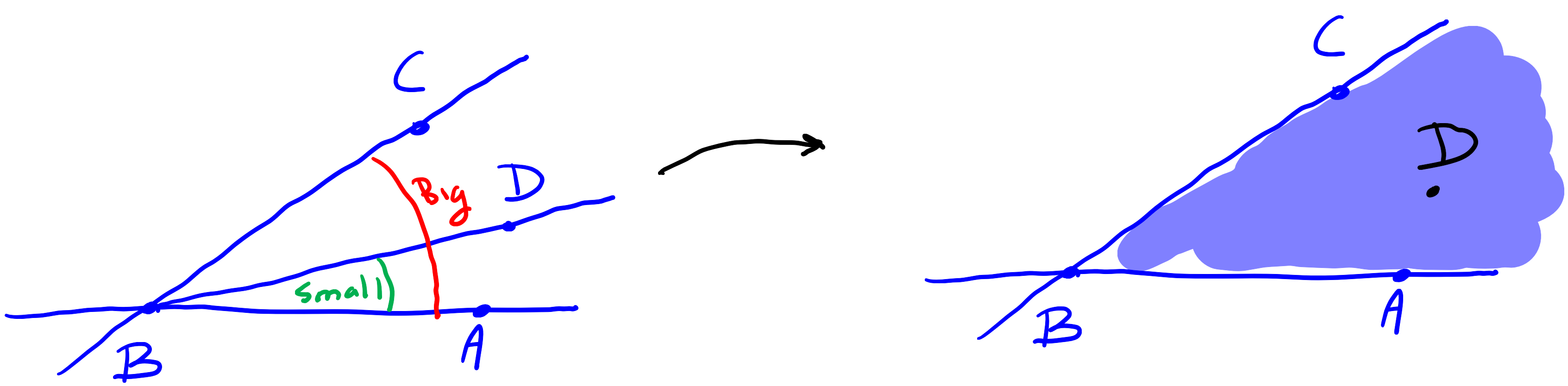
It turns out that the converse of the unstated Corollary is true, and its proof is not so simple. The converse statement is presented in the book as Theorem 5.3.1

Theorem 5.3.1

In a protractor geometry, given points C, D in the same half plane of $\overleftrightarrow{AB}$,

if $m(\angle ABD) < m(\angle ABC)$ then $D \in int(\angle ABC)$

Illustration



The book does a proof by contradiction. I don't feel that a proof by contradiction is necessary, and I feel that it makes the proof harder to understand. I find it simpler to just prove the contrapositive. That is, I will prove the following:

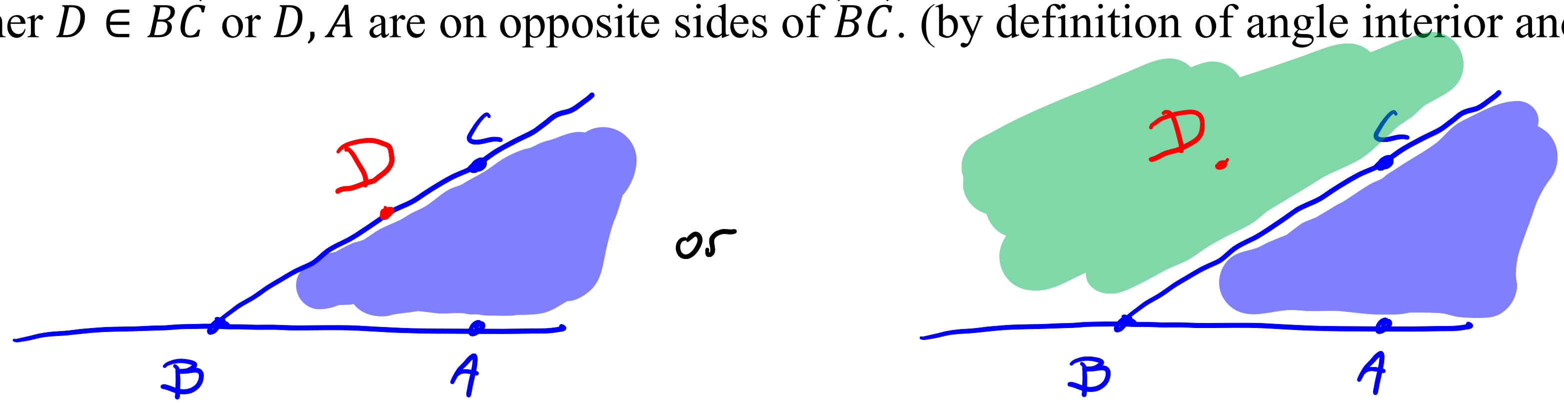
> **Theorem 5.3.1 (Contrapositive Version)**
>
> In a protractor geometry, given points C, D in the same half plane of $\overleftrightarrow{AB}$,
>
> if $D \notin int(\angle ABC)$ then $m(\angle ABD) \not< m(\angle ABC)$

Proof

(1) Suppose that in a protractor geometry, points C, D are in the same half plane of $\overleftrightarrow{AB}$, (that is, $D \in H_{\overleftrightarrow{BA},C}$) and that $D \notin int(\angle ABC)$.

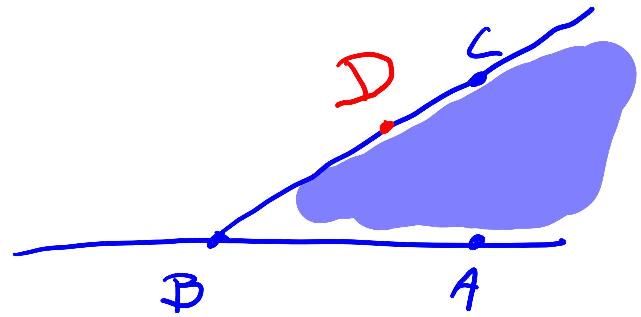
(2) Then either $D \in \overrightarrow{BC}$ or D, A are on opposite sides of $\overleftrightarrow{BC}$. (by definition of angle interior and by PSA (i))

(Illustrate.)



(3) **(Case 1)** Suppose that $D \in \overrightarrow{BC}$.

(Illustrate)

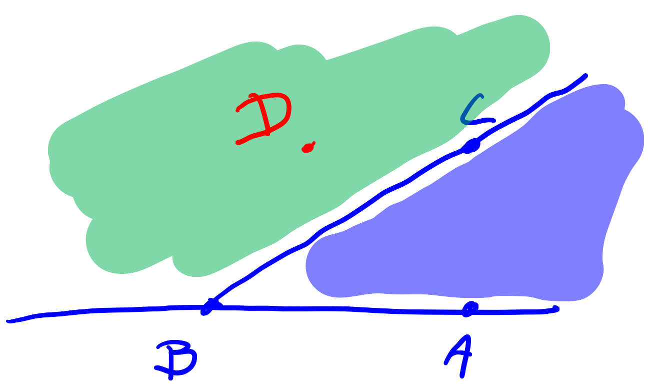


$m(\angle ABD)$

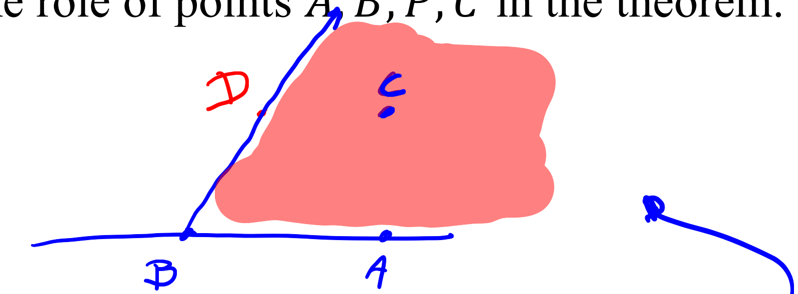
(4) Then ray $\overrightarrow{BC}$ is the same ray as $\overrightarrow{BD}$, so $m(\angle ABC) = m(\angle ABC)$.

(5) So $m(\angle ABD) \nless m(\angle ABC)$ in this case.

(6) **(Case 2)** Suppose that D, A are on opposite sides of $\overleftrightarrow{BC}$. **(Illustrate)**

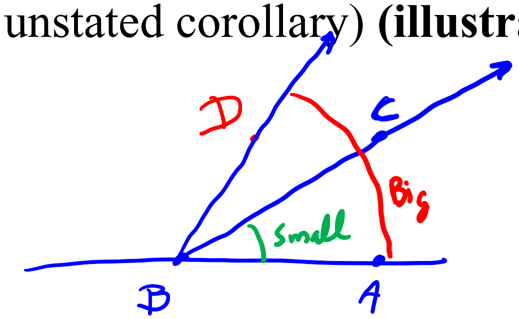


(7) Then A, C are on the same side of line $\overleftrightarrow{BD}$. (By Theorem 4.4.8(ii) ➔ (i)), with points A, B, C, D in our current situation playing the role of points A, B, P, C in the theorem. **(Illustrate)**



(8) So $C \in \text{int}(\angle ABD)$ (by (1), (7), and definition of angle interior) **(Illustrate)**

(9) So $m(\angle ABC) < m(\angle ABD)$ (by the unstated corollary) **(illustrate)**



(10) Therefore, $m(\angle ABD) \nless m(\angle ABC)$ in this case, as well.

(11)(**Conclusion of cases**) Conclude that $m(\angle ABD) \nless m(\angle ABC)$ (because it is true in all cases.)

End of Proof

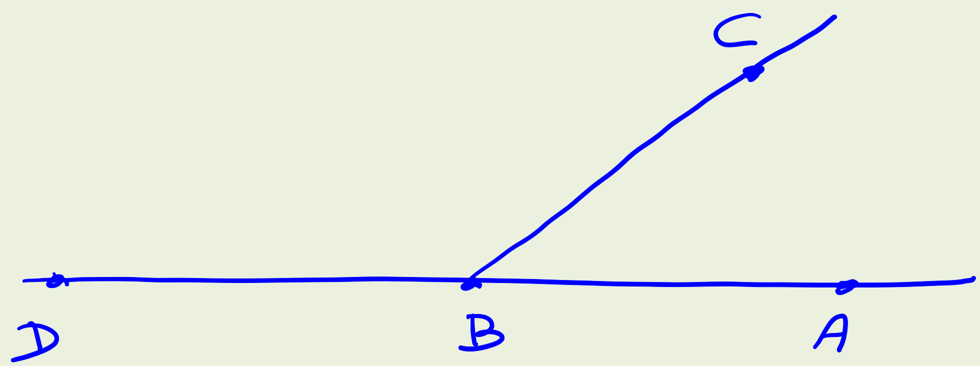
The Linear Pair Theorem

Definition of Linear Pair

Words: *Two angles from a linear pair.*

Meaning: The two angles can be labeled $\angle ABC$ and $\angle CBD$ with $A - B - D$.

Illustration:

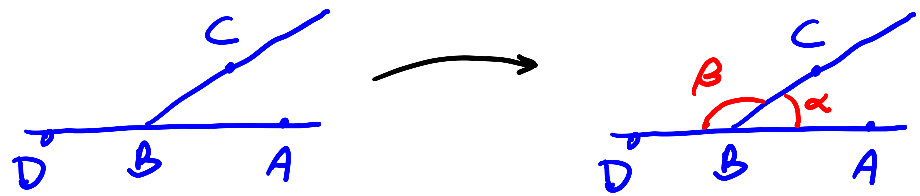


Theorem 5.3.2 The Linear Pair Theorem

In a protractor geometry,

if $\angle ABC$ and $\angle CBD$ form a linear pair, then $m(\angle ABC) + m(\angle ABC) = 180$

Remark on the Word "Supplementary"



$\alpha + \beta = 180$

In the book, the word supplementary is defined as follows:

Two angles are said to be supplementary if their measures add up to 180.

Then the Linear Pair Theorem is stated as follows

If two angles form a linear pair, then they are supplementary.

I don't find any advantage to using the word *supplementary*. To me, it it is just an unnecessary extra layer of notation. That's why I have worded the Linear Pair Theorem the way I have, without using the word *supplementary*.

Proof Structure:

Proof Part 1: Prove that $\alpha + \beta < 180$ cannot happen.

I will expand, justify & illustrate the book's proof in in this video.

Proof Part 2: Prove that $\alpha + \beta > 180$ cannot happen.

You will expand, justify & illustrate the book's proof in your Homework.

Conclusion: Therefore $\alpha + \beta = 180$

Book's Proof of Part 1

Let $m(\angle ABC) = \alpha$ and $m(\angle CBD) = \beta$. We must show that $\alpha + \beta = 180$. We do this by showing that both $\alpha + \beta < 180$ and $\alpha + \beta > 180$ lead to contradictions.

Suppose $\alpha + \beta < 180$. By the Angle Construction Axiom, there is a unique ray $\overrightarrow{BE}$ with E on the same side as C and with $m(\angle ABE) = \alpha + \beta$. By Theorem 5.3.1, $C \in \text{int}(\angle ABE)$ so that $m(\angle ABC) + m(\angle CBE) = m(\angle ABE)$. Thus,

$$\alpha + m(\angle CBE) = \alpha + \beta \quad \text{or } m(\angle CBE) = \beta$$

On the other hand, $E \in int(\angle CBD)$ (Why?) so that $m(\angle CBE) + m(\angle EBD) = m(\angle CBD)$. Thus

$$\beta + m(\angle EBD) = \beta \quad \text{or } m(\angle EBD) = 0$$

which is impossible. Thus, $\alpha + \beta < 180$ cannot occur.

(Book's Illustration).

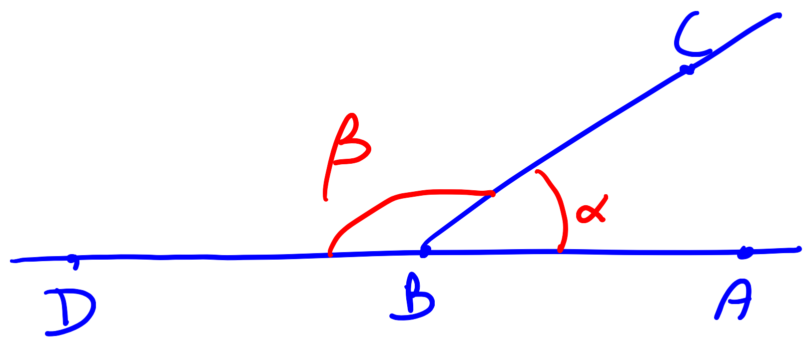
My Expanded, Justified, and Illustrated Version of Book's Proof Part 1.

(Statements and justifications that I have added are in **boldface**.)

Proof Part 1

(1) **Suppose that $\angle ABC$ and $\angle CBD$ form a linear pair.** Let $m(\angle ABC) = \alpha$ and $m(\angle CBD) = \beta$.

(Added Illustration).



(2) **We know that $\alpha + \beta > 0$ (because Angle Measure Axiom (i), says that angle measure is positive.)**

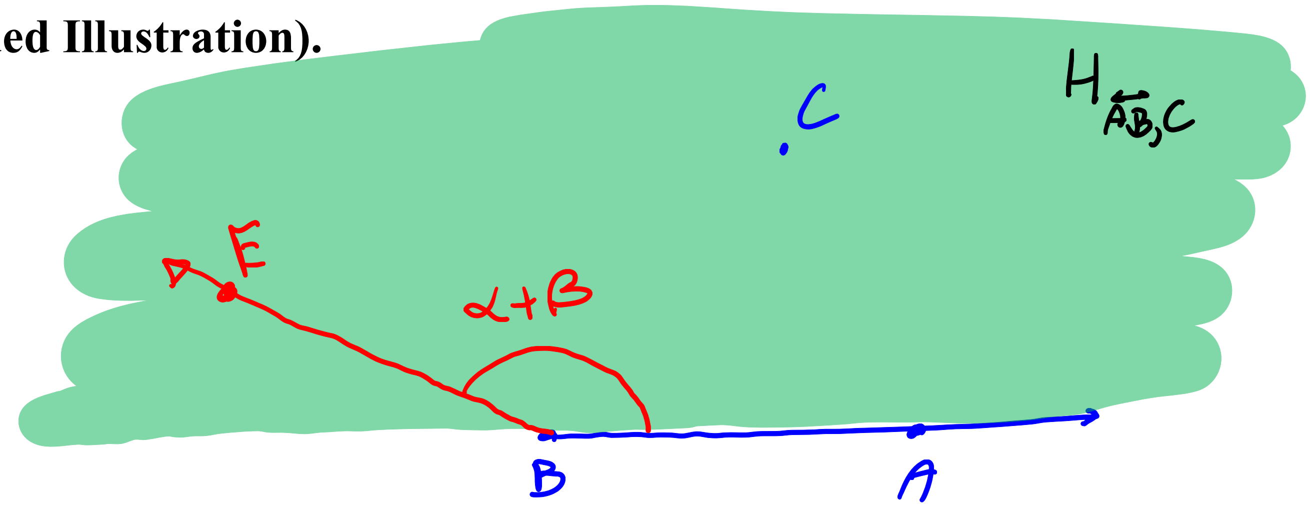
(3) Suppose that $\alpha + \beta < 180$. **(Assumption for Proof by Contradiction.)**

of line $\overleftrightarrow{AB}$

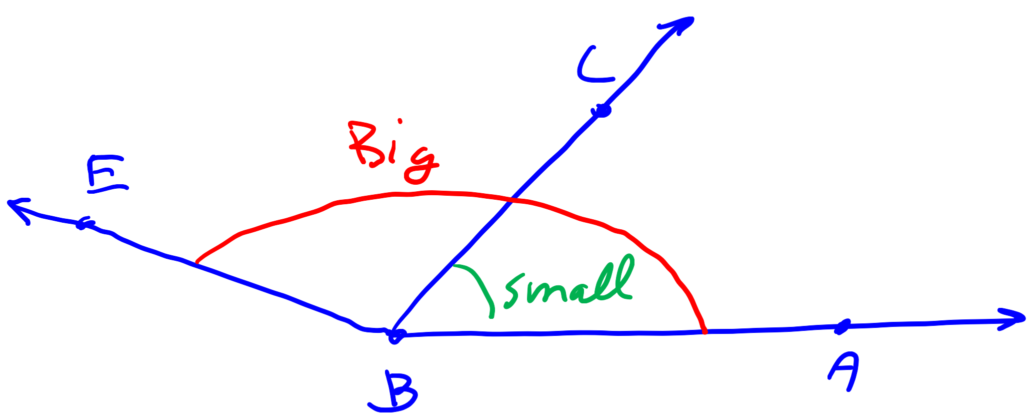
(4) There is a unique ray $\overrightarrow{BE}$ with E on the same side as C and with $m(\angle ABE) = \alpha + \beta$.

(by the Angle Construction Axiom **applied to ray $\overrightarrow{BA}$ on the edge of half plane $H_{\overleftrightarrow{BA},C}$ and given number $\alpha + \beta$ that has the property that $0 < \alpha + \beta < 180$ by statements (2) and (3).**)

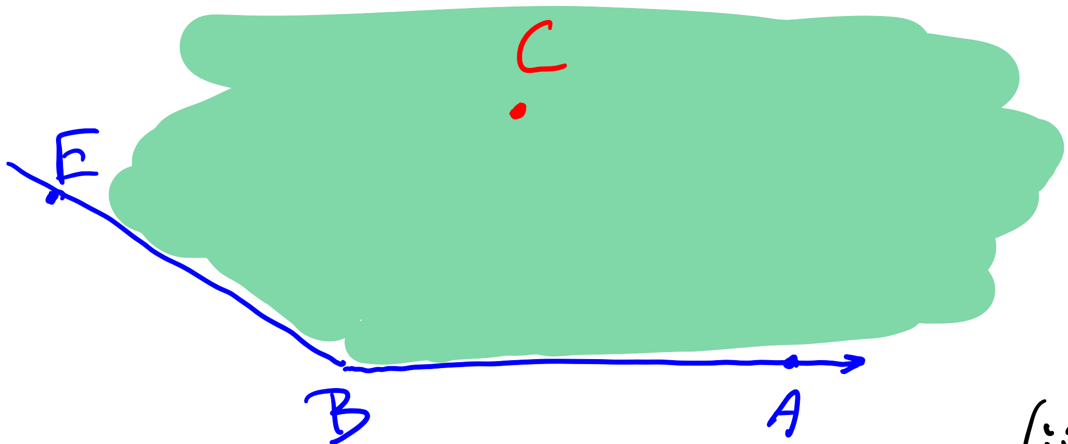
(Added Illustration).



(5) **Observe that C, E are on the same side of line $\overleftrightarrow{AB}$ and that $m(\angle ABC) = \alpha$ and $m(\angle ABE) = \alpha + \beta$, where $\beta > 0$, so that $m(\angle ABC) < m(\angle ABE)$ (Added Illustration)**

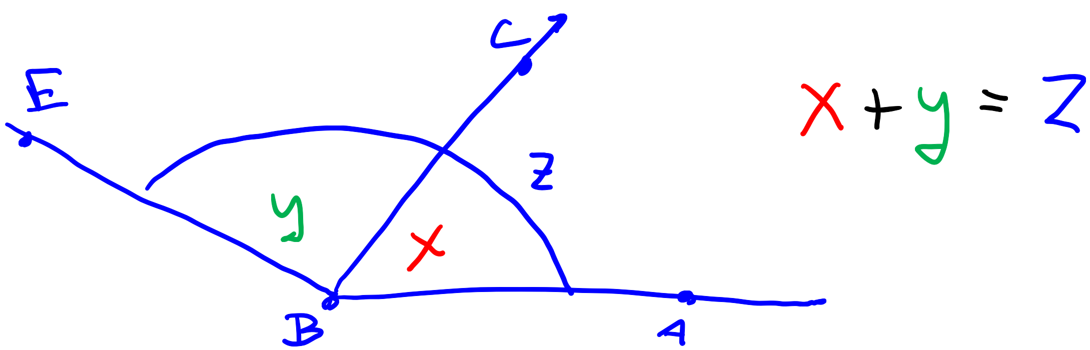


(6) $C \in \operatorname{int}(\angle ABE)$ (by Theorem 5.3.1 **applied to points C, E on the same side of line $\overleftrightarrow{AB}$ such that $m(\angle ABC) < m(\angle ABE)$) (Added Illustration)**



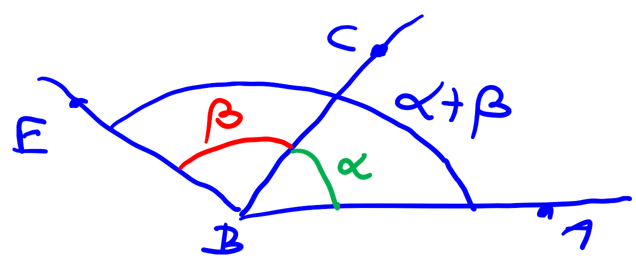
(iii)

(7) $m(\angle ABC) + m(\angle CBE) = m(\angle ABE)$. **(by Angle Measure Axiom (ii), the Angle Addition Axiom, applied to point $C \in \operatorname{int}(\angle ABE)$) (Added Illustration)**

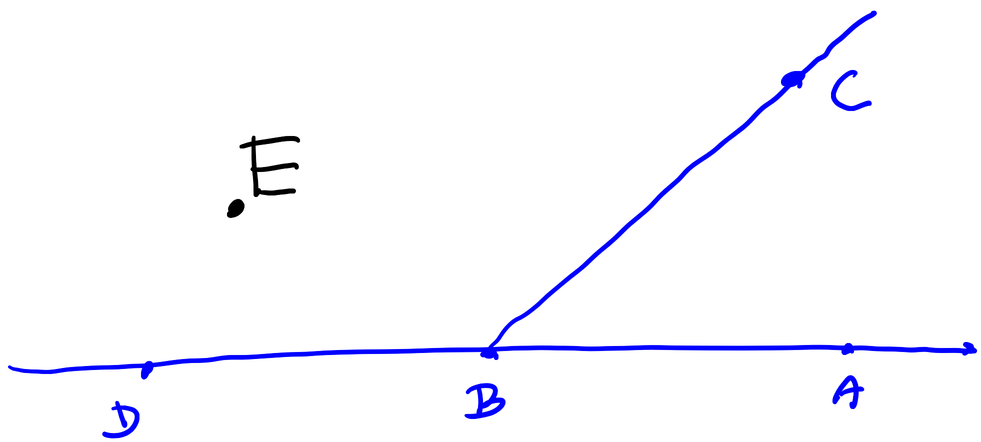


(8) $\alpha + m(\angle CBE) = \alpha + \beta$ **(substituted angle measures from (1),(4) into equation in (7))**

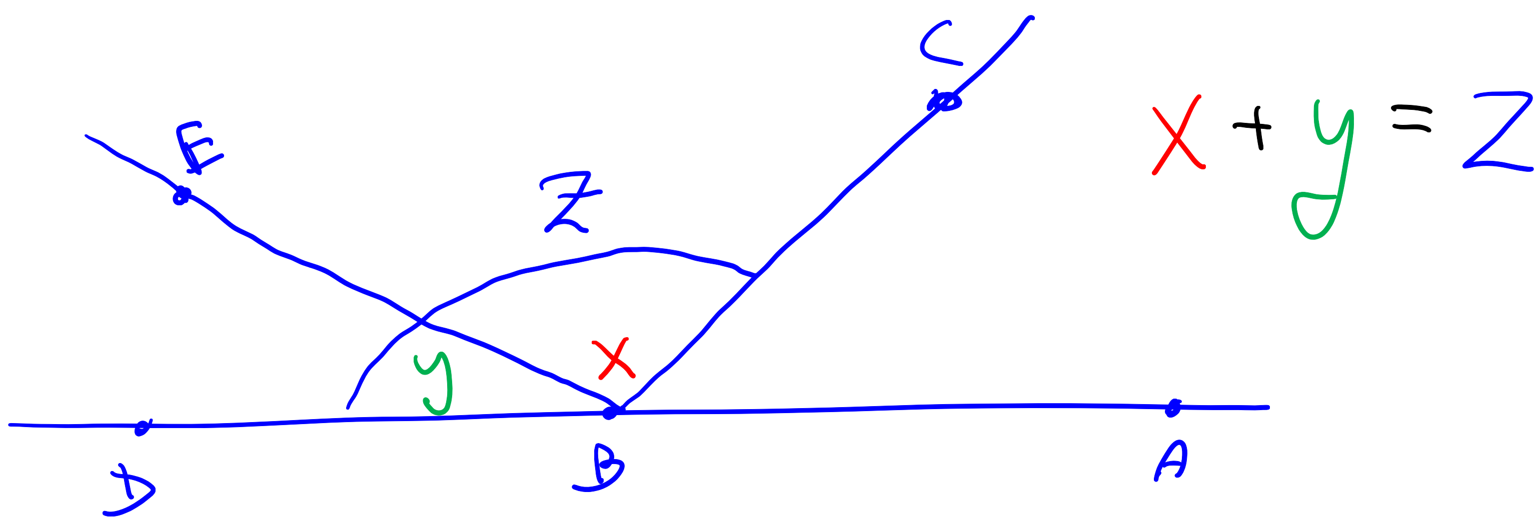
(9) $m(\angle CBE) = \beta$ **(by (8) and arithmetic)**



(10) $E \in int(\angle CBD)$ **(Because $\boldsymbol{A-B-D}$ (by definition of Linear Pair), and $\boldsymbol{C} \in \text{int}(\angle \boldsymbol{ABE})$ (by (6)), Theorem 4.4.9 tells us that $\boldsymbol{E} \in \text{int}(\angle \boldsymbol{CBD})$. (Added Illustration).**

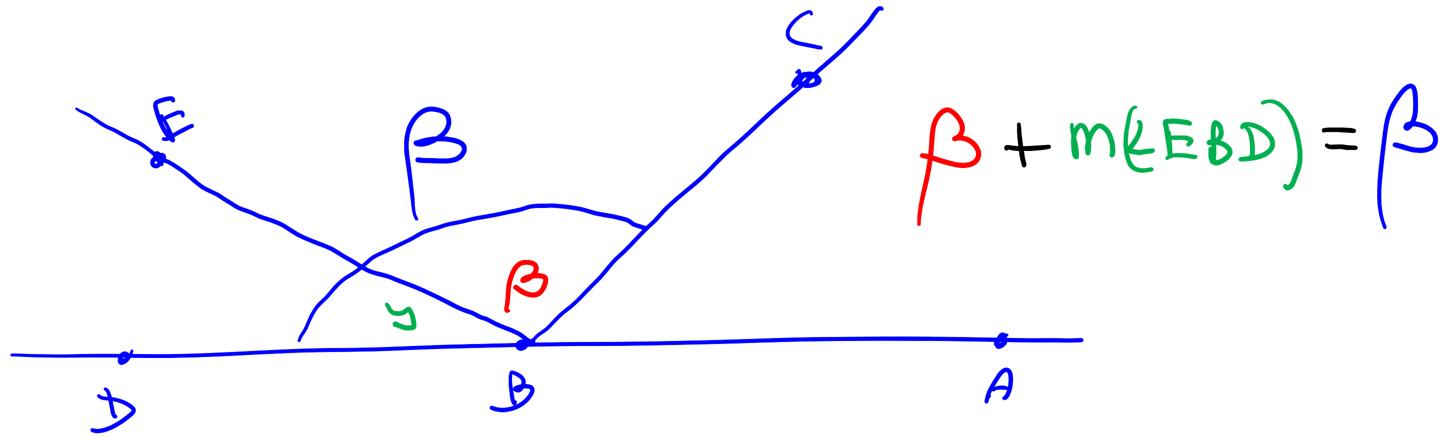


(11) $m(\angle CBE) + m(\angle EBD) = m(\angle CBD)$ **(by Angle Measure Axiom (ii), the Angle Addition Axiom, applied to point $\boldsymbol{E} \in \text{int}(\angle \boldsymbol{CBD})$) (Added Illustration)**



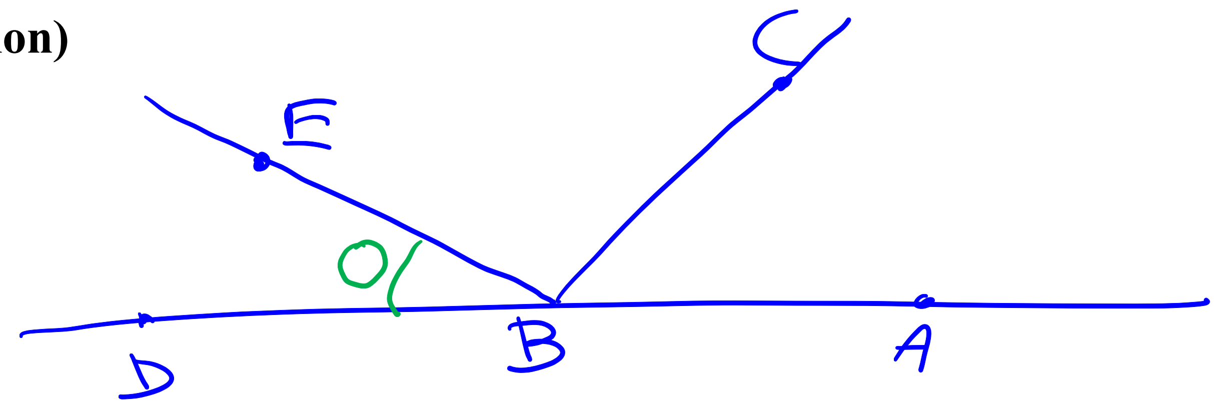
(12) $\beta + m(\angle EBD) = \beta$ **(substituted angle measures from (1),(9) into equation in (11))**

(Added Illustration)



(13) $m(\angle EBD) = 0$ **(by (12) and arithmetic)**

(Added Illustration)



(14) Statement (13) is impossible **(It contradicts Angle Measure Axiom (i), which says that angle measure must be positive.) Therefore, our assumption in statement (3) was wrong.** Thus $\alpha + \beta < 180$ cannot occur.

End of Proof Part 1

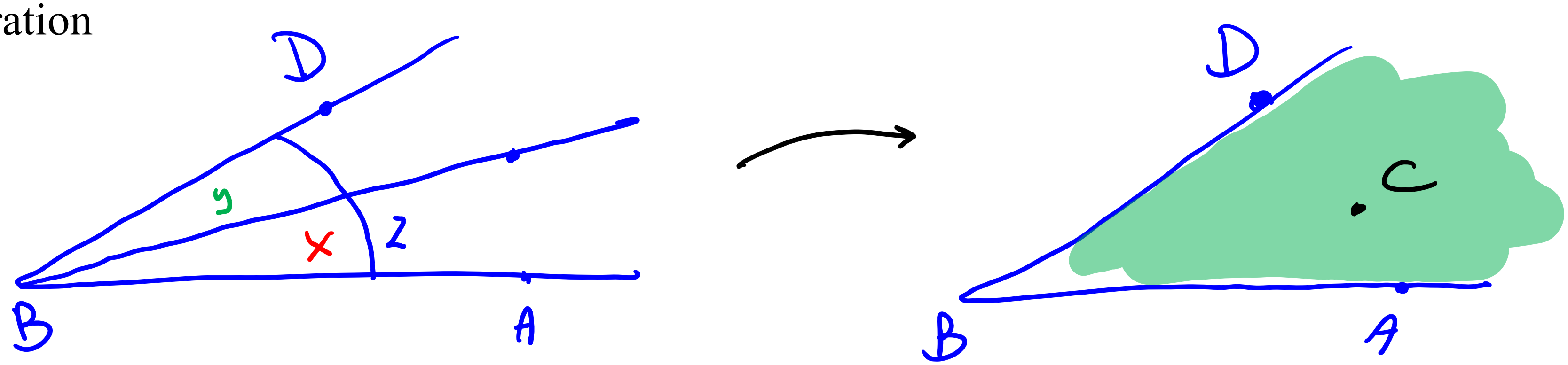
Converse Statements that are Also Theorems

Remember from earlier in the video and (from previous videos), when a theorem (or axiom) has the form of a conditional statement, the converse statement is *not* automatically true. If the converse statement is true, then it constitutes a separate theorem, and it has to be proved.

It turns out that the converse of the statement of the Angle Measurement Axiom (iii) (Angle Addition) is true. The converse statement is presented in the book as Theorem 5.3.3.

> **Theorem 5.3.3 (Converse of the Statement of the Angle Addition Axiom)**
>
> In a protractor geometry, if $m(\angle ABC) + m(\angle CBD) = m(\angle ABD)$ then $C \in int(\angle ABD)$

Illustration



$x + y = z$

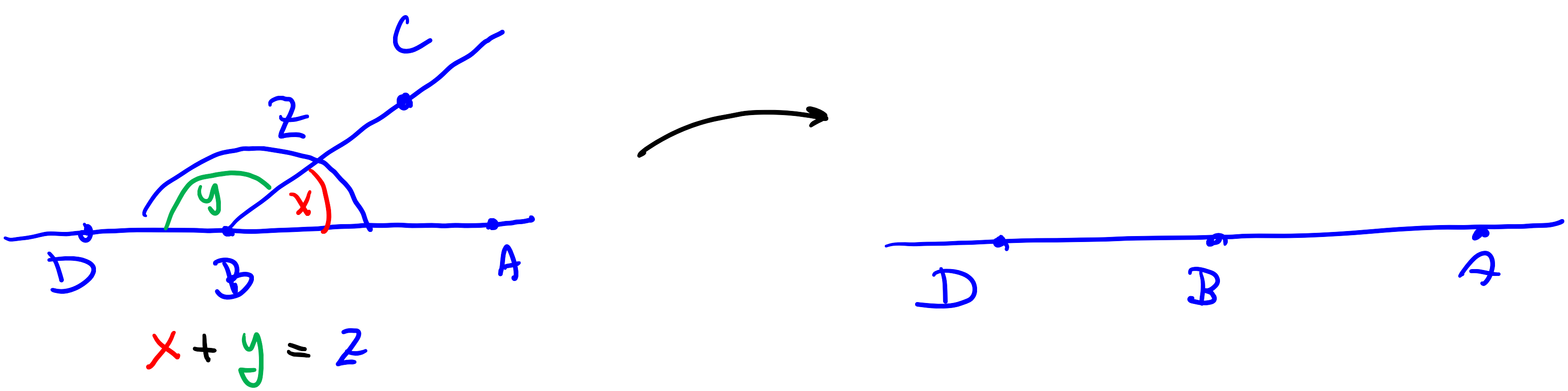
The Proof of Theorem 5.3.3 is comparable in difficulty to the proof of the Linear Pair Theorem. We won't discuss the proof in MATH 3110, and will just accept the Theorem as given.

It turns out that the converse of the statement of the Linear Pair Theorem is also true. The converse statement is presented in the book as Theorem 5.3.4.

Theorem 5.3.4 (Converse of the Statement of the Linear Pair Theorem)

In a protractor geometry,

if $m(\angle ABC) + m(\angle CBD) = 180$, then $A - B - D$ (and so the angles form a Linear Pair)



You will justify and illustrate a given proof of this Theorem in a homework exercise.

End of Video