Video 5.3b: The Angle Construction Axiom, Perpendicularity, and Angle Bisectors

produced by Mark Barsamian, 2021.03.24

for Ohio University MATH 3110/5110 College Geometry

Topics

- **Definitions of Terms: Acute, Right, Obtuse, Vertical, Perpendicular**
- **Existence and Uniqueness of Certain Perpendicular Lines**
- **Angle Bisectors**
- **Angle Congruence**
- **Basic Theorems about Angle Congruence**

Reading: Section 5.3

in *Geometry: A Metric Approach with Models, Second Edition* by Millman & Parker

Homework: Section 5.3 # 2, 3, 4, 5, 6, 8, 9, 10, 11, 15, 16, 18

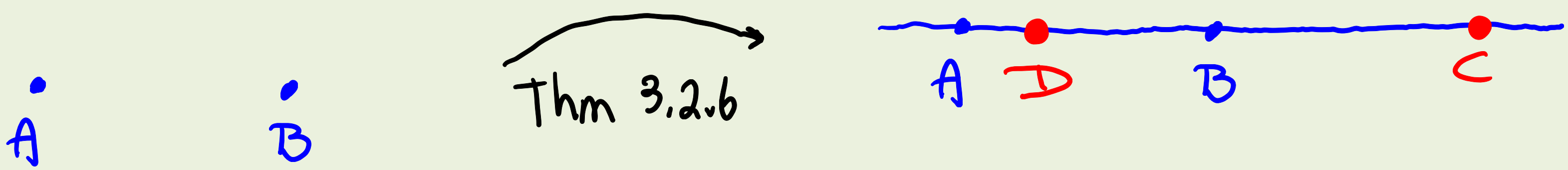
Recall Two Important Results From Chapter 3

Theorem 3.2.6 Existence of Points with Certain Betweenness Relationships

Given: Distinct points A, B in a *metric geometry*

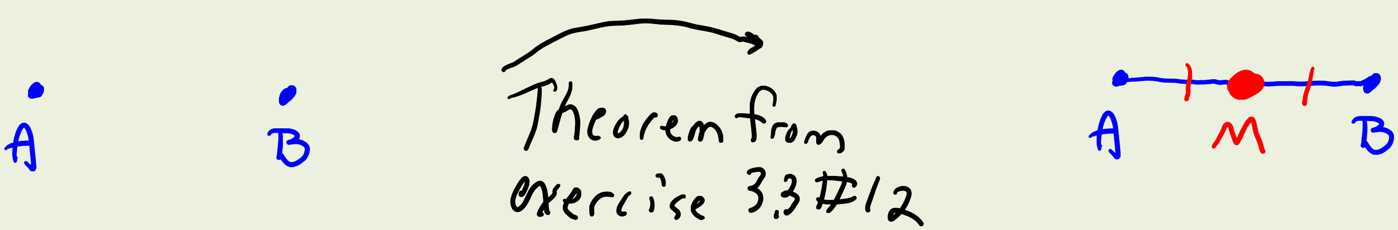
Claim: (i) There exists a point C with $A - B - C$

(ii) There exists a point D with $A - D - B$



Theorem Proven in Exercise 3.3#12: Existence and Uniqueness of the Midpoint of a Segment

If A, B are distinct points in a metric geometry, then segment $\overline{AB}$ has exactly one midpoint.



Recall from Section 4.1 the Definition of the Plane Separation Axiom

Definition: The Plane Separation Axiom (PSA) (**My version of the definition**)

- **Words:** A metric Geometry $(\mathcal{P}, \mathcal{L}, d)$ satisfies the **plane separation axiom** (PSA)
- **Meaning:** For every line $l \in \mathcal{L}$, there are two associated sets of points called *half planes*, denoted H_1 and H_2, with the following properties:

  (i) The three sets l, H_1, H_2 form a partition of the set $\mathcal{P}$ of all points.

  (ii) Each of the *half planes* is convex.

  (iii) If $A \in H_1$ and $B \in H_2$, then $\overline{AB}$ *intersects* line l.
- **Additional Terminology:**
  - Line l is called the ***edge*** of *half planes* H_1 and H_2.
  - **Words:** *Points* A, B *lie on the* ***same side*** *of line* l.
    - **Meaning:** Points A, B are elements of the same half plane associated to l.
  - **Words:** *Points* A, B *lie on* ***opposite sides*** *of line* l.
    - **Meaning:** Points A, B are elements of different half planes associated to l.

And recall from Section 4.1 that it is useful to know the Contrapositives of PSA (ii) and (iii)

PSA (ii) and (iii) and their Contrapositives

PSA (ii): If distinct points P, Q are in the same *half plane*,

then $\overline{PQ}$ does not intersect line l.

PSA (ii) (contrapositive): If $\overline{PQ}$ does intersect line l,

then P, Q are *not* in the same *half plane*.

PSA (iii) If P, Q are not in the same *half plane*,

then $\overline{PQ}$ intersects line l.

PSA (iii) (contrapositive) If $\overline{PQ}$ does not intersect line l,

then P, Q are distinct points in the same *half plane*.

Recall Definition of Angle Measure from Section 5.1

Definition: The symbol $\mathcal{A}$ denotes the set of all angles in a Pasch Geometry

Definition of Angle Measure

Words: Angle measure (or protractor) based on r_0

Usage: There is a Pasch geometry in the discussion, and r_0 is a fixed positive real number

Meaning: a function $m: \mathcal{A} \rightarrow \mathbb{R}$ that has these three properties (the *Axioms of Angle Measure*)

(i) $0 < m(\angle ABC) < r_0$

(ii) (This statement is called the *Angle Constuction Axiom*)

Given

- a half plane H
- a ray $\overrightarrow{BC}$ on the edge of that half plane
- a number θ such that $0 < \theta < r_0$

There exists a unique ray $\overrightarrow{BA}$ with $A \in H$ such that $m(\angle ABC) = \theta$.

(iii) (This statement is called the *Angle Addition Axiom*)

Angle measure is "additive" in the following sense:

If $D \in int(\angle ABC)$, then $m(\angle ABD) + m(\angle DBC) = m(\angle ABC)$

And Recall the Definition Protractor Geometry from Section 5.1

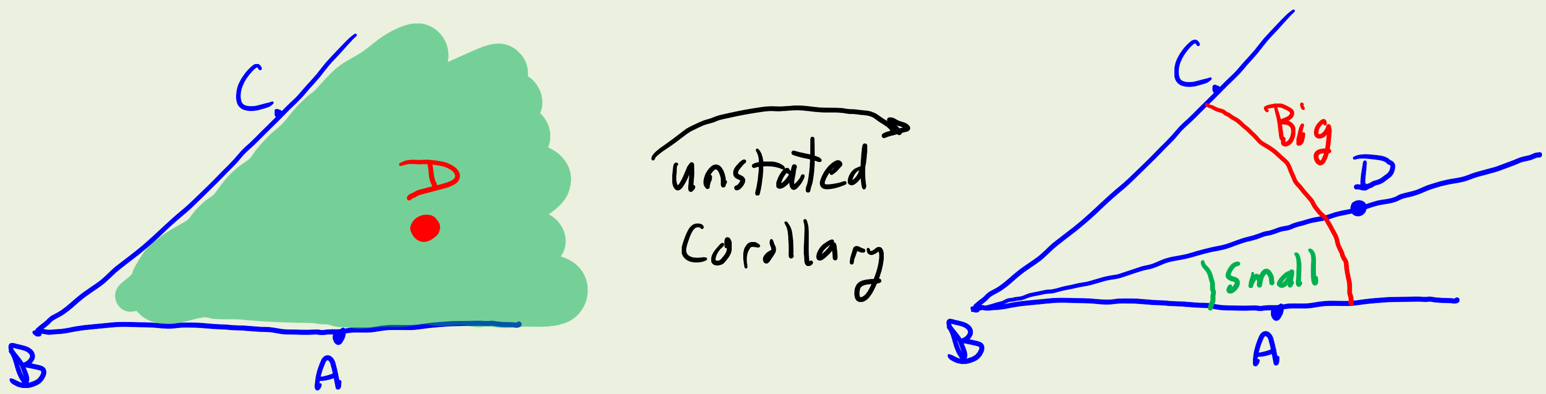
> **Definition:** A **Protractor Geometry** is an ordered quadruple $(\mathcal{P},\mathcal{L},d,m)$ such that the ordered triple $(\mathcal{P},\mathcal{L},d)$ is a *Pasch Geometry* and m is an *angle measure* for $(\mathcal{P},\mathcal{L},d)$.

Recall Topics from Video 5.3a

We are discussing Section 5.3 in two videos. The section is titled *Perpendicularity and Angle Congruence*. Those topics will be taken up in this video, which is the second of the two videos. But before discussing those topics, we should quickly recall that in the previous video about Section 5.3, we discussed some important consequences of Angle Measure Axiom (iii), the *Angle Addition Axiom*.

Statements about Points in the Interior of Angles and Big & Small Angle Measures

Unstated Corollary: If $D \in int(\angle ABC)$ then $m(\angle ABD) < m(\angle ABC)$.



Remember that when a theorem has the form of a conditional statement, the contrapositive statement is automatically true, and is not a separate theorem.

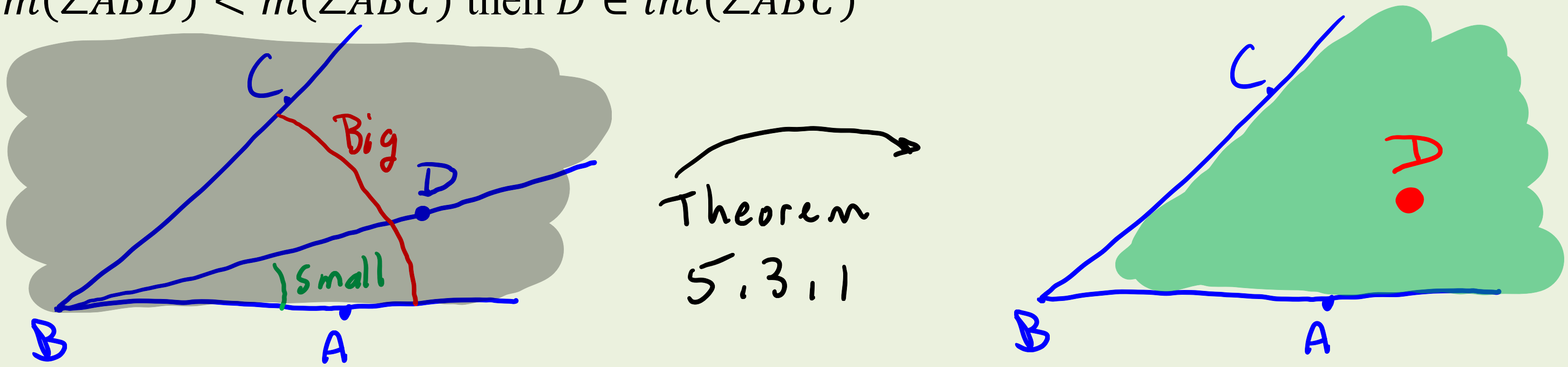
Contrapositive of Unstated Corollary: If $m(\angle ABD) \not< m(\angle ABC)$, then $D \notin int(\angle ABC)$.

But remember that the converse of a conditional statement does not mean the same thing as the original conditional statement. So when a theorem (or axiom) has the form of a conditional statement, the converse statement is *not* automatically true. If the converse statement is true, then it constitutes a separate theorem, and it has to be proved.

Theorem 5.3.1 (Converse of the Statement of the Unstated Corollary)

In a protractor geometry, given points C, D in the same half plane of $\overleftrightarrow{AB}$,

if $m(\angle ABD) < m(\angle ABC)$ then $D \in int(\angle ABC)$



Theorem 5.3.1 (Contrapositive Version)

In a protractor geometry, given points C, D in the same half plane of $\overleftrightarrow{AB}$,

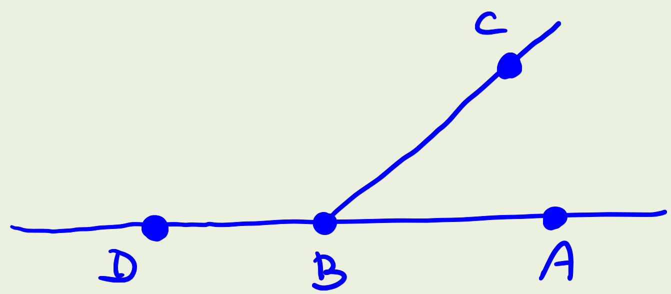
if $D \notin int(\angle ABC)$ then $m(\angle ABD) \not< m(\angle ABC)$

Definition of Linear Pair

Words: *Two angles from a linear pair.*

Meaning: The two angles can be labeled $\angle ABC$ and $\angle CBD$ with $A - B - D$.

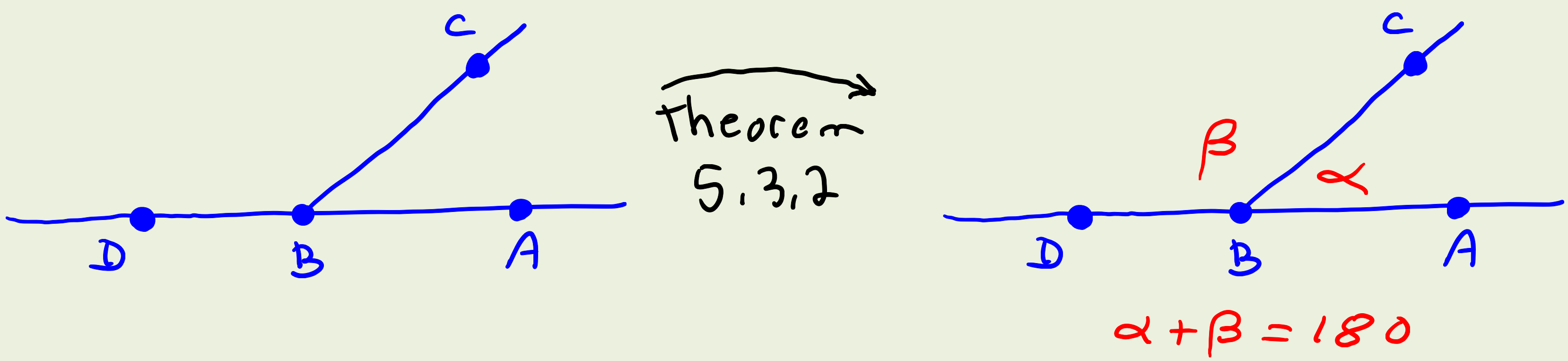
Illustration:



Theorem 5.3.2 The Linear Pair Theorem

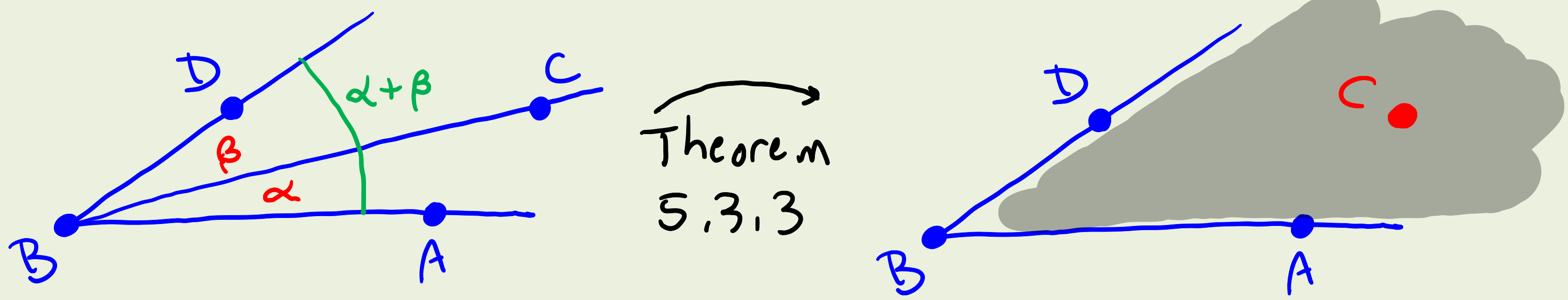
In a protractor geometry,

if $\angle ABC$ and $\angle CBD$ form a linear pair, then $m(\angle ABC) + m(\angle ABC) = 180$



Theorem 5.3.3 (Converse of the Statement of the Angle Addition Axiom)

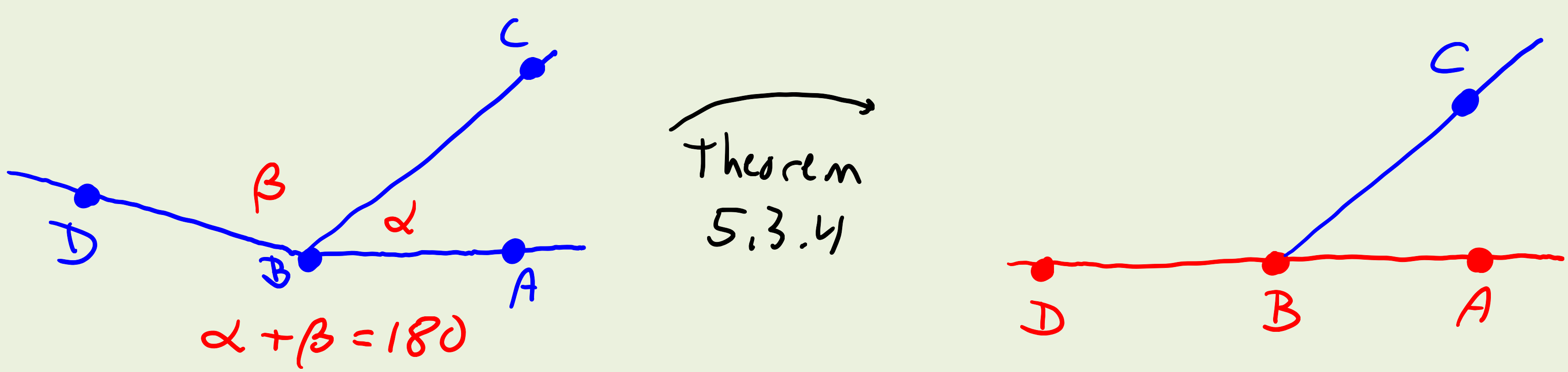
In a protractor geometry, if $m(\angle ABC) + m(\angle CBD) = m(\angle ABD)$ then $C \in int(\angle ABD)$



Theorem 5.3.4 (Converse of the Statement of the Linear Pair Theorem)

In a protractor geometry,

if $m(\angle ABC) + m(\angle CBD) = 180$, then $A - B - D$ (and so the angles form a Linear Pair)



Video 5.3b Perpendicularity and Angle Congruence

Whereas Video 5.3a was primarily about some important consequences of Angle Measure Axiom (iii), the *Angle Addition Axiom*, the current Video 5.3b will be about important consequences of Angle Measure Axiom (ii), the *Angle Construction Axiom*.

We will start with some definitions.

Definitions of Terms: Acute, Right, Obtuse, Vertical, Perpendicular

Definition of Acute, Right, Obtuse Angles

An **acute angle** is an angle whose measure is less than 90.

A **right angle** is an angle whose measure is 90. (Right angles are marked with a box.)

An **obtuse angle** is an angle whose measure is greater than 90.

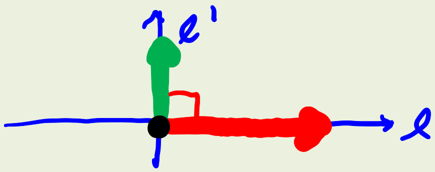
Definition of Perpendicular Lines

Words: l and l' are perpendicular

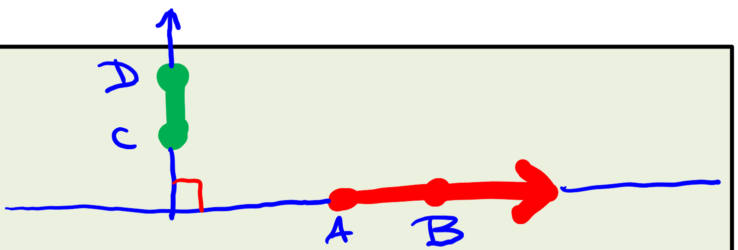
Symbol: $l \perp l'$ (In a drawing, perpendicular lines are marked with a box at the angle.)

Usage: l, l' are lines in a protractor geometry

Meaning: $l \cup l'$ contains a right angle



Definition of Perpendicular Rays, Segments



Words: $\overrightarrow{AB}$ (or $\overline{AB}$) and $\overline{CD}$ (or $\overrightarrow{CD}$) are perpendicular

Usage: $\overrightarrow{AB}$ (or $\overline{AB}$) and $\overline{CD}$ (or $\overrightarrow{CD}$) are segments or rays in a protractor geometry

Meaning: The lines containing those objects are perpendicular. That is, $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$.

Remark: Notice that segments and rays can be perpendicular without intersecting.

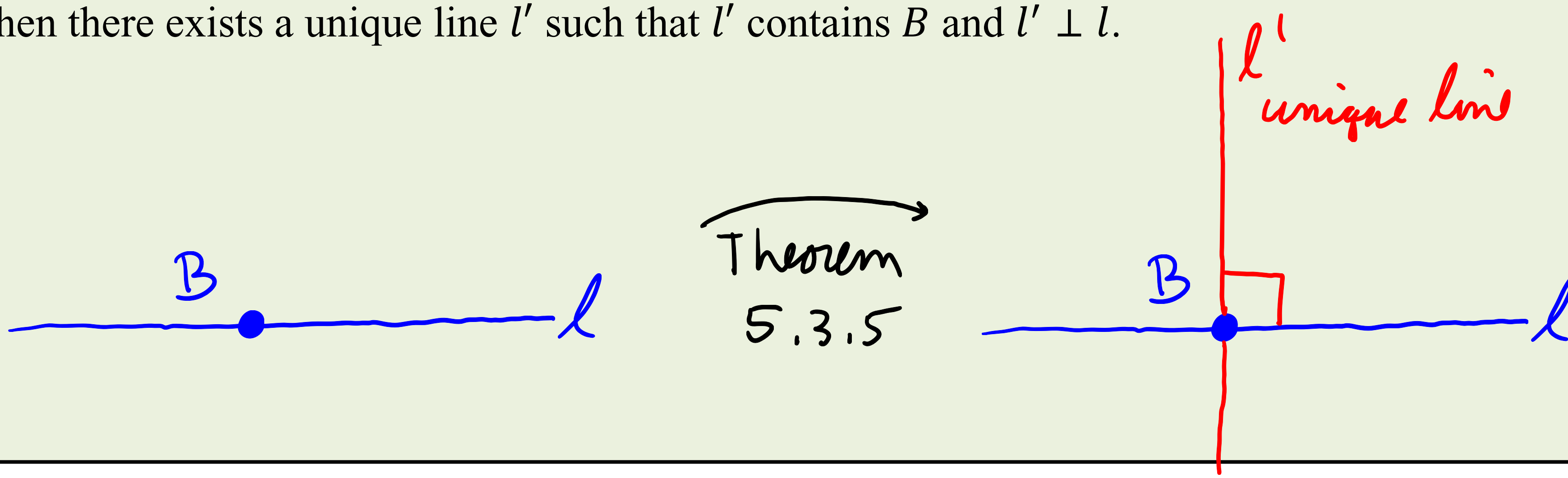
Existence and Uniqueness of Certain Perpendicular Lines

As mentioned above, the current Video 5.3b will be about important consequences of Angle Measure Axiom (ii), the *Angle Construction Axiom*. The first two consequences are about the existence and uniqueness of lines that are perpendicular to given lines or segments.

Theorem 5.3.5 Existence of a Unique Perpendicular to a Line through a Point On the Line

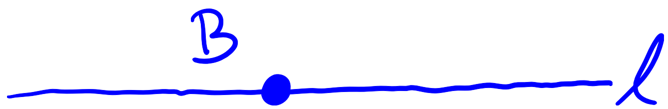
In a protractor geometry, if B is a point on a line l,

then there exists a unique line l' such that l' contains B and $l' \perp l$.



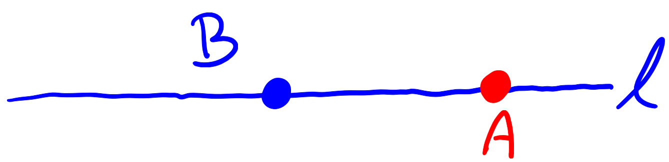
Proof

(1) Suppose that B is a point on line l in a protractor geometry.

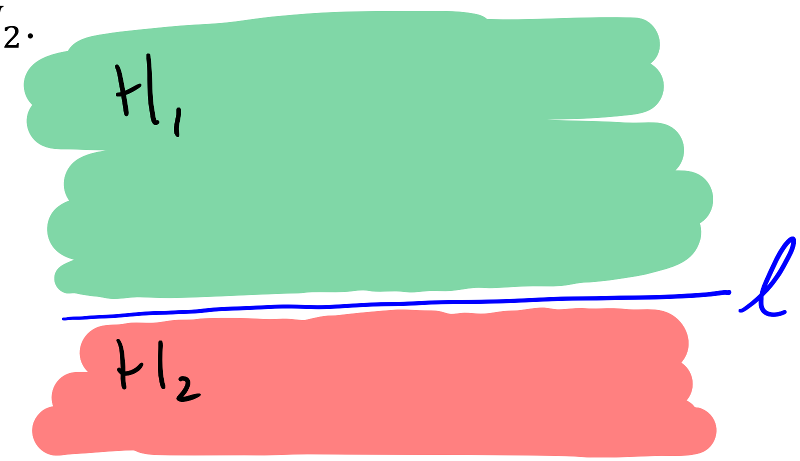


Part 1 Prove that a Perpendicular Line Exists

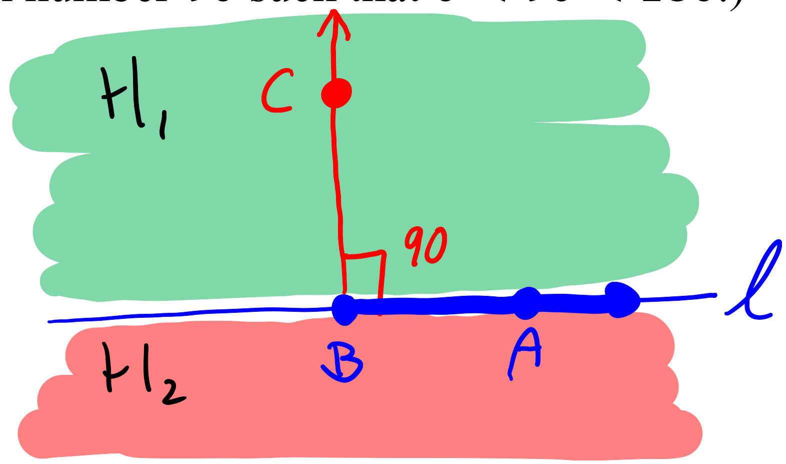
(2) There exists another point on line l. (by Abstract Geometry Axiom (ii) Call it A.



(3) There exist two half planes associated to line l (by the Plane Separation Axiom). Call those half planes H_1, H_2.

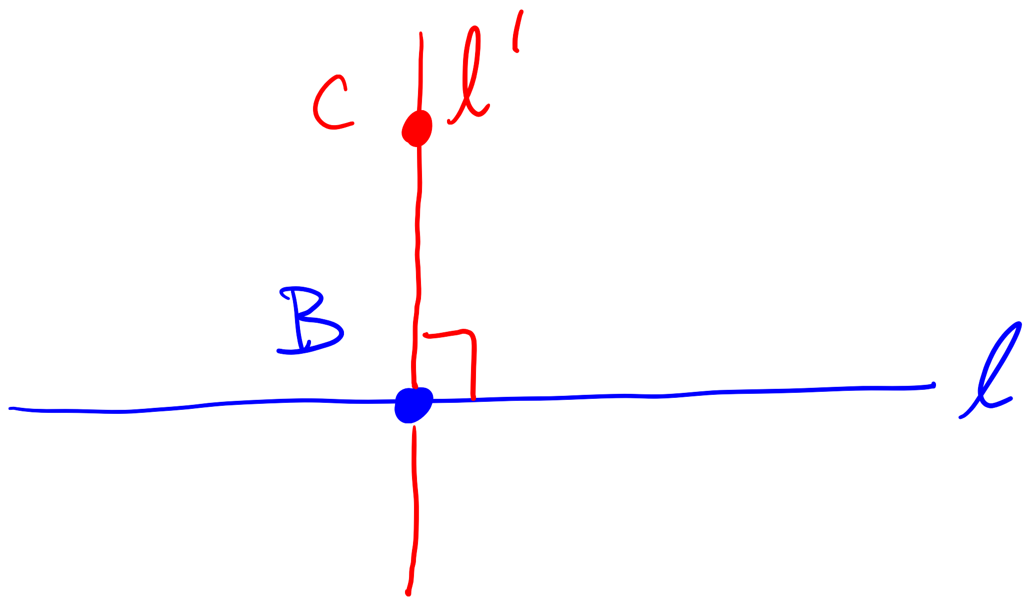


(4) There exists a ray $\overrightarrow{BC}$, with $C \in H_1$, such that $m(\angle ABC) = 90$. (by Angle Measurement Axiom (ii), the *Angle Construction Axiom* (the *existence* part of the axiom), applied to given ray $\overrightarrow{BA}$ on the edge of given half plane H_1 and given number 90 such that $0 < 90 < 180$.)



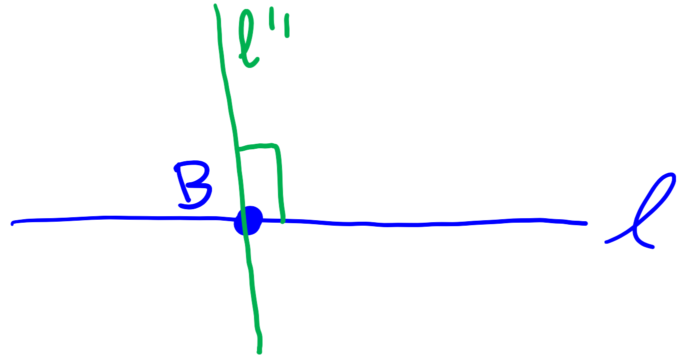
(5) Define l' to be line $\overleftrightarrow{BC}$. Then l' is perpendicular to l. (by (4) and definition of perpendicular)

We have shown that a perpendicular line exists.

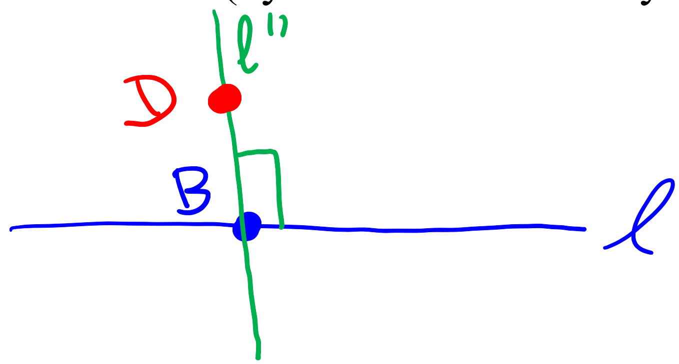


Part 2 Prove that the Perpendicular Line is Unique

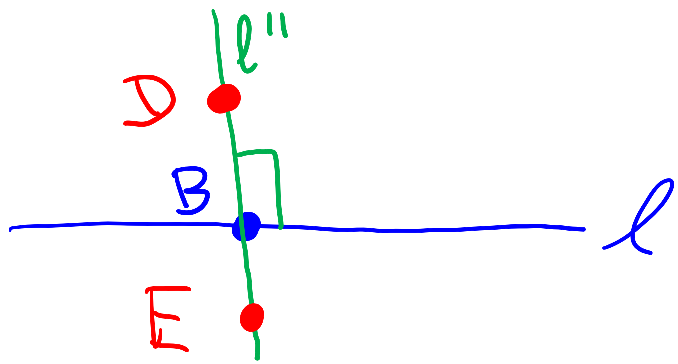
(6) Suppose that line l'' is a line that contains point B and is perpedicular to l. (We must show that line l'' is actually line l'.



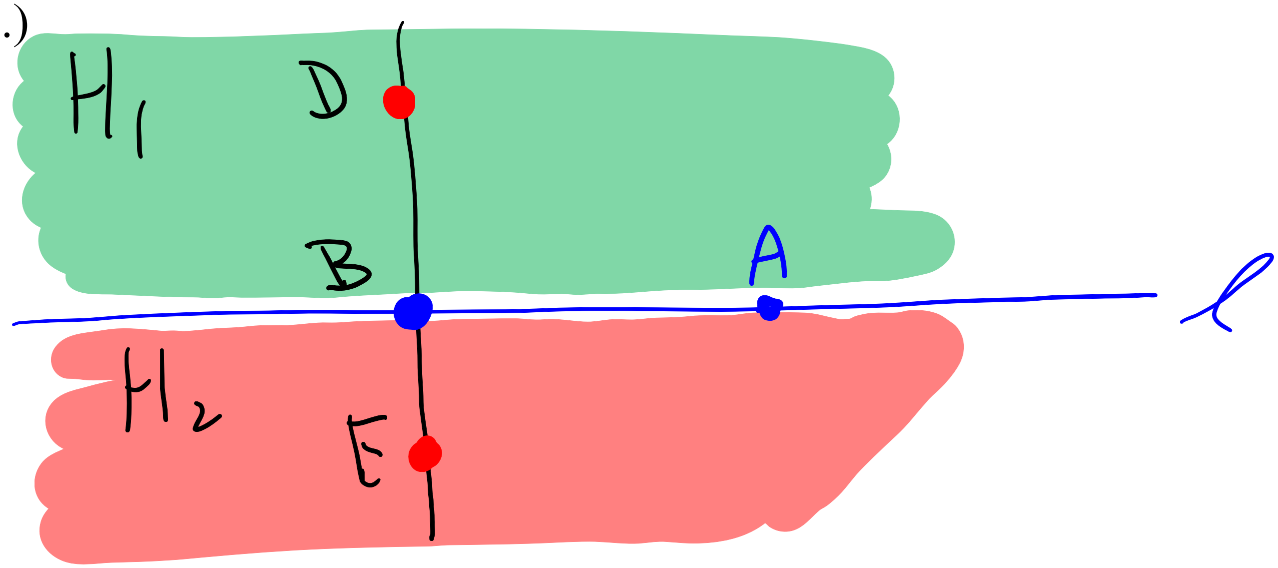
(7) There exists a second point D on line l'' (by Abstract Geometry axiom (ii)).



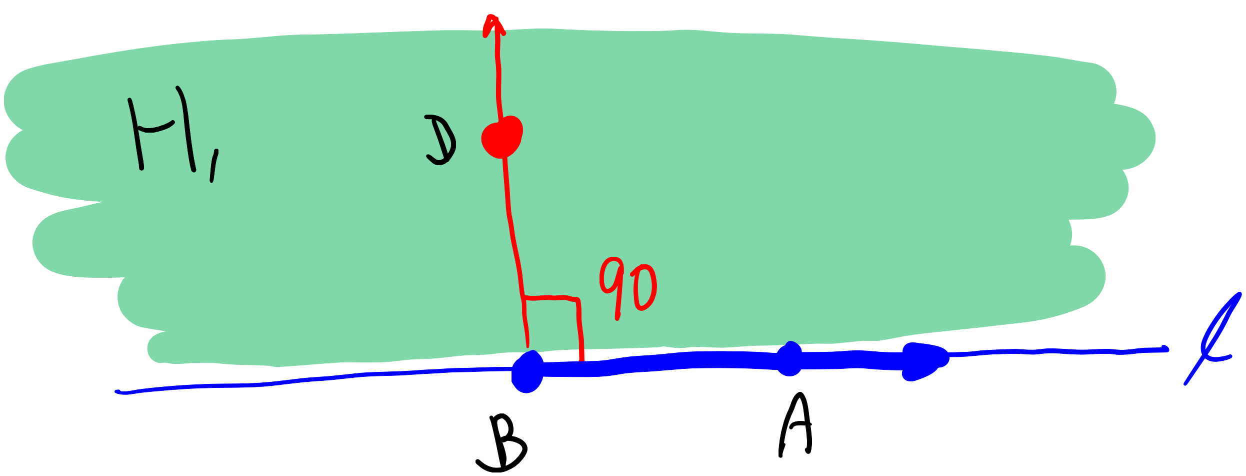
(8) There exists a point E such that $D - B - E$ (by Theorem 3.2.6 (i) applied to given points D, B)



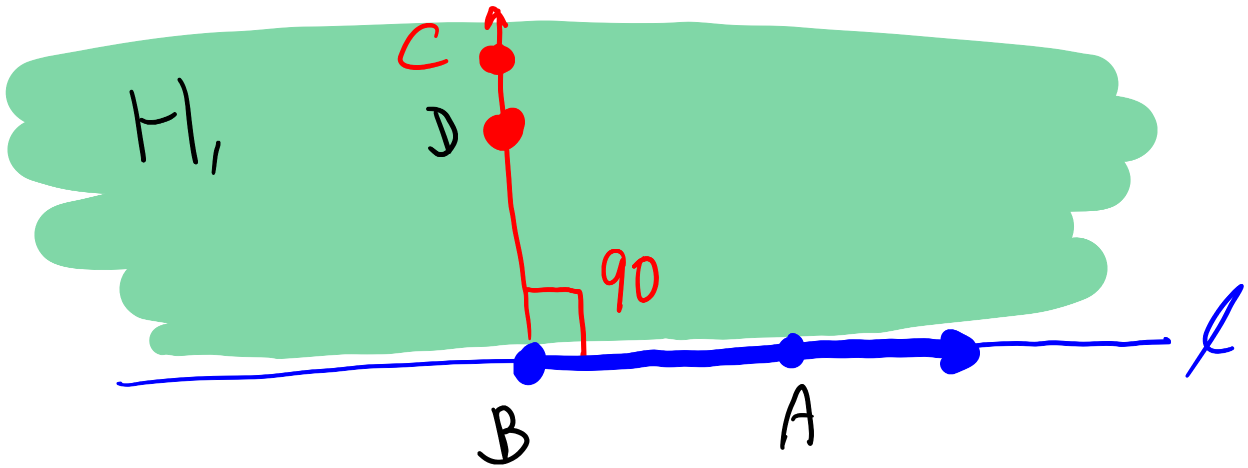
(9) Points D, E are on opposite sides of line $l = \overleftrightarrow{AB}$. (because line $\overleftrightarrow{AB}$ intersects segement $\overline{DE}$ at point B such that $D - B - E$, PSA (ii) contrapositive tells us that points D, E are not in the same half plane) We can assume that point D is in half plane H_1. (If it is not, simply rename D, E so that D is the point that is in H_1.)



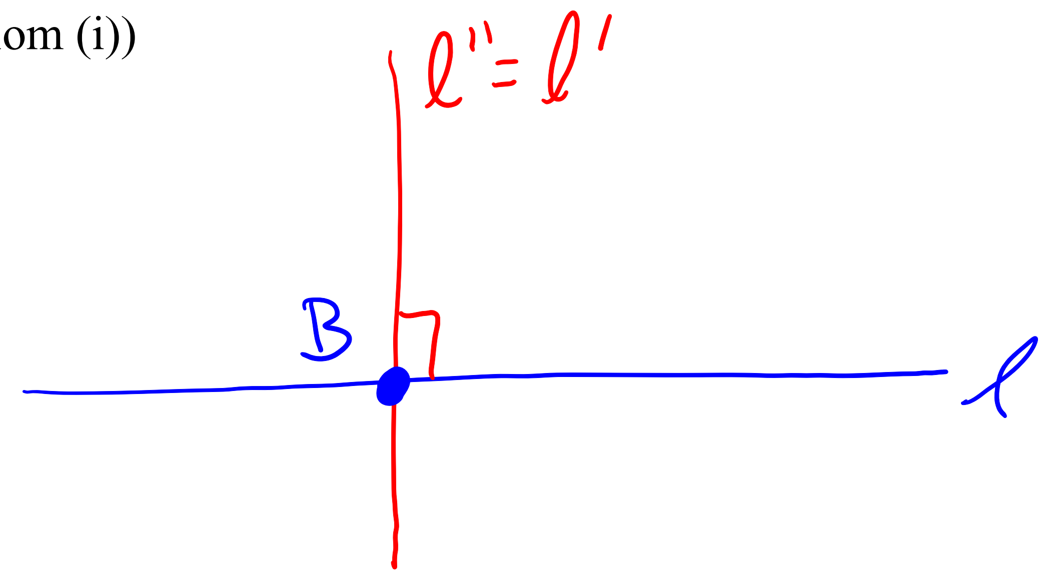
(10) Observe that $D \in H_1$ and that $m(\angle ABD) = 90$. (by (6) and definition of perpendicular)



(11) Ray $\overrightarrow{BD}$ must be the same ray as ray $\overrightarrow{BC}$. (by Angle Measurement Axiom (ii), the *Angle Construction Axiom* (the *uniqueness* part of the axiom), applied to given ray $\overrightarrow{BA}$ on the edge of given half plane H_1 and given number 90 such that $0 < 90 < 180$.)



(12) In other words, point D must be on line l', so that line l" is actually the same line as l'. (By Incidence Geometry axiom (i))



End of Proof

Notice that the proof of Theorem 5.3.5 used a variety of axioms and previous theorems, but that at its core, the proof of the existence and uniqueness of the perpendicular was simply an application of the *angle construction axiom* about the existence and uniqueness of a ray that forms an angle of a given size. (Observe that *both* the *existence* and *uniqueness* claims of that axiom were used.)

Also notice that Theorem 5.3.5 tells us that a perpendicular exists and is unique, but it tells us nothing about how to *find* the perpendicular. The details of finding the perpendicular will depend on which *model* of protractor geometry one is working in. In book Example 5.3.6 (on page 107), the authors discuss how to find a line perpendicular to a point on a given line in the Poincaré plane.

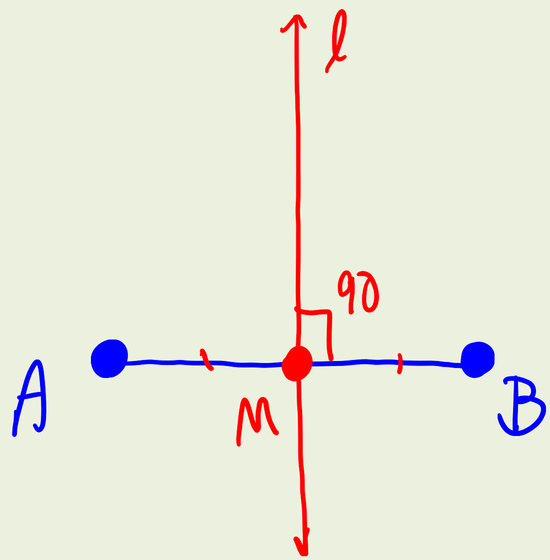
A simple corollary of Theorem 5.3.5 has to do with perpendicular bisectors of line segments.

Definition of Perpendicular Bisector of a Segment

Words: *a perpendicular bisector of* $\overline{AB}$

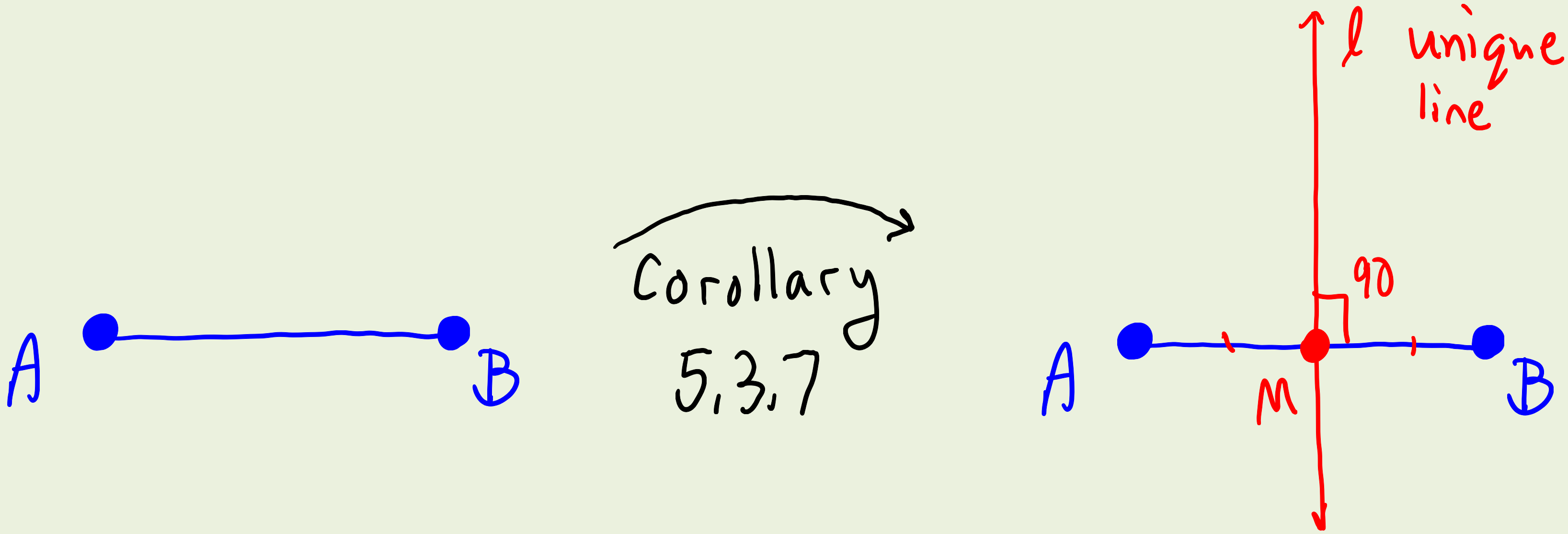
Usage: $\overline{AB}$ is a segment in a protractor geometry

Meaning: a line l such that $l \perp \overleftrightarrow{AB}$ and $l \cap \overleftrightarrow{AB} = \{M\}$ where M is the midpoint of $\overline{AB}$.



Corollary 5.3.7 Existence of a Unique Perpendicular Bisector

If $\overline{AB}$ is a segment in a protractor geometry, then $\overline{AB}$ has a *unique perpendicular bisector*.



You will prove Corollary 5.3.7 in a homework exercise.

Angle Bisectors

As mentioned earlier, the current Video 5.3b is about important consequences of Angle Measure Axiom (ii), the Angle Construction Axiom. The first two consequences were about the existence and uniqueness of lines that are perpendicular to given lines or segments.

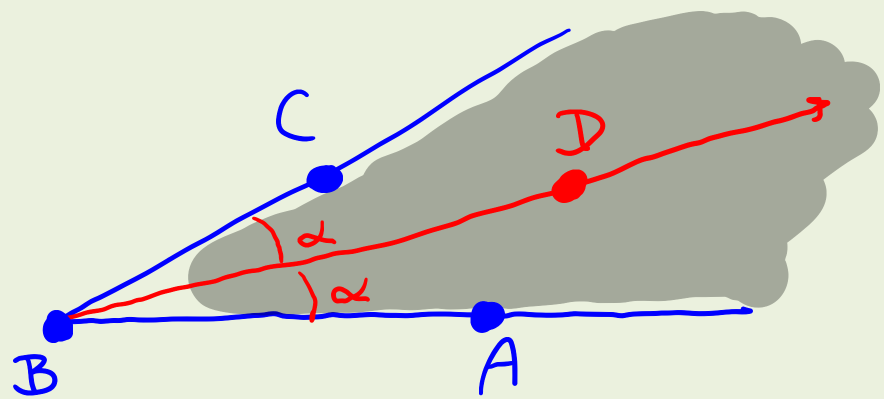
The third consequence of the Angle Construction Axiom that we will study has to do with *bisectors of angles.*

Definition of Angle Bisector

Words: *a bisector of* $\angle ABC$

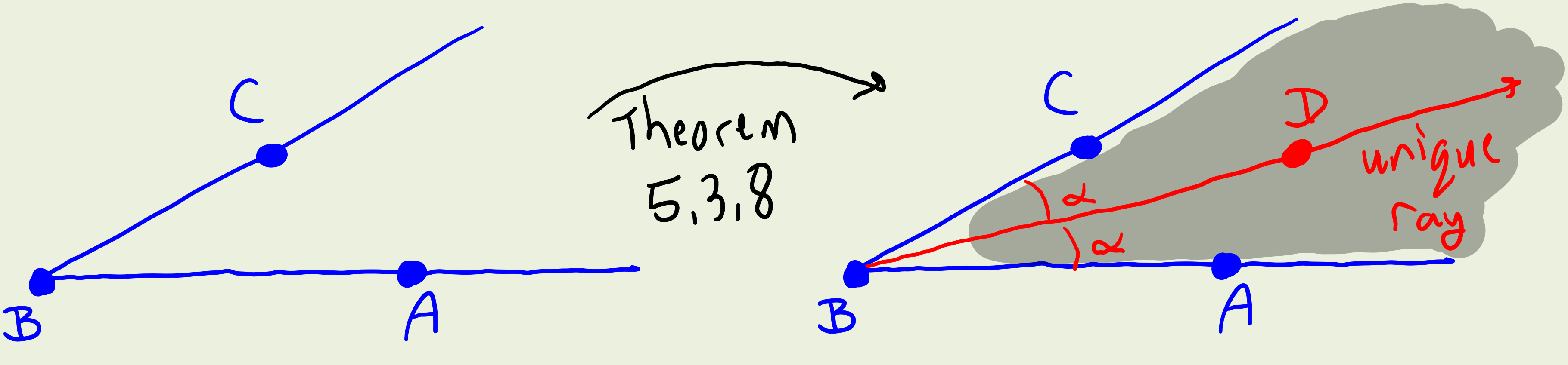
Usage: $\angle ABC$ is an angle in a protractor geometry

Meaning: a ray $\overrightarrow{BD}$ such that $D \in \text{int}(\angle ABC)$ and $m(\angle ABD) = m(\angle DBC)$.



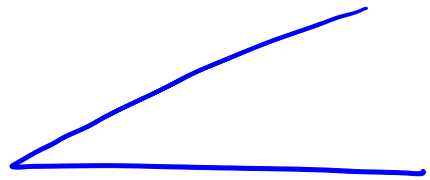
Theorem 5.3.8 Existence of a Unique Angle Bisector

If $\angle ABC$ is an angle in a protractor geometry, then $\angle ABC$ has a *unique angle bisector*.

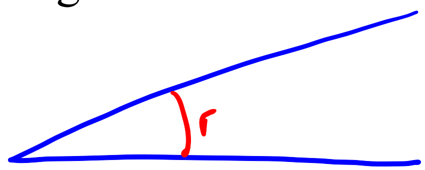


Proof Strategy

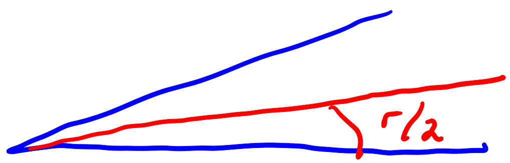
Given an angle



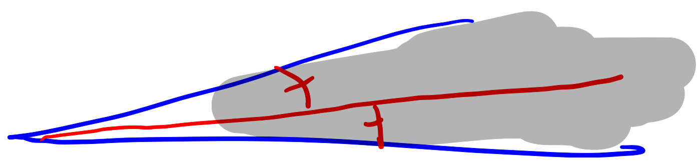
Measure the existing angle



Show that a ray exists that creates an angle whose measure is half as big.



Show that that ray is an angle bisector.



Show that the angle bisector is unique.

Here are the steps of a proof of Theorem 5.3.8 following the strategy outlined above. You will justify and illustrate the steps in this proof in a homework exercise.

Proof

(1) Suppose that $\angle ABC$ is an angle in a protractor geometry.

(2) The number $m(\angle ABC)$ exists and has the property that $0 < m(\angle ABC) < 180$ **(Justify)**

(3) Let $r = \frac{1}{2}m(\angle ABC)$. Then $0 < r < 90$, so $0 < r < 180$ is certainly true.

Part 1 Prove that an angle bisector exists

(4) There exists a ray $\overrightarrow{BD}$ such that $D \in H_{\overleftrightarrow{AB},C}$ and that $m(\angle ABD) = r$. **(Justify)(Illustrate)**

(5) $D \in \text{int}(\angle ABC)$ **(Justify)(Illustrate)**

(6) $m(\angle ABD) + m(\angle DBC) = m(\angle ABC)$**(Justify)(Illustrate)**

(7) $\frac{1}{2}m(\angle ABC) + m(\angle DBC) = m(\angle ABC)$ (by (3),(4),(7))

(8) $m(\angle DBC) = \frac{1}{2}m(\angle ABC)$ (by (7), arithmetic)

(9) $m(\angle ABD) = m(\angle DBC)$ (by (4),(8))

(10) Ray $\overrightarrow{BD}$ is a bisector of angle $\angle ABC$. **(Justify)(Illustrate)**

Part 2 Prove that the angle bisector is unique

(11) Suppose that ray $\overrightarrow{BE}$ is a bisector of angle $\angle ABC$. **(Illustrate)** (We must show that ray $\overrightarrow{BE}$ is actually ray $\overrightarrow{BD}$.)

(12) $E \in \text{int}(\angle ABC)$ and $m(\angle ABE) = m(\angle EBC)$ **(Justify)(Illustrate)**

(13) $m(\angle ABE) + m(\angle EBC) = m(\angle ABC)$**(Justify)(Illustrate)**

(14) $m(\angle ABE) + m(\angle ABE) = m(\angle ABC)$**(Justify)**

(15) $m(\angle ABE) = \frac{1}{2}m(\angle ABC) = r$ **(Justify)**

(16) $E \in H_{\overleftrightarrow{AB},C}$ **(Justify)(Illustrate)**

(17) ray $\overrightarrow{BE}$ is actually ray $\overrightarrow{BD}$. **(Justify)(Illustrate)**

End of Proof

Angle Congruence

Our final topics for the current video (and for Section 5.3) involve the concept of *angle congruence*.

Recall that *line segments* are defined to be *congruent* if they have the same *length*. *Congruent angles* are defined in an analogous way.

Definition of Congruent Angles

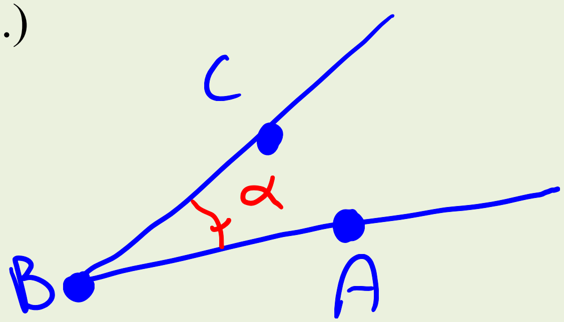
Symbol: $\angle ABC \simeq \angle DEF$

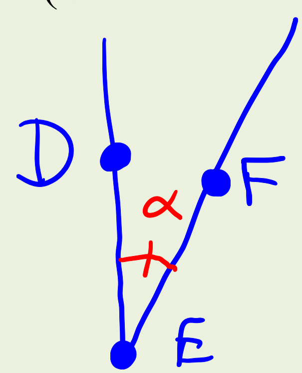
Words: $\angle ABC$ *is congruent to* $\angle DEF$

Usage: $\angle ABC$ and $\angle DEF$ are angles in a protractor geometry

Meaning: $m(\angle ABC) = m(\angle DEF)$

Illustration: Congruent angles are marked with tic marks. (Or with a matching number of tic marks.)





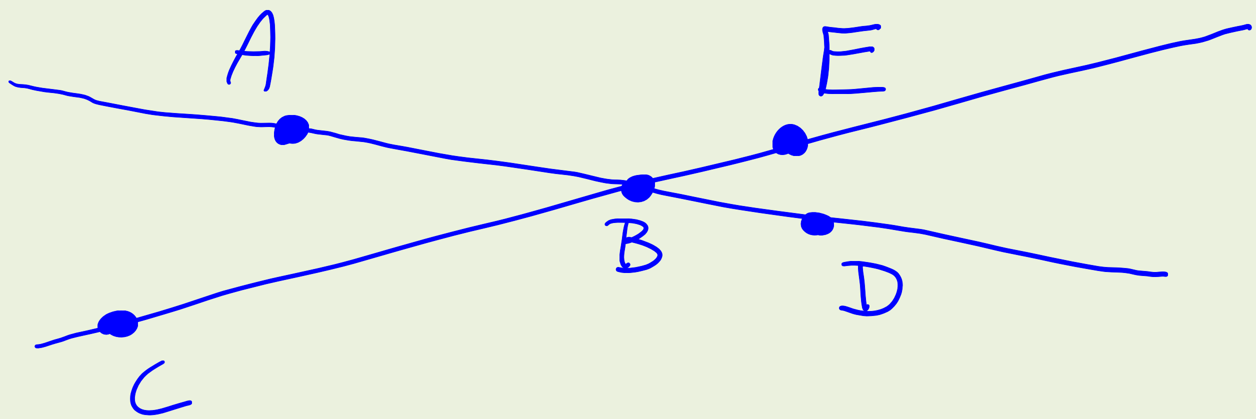
Vertical Angles (Bowtie Angles)

Definition of Vertical Pair (Bowtie Pair)

Words: *Two angles from a vertical pair (bowtie pair).*

Meaning: The two angles can be labeled $\angle ABC$ and $\angle DBE$ with $A - B - D$ and $C - B - E$.

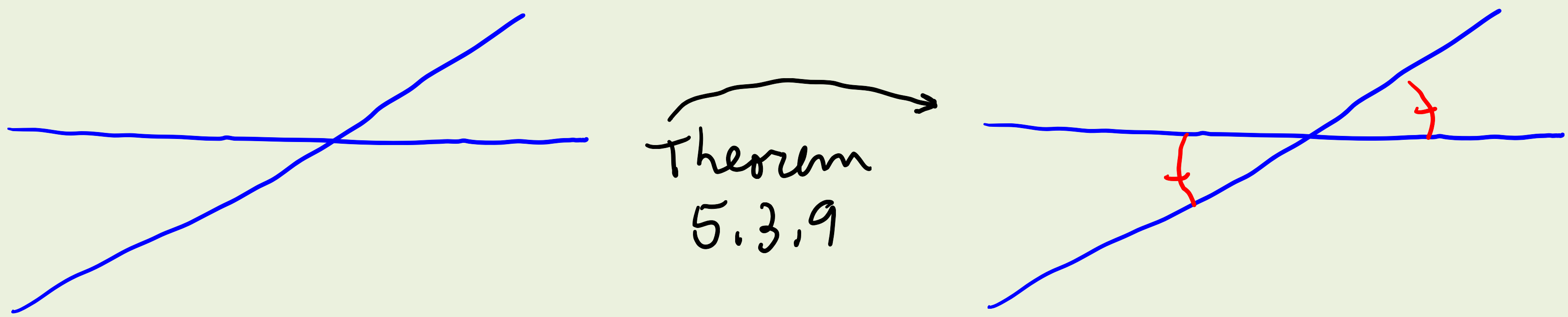
Illustration:



Theorem 5.3.9 (Vertical Angle (Bowtie Angle) Theorem)

In a protractor geometry,

if two angles form a vertical pair (a bowtie pair), then they are congruent.



You will Prove Theorem 5.3.9 in a homework exercise.

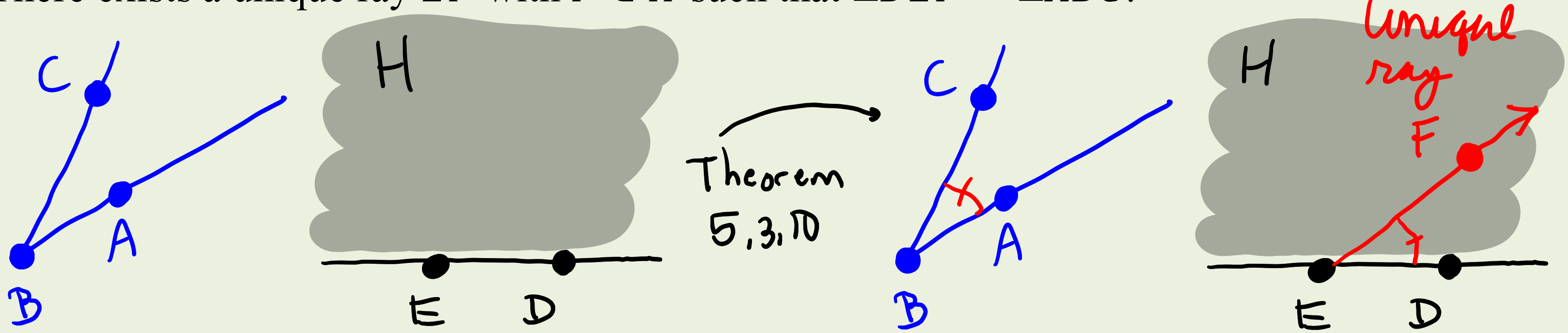
The following theorem is called the *Angle Construction Theorem* in the book. That name is bad on two counts:

- We already have an *axiom* called the *Angle Construction Axiom*.
- The theorem claims that something *exists*, so it should be called an *existence theorem*. (For that matter, the *angle construction axiom* really should be called the *angle existence axiom*.)

Furthermore, the proof of the theorem is so short that the theorem really ought to be called a corollary of the angle measurement axiom and the definition of congruent angles.

Theorem 5.3.10 (Congruent Angle Existence Theorem)

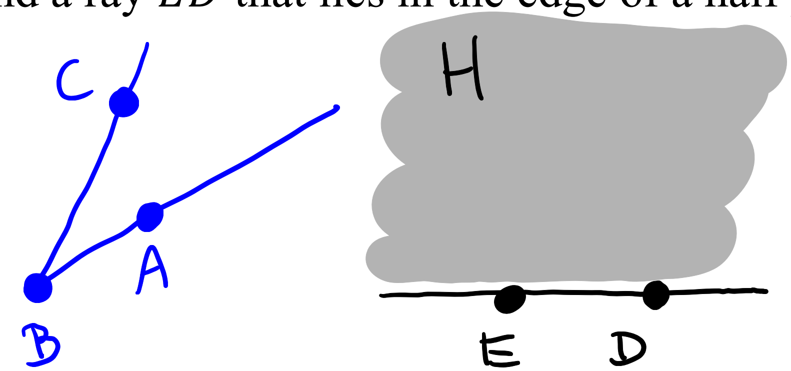
In a protractor geometry, given $\angle ABC$ and a ray $\overrightarrow{ED}$ that lies in the edge of a half plane H, There exists a unique ray $\overrightarrow{EF}$ with $F \in H$ such that $\angle DEF \simeq \angle ABC$.



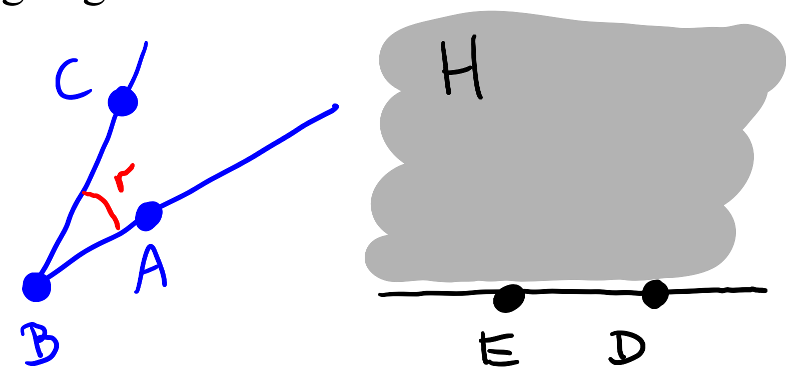
You will prove Theorem 5.3.10 in a homework exercise.

Proof Strategy

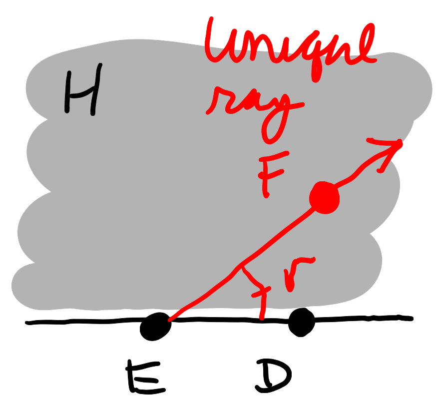
Given an $\angle ABC$ and a ray $\overrightarrow{ED}$ that lies in the edge of a half plane H,



Measure the existing angle $\angle ABC$. Call its measure r.



Show that there exists a unique ray $\overrightarrow{EF}$ with $F \in H$ such that $m(\angle DEF) = r$.



Two more theorems presented in the book could also be considered corollaries of the angle measurement axiom, some earlier theorems from Section 5.1, and the definition of congruent angles.

Theorem 5.3.11 (Congruent Angle Addition Theorem)

In a protractor geometry,

if $D \in int(\angle ABC)$ and $S \in int(\angle PQR)$ and $\angle ABD \simeq \angle PQS$ and $\angle DBC \simeq \angle SQR$,

then $\angle ABC \simeq \angle PQR$.

Theorem 5.3.12 (Congruent Angle Subtraction Theorem)

In a protractor geometry,

if $D \in int(\angle ABC)$ and $S \in int(\angle PQR)$ and $\angle ABD \simeq \angle PQS$ and $\angle ABC \simeq \angle PQR$,

then $\angle DBC \simeq \angle SQR$.

You will prove these theorems in homework exercises.

End of Video.