Video 6.1a: Triangle Congruence, The Side-Angle-Side Axiom, Neutral Geometry

produced by Mark Barsamian, 2021.03.26

for Ohio University MATH 3110/5110 College Geometry

Topics

- **Definition of Triangle Congruence**
- **Desirable Triangle Congruence Behavior**
- **Triangle Congruence Axioms**
- **Neutral Geometry**
- **Digression About the Names of Axioms and Theorems**
- **First Proofs about Congruences in Neutral Geometry**

Reading: Page 124 – top of page 128 of Section 6.1 The Side-Angle Side Axiom
in *Geometry: A Metric Approach with Models, Second Edition* by Millman & Parker

Homework: Section 6.1 # 1, 2, 8, 9

Definition of Triangle Congruence

When comparing drawn triangles, one of the things that we can do is slide one drawn triangle on top of another and seeing if they fit. When they do fit, all of the sides are the same size and all of the angles are the same size. (That is, all of the corresponding parts are the same size.) We want to have an analogous notion of being able to compare triangles in axiomatic geometry. That is the idea of *triangle congruence*.

In order to formulate a definition of triangle congruence in an axiomatic geometry, it helps to have a notion of corresponding parts of triangles.

Definition of Corresponding Parts of Two Triangles

Let ΔABC and ΔDEF be two triangles in a protractor geometry, and let $f:\{A,B,C\} \to \{D,E,F\}$ be a bijection from the set of vertices of ΔABC to the set of vertices of ΔDEF. Then associated to the bijection f is an automatic correspondence of six pairs of parts of the two triangles.

Segment $\overline{AB}$ of ΔABC corresponds to segment $\overline{f(A)f(B)}$ of ΔDEF.

Segment $\overline{BC}$ of ΔABC corresponds to segment $\overline{f(B)f(C)}$ of ΔDEF.

Segment $\overline{CA}$ of ΔABC corresponds to segment $\overline{f(A)f(A)}$ of ΔDEF.

Angle $\angle ABC$ of ΔABC corresponds to angle $\angle f(A)f(B)f(C)$ of ΔDEF.

Angle $\angle BCA$ of ΔABC corresponds to angle $\angle f(B)f(C)f(A)$ of ΔDEF.

Angle $\angle CAB$ of ΔABC corresponds to angle $\angle f(C)f(A)f(B)$ of ΔDEF.

Definition of Congruence between Triangles

Words: *A congruence between* ΔABC *and* ΔDEF

Usage: ΔABC and ΔDEF are two triangles in a protractor geometry,

Meaning: A bijection $f: \{A, B, C\} \to \{D, E, F\}$ from the set of vertices of ΔABC to the set of vertices of ΔDEF such that each pair of corresponding parts is congruent. That is,

$$\overline{AB} \simeq \overline{f(A)f(B)}$$

$$\overline{BC} \simeq \overline{f(B)f(C)}$$

$$\overline{CA} \simeq \overline{f(C)f(A)}$$

$$\angle ABC \simeq \angle f(A)f(B)f(C)$$

$$\angle BCA \simeq \angle f(B)f(C)f(A)$$

$$\angle CAB \simeq \angle f(C)f(A)f(B)$$

Definition of Congruent Triangles

Words: ΔABC *and* ΔDEF *are congruent.*

Usage: ΔABC and ΔDEF are two triangles in a protractor geometry,

Meaning: There exists a congruence between ΔABC and ΔDEF.

Definition of Symbol to Indicate a Particular Congruence

Symbol: $\Delta ABC \simeq \Delta DEF$

Usage: ΔABC and ΔDEF are two triangles in a protractor geometry,

Meaning: The particular bijection $f:\{A,B,C\} \to \{D,E,F\}$ defined by

$$f(A) = (D), f(B) = (E), f(C) = (F)$$

is a congruence.

Subtlety in the $\simeq$ symbol

Observe that if $\Delta ABC \simeq \Delta DEF$, then ΔABC and ΔDEF are congruent. But realize that the symbol

$$\Delta ABC \simeq \Delta DEF$$

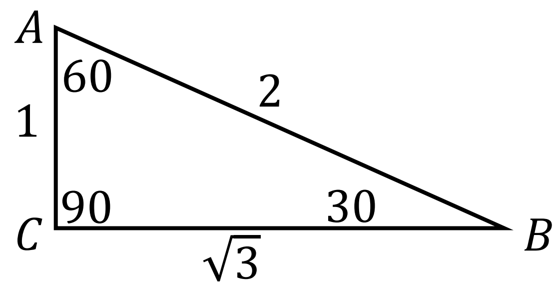
means *more* than the sentence

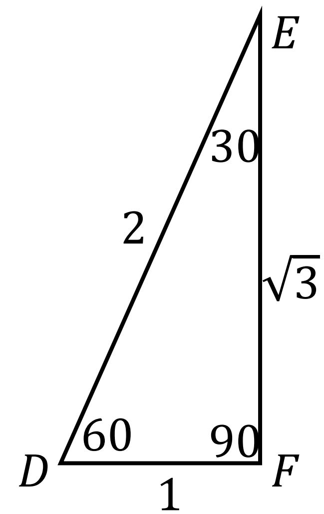
$$\Delta ABC \text{ and } \Delta DEF \text{ are congruent}$$

That is, it is possible for ΔABC and ΔDEF to be congruent and yet have $\Delta ABC \not\simeq \Delta DEF$.

[Example 1]

Consider the triangles shown.





The statement "ΔABC is congruent to ΔDEF" is true.

Proof: The function $(A, B, C) \longmapsto (D, E, F)$ is a congruence.

The statement "$\Delta ABC \simeq \Delta DEF$" is true.

Proof: The function $(A, B, C) \longmapsto (D, E, F)$ is a congruence.

The statement "ΔABC is congruent to ΔDFE" is true.

Proof: The function $(A, B, C) \longmapsto (D, E, F)$ is a congruence. This is the same congruence from the previous example. Since there exists a congruence, we say that the triangles are congruent.

The statement "$\Delta ABC \simeq \Delta DFE$" is false, because he function $(A, B, C) \longmapsto (D, F, E)$ is not a congruence. Observe that $m(\angle ABC) = 30$ while the corresponding angle has $m(\angle DFE) =$ ~~60~~ 90.

End of [Example 1]

[Example 2] Verifying a Congruence

In your homework exercise 6.1#2, you are are given four points in $\mathbb{H}$

$$A = (0,1), B = (0,2), C = (0,4), D = \left(1, \sqrt{3}\right)$$

and you are asked to prove that for Poincaré triangles ΔABD and ΔCBD, the following is true:

$$\Delta ABD \simeq \Delta CBD$$

This will entail verifying that

$$\overline{AB} \simeq \overline{CB}$$

$$\overline{BD} \simeq \overline{BD}$$

$$\overline{DA} \simeq \overline{DC}$$

$$\angle ABD \simeq \angle CBD$$

$$\angle BDA \simeq \angle BDC$$

$$\angle DAB \simeq \angle DCB$$

That is six congruences to check.

It will be a tedious exercise.

End of [Example 2]

Equivalence

Triangle Congruence Is An Equivaence Relation

In your homework, you will prove the following

> **Theorem (Exercise 6.1#1)**
>
> In a protractor geometry, congruence is an equivalence relation on the set of all triangles

Remember the proof structure that will be required.

You will need to prove that congruence is, *reflexive* using the following kind of proof structure:

Proof that congruence is reflexive

Suppose that a triangle ΔABC is given

Some steps here

Therefore the triangle is congruent to itself

End of proof

Realize to show that triangle ΔABC is congruent to itself entails showing that the requirements of the definition of triangle congruence are satisfied. That is, one must provide an example of a bijection $f: \{A, B, C\} \rightarrow \{A, B, C\}$ and show that it is a congruence.

You will need to prove that congruence is *symmetric*, using thefollowing kind of proof structure:

Proof that congruence is symmetric

Suppose that $\Delta ABC$is congruent to ΔDEF

Some steps here

Therefore a $\Delta DEF$is congruent to ΔABC

End of proof

Again, realize to show that $\Delta DEF$is congruent to ΔABC entails showing that the requirements of the definition of triangle congruence are satisfied. That is, one must provide an example of a bijection $g: \{D, E, F\} \rightarrow \{A, B, C\}$ and show that it is a congruence.

In order to come up with an example of a bijection g that works, you will need to use the given fact that $\Delta ABC$is congruent to ΔDEF. This is a defined statement. You will need to unpack that defined statement, follow it with a statement that expresses what it really means. This unpacked statement will involve the existence of a bijection $f: \{A, B, C\} \rightarrow \{A, B, C\}$ that is a congruence. Hint: Use that known bijection f in some way to define the bijection g that you need.

You will need to prove that congruence is *transitive*, using thefollowing kind of proof structure:

Proof that congruence is transitive

Suppose that $\Delta ABC$is congruent to ΔDEF and that $\Delta DEF$is congruent to ΔGHI

Some steps here

Therefore a $\Delta ABC$is congruent to ΔGHI

End of proof

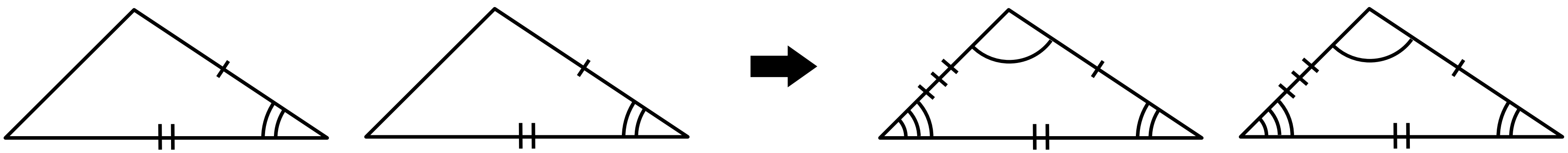
Desirable Congruence Behavior

When comparing drawn triangles, we know that if enough parts of one drawing fit on top of the corresponding parts of another drawing, then all of the other parts will fit, as well.

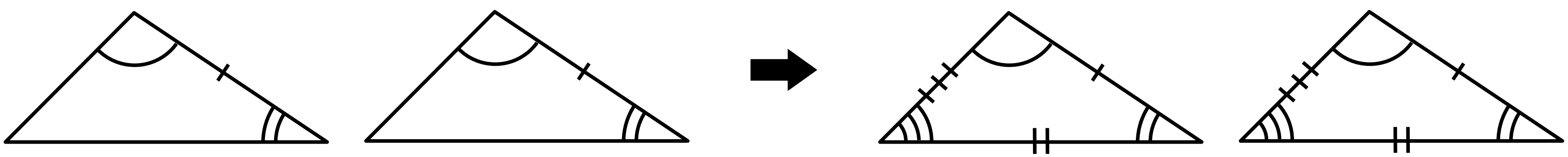
This can be said more precisely. To determine whether or not two drawn triangles fit on top of each other, one would officially have to verify that every pair of corresponding drawn line segments fit on top of each other and also that every pair of corresponding drawn angles fit on top of each other. That is total of six fits that must be checked. But we know that with drawings, one does not really need to check all six fits. Certain combinations of three fits are enough.

Here are four examples of sets of three fits that will guarantee that two drawn triangles will fit perfectly on top of each other:

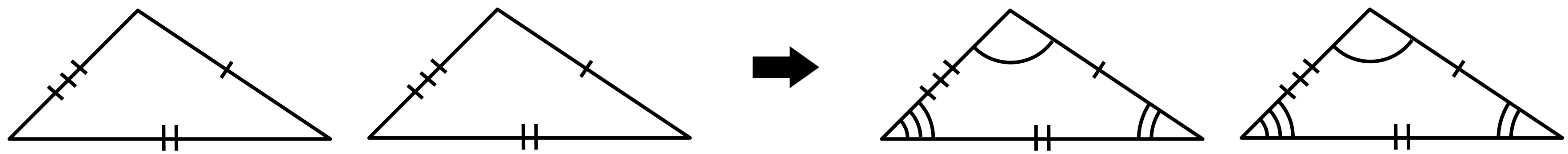
(1) If two sides and the included angle of the first drawn triangle fit on top of the corresponding parts of the second drawn triangle, then all the remaining corresponding parts always fit, as well.



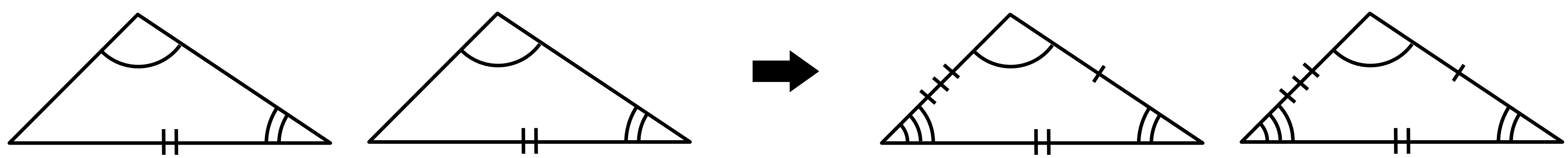
(2) If two angles and the included side of the first drawn triangle fit on top of the corresponding parts of the second drawn triangle, then all the remaining corresponding parts always fit, as well.



(3) If all three sides of the first drawn triangle fit on top of the corresponding parts of the second drawn triangle, then all the remaining corresponding parts always fit, as well.



(4) If two angles and some non-included side of the first drawn triangle fit on top of the corresponding parts of the second drawn triangle, then all the remaining corresponding parts always fit, as well.



We would like our axiomatic geometry line segment congruence and angle congruence and triangle congruence to have this same sort of behavior. We could refer to it as *desirable triangle congruence behavior*.

Desirable Triangle Congruence Behavior

(1) If two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle, then it should be that all the remaining corresponding parts are congruent, as well (so the triangles are congruent).

(2) If two angles and the included side of the first drawn triangle are congruent to the corresponding parts of the second triangle, then it should be that all the remaining corresponding parts are congruent, as well (so the triangles are congruent).

(3) If all three sides of the first triangle are congruent to the corresponding parts of the second triangle, then it should be that all the remaining corresponding parts are congruent, as well (so the triangles are congruent).

(4) If two angles and some non-included side of the first triangle are congruent to the corresponding parts of the second triangle, are congruent to the corresponding parts of the second triangle, then it should be that all the remaining corresponding parts are congruent, as well (so the triangles are congruent).

An obvious question is

Question: Does every protractor geometry have the *desirable triangle congruence behavior*?

[Example 3]

In the Taxicab plane let

$$A = (-1,1), B = (0,0), C = (1,1)$$

and let

$$D = (2,2), E = (2,0), F = (4,0)$$

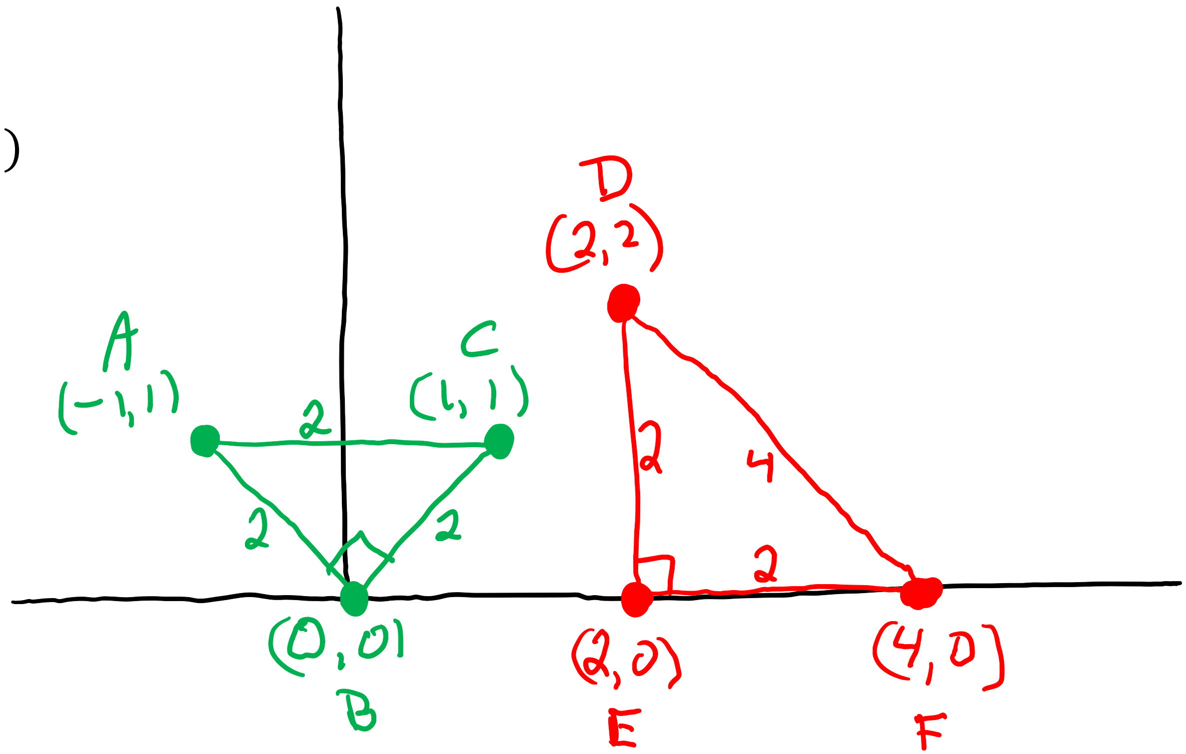
Observe that

$$\overline{AB} \simeq \overline{DE}$$

$$\angle ABC \simeq \angle DEF$$

$$\overline{BC} \simeq \overline{EF}$$

But $\overline{CA} \not\simeq \overline{FD}$



End of [Example 3]

So the Taxicab plane (which is a protractor geometry) *does not* have the *desirable triangle congruence behavior*. That answers our question.

Question: Does every protractor geometry have the *desirable triangle congruence behavior*?

Answer: No, because there is an example of a protractor geometry that does not.

Triangle Congruence Axioms

We would like to focus on protractor geometries that *do* have the *desirable triangle congruence behavior*. But if we want them to have that behavior, we must require it by specifying that behavior in some additional axioms.

Definition of the Side-Angle-Side Axiom

Words: *A protractor geometry satisfies the Side-Angle-Side (SAS) Axiom.*

Meaning: If there is a bijection between the vertices of two triangles, and two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle, then all the remaining corresponding parts are congruent as well, so the bijection is a congruence and the triangles are congruent.

Definition of the Angle-Side-Angle Axiom

Words: *A protractor geometry satisfies the Angle-Side-Angle (ASA) Axiom.*

Meaning: If there is a bijection between the vertices of two triangles, and two angles and the included side of one triangle are congruent to the corresponding parts of the other triangle, then all the remaining corresponding parts are congruent as well, so the bijection is a congruence and the triangles are congruent.

Definition of the Side-Side-Side Axiom

Words: *A protractor geometry satisfies the Side-Side-Side (SSS) Axiom.*

Meaning: If if there is a bijection between the vertices of two triangles, and the three sides of one triangle are congruent to the corresponding parts of the other triangle, then all the remaining corresponding parts are congruent as well, so the bijection is a congruence and the triangles are congruent.

Definition of the Angle-Angle-Side Axiom

Words: *A protractor geometry satisfies the Angle-Angle-Side (AAS) Axiom.*

Meaning: If if there is a bijection between the vertices of two triangles, and two angles and a non-included side of one triangle are congruent to the corresponding parts of the other triangle, then all the remaining corresponding parts are congruent as well, so the bijection is a congruence and the triangles are congruent.

Neutral Geometry

One might think that it would be necessary to add all four axioms as requirements, to guarantee that the four kinds of *desirable triangle congruence behavior* will also be observed in axiomatic triangles. But the amazing thing is that we don't need to require that all four axioms be satisfied. We can require that just *one* axiom, about just *one kind* of *desirable behavior*, be satisfied. We can then prove *theorems* that show that triangles will *also* have the other three kinds of *desirable behavior*. That is the idea behind the definition of *neutral geometry*.

Definition of Neutral Geometry

A **neutral geometry** (or **absolute geometry**) is a protractor geometry that satisfies *SAS*.

In Chapter 6, we will prove three theorems about *desirable triangle congruence behavior*.

Theorem: Every neutral geometry satisfies *ASA*.

Theorem: Every neutral geometry satisfies *SSS*.

Theorem: Every neutral geometry satisfies *AAS*.

Note that the statements of the three theorems have been mentioned here just as an introduction to the coming material. The three theorems have not yet been proven and they do not yet have theorem numbers, so we may not yet use any of them in proofs. Soon, but not yet.

Digression about the names of theorems

Names of axioms and theorem often follow an informal convention: They are named for the situation described in their *hypotheses*.

For example, suppose two theorems are stated as follows.

Theorem 1: If the dog is blue, then the car is red.

Theorem 2: If the car is red, then the bear is hungry.

Following the naming convention,

Theorem 1 would be called “The Blue Dog Theorem”

Theorem 2 would be called “The Red Car Theorem”.

It is important to realize that "The Red Car Theorem" could never be used to prove that a car is red! The Red Car Theorem tells us something about the situation in which we *already know* that the car is red. (The theorem tells us that in that situation, the bear is hungry.) If we do not know that the car is red, and we want to prove that the car is red, then we will need a theorem that has the statement "the car is red" as part of the conclusion. We see that the Blue Dog Theorem would work. So one strategy for proving that the car is red would be to

> First prove somehow that the dog is blue.
> Then use the Blue Dog Theorem to prove that the car is red.

This has a direct bearing on the use of the *Side-Angle-Side (SAS) axiom*. The axiom is named for the situation in the *hypothesis* of the axiom. The *SAS axiom* could *never* be used to prove that two triangles have congruent corresponding Side, Angle, and Side. No, the *SAS axiom* tells us about the situation in which we *already know* that two triangles have congruent corresponding Side, Angle, and Side. (The axiom tells us that in that situation, the triangles are *congruent*.)

First Proofs about Congruences in Neutral Geometry

We’ll consider two examples of the form

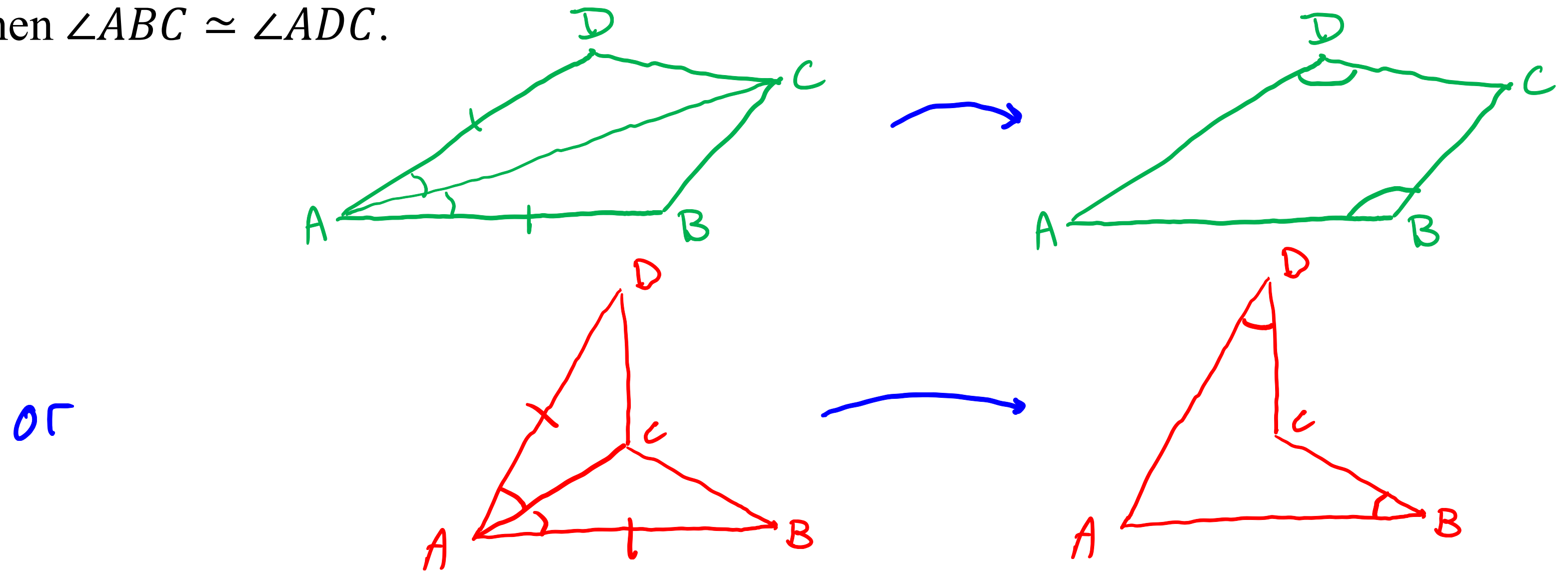
In neutral geometry, given some configuration of objects with some congruent objects, prove that some other objects are congruent.

Since, at the current time, our only available tool for proving that anything is congruent is the Side-Angle-Side axiom, our only possible approach is the following.

- *In neutral geometry, given some configuration of objects with some congruent objects.*
- *Identify two triangles whose parts include the other objects and that are known (or can be shown) to have congruent Side-Angle-Side.*
- *Use the Side-Angle-Side axiom to justify saying that the two triangles are congruent (which means that all of their corresponding parts are congruent).*
- *Therefore, the other objects are congruent.*

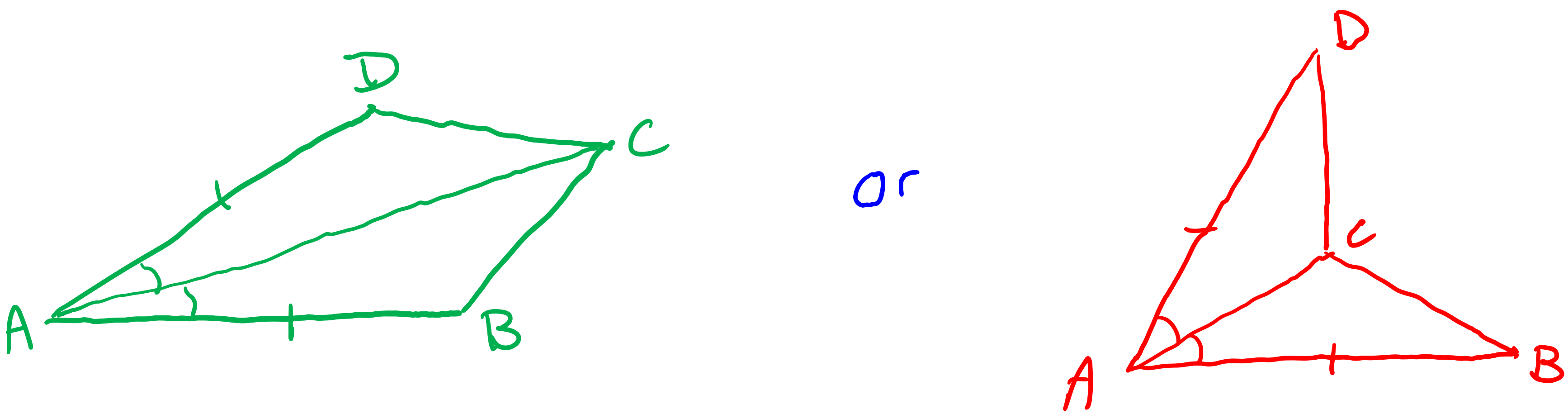
[Example 4] Prove the following:

In Neutral Geometry, given quadrilateral $\square ABCD$ such that $\overline{AB} \simeq \overline{AD}$ and that $\overrightarrow{AC}$ is the bisector of angle $\angle BAD$, then $\angle ABC \simeq \angle ADC$.

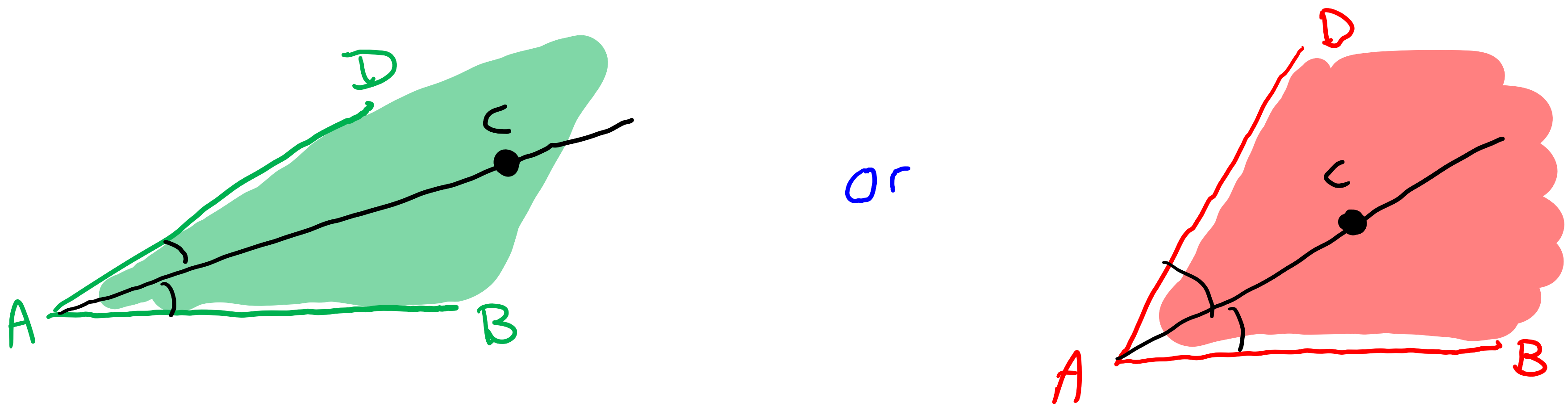


Proof

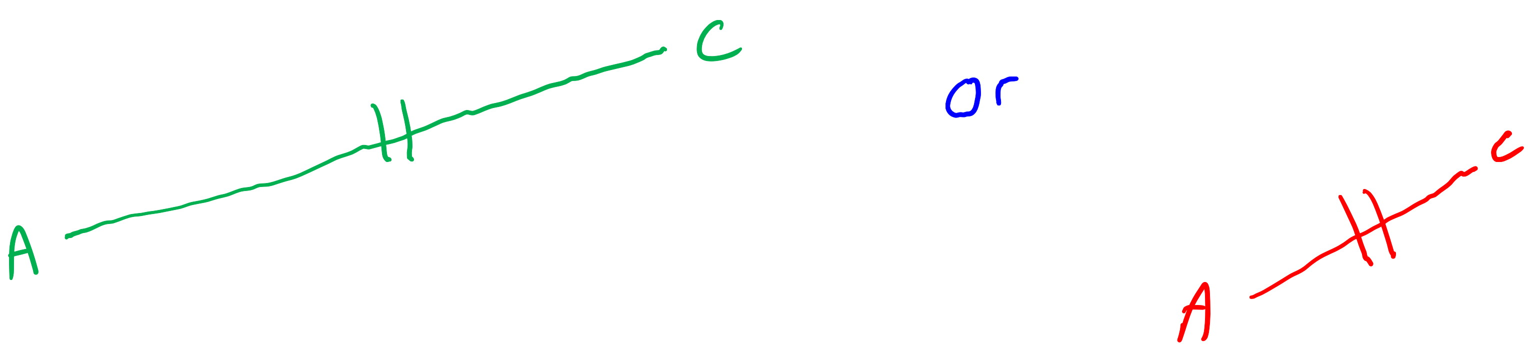
(1) In Neutral Geometry, suppose that quadrilateral $\square ABCD$ has $\overline{AB} \simeq \overline{AD}$ and that $\overrightarrow{AC}$ is the bisector of angle $\angle BAD$.



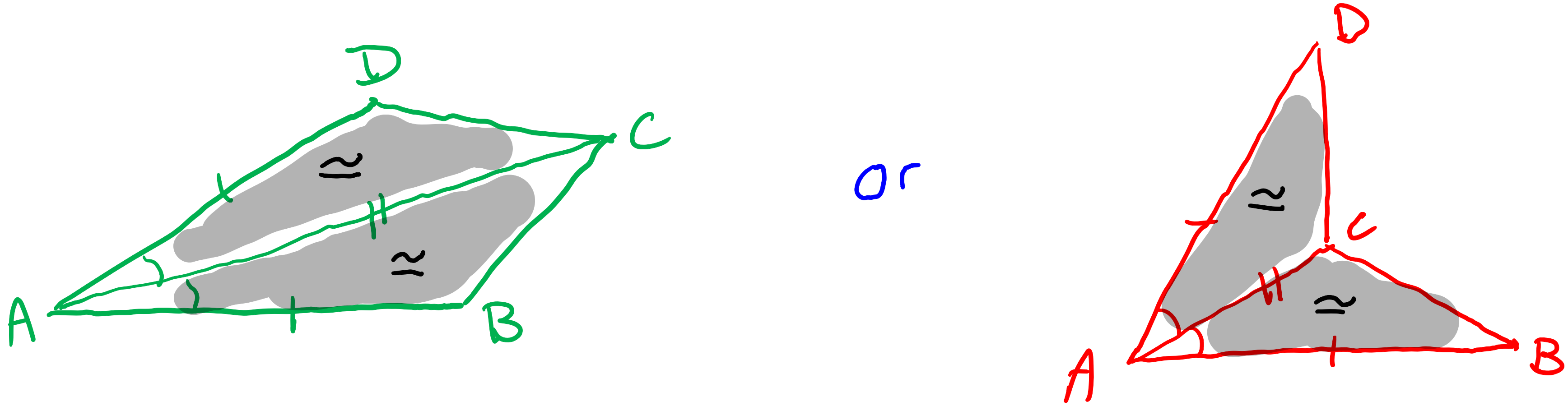
(2) C is in the interior of angle $\angle BAD$ and $\angle CAB \simeq \angle CAD$. (by (1) and definition of angle bisector)



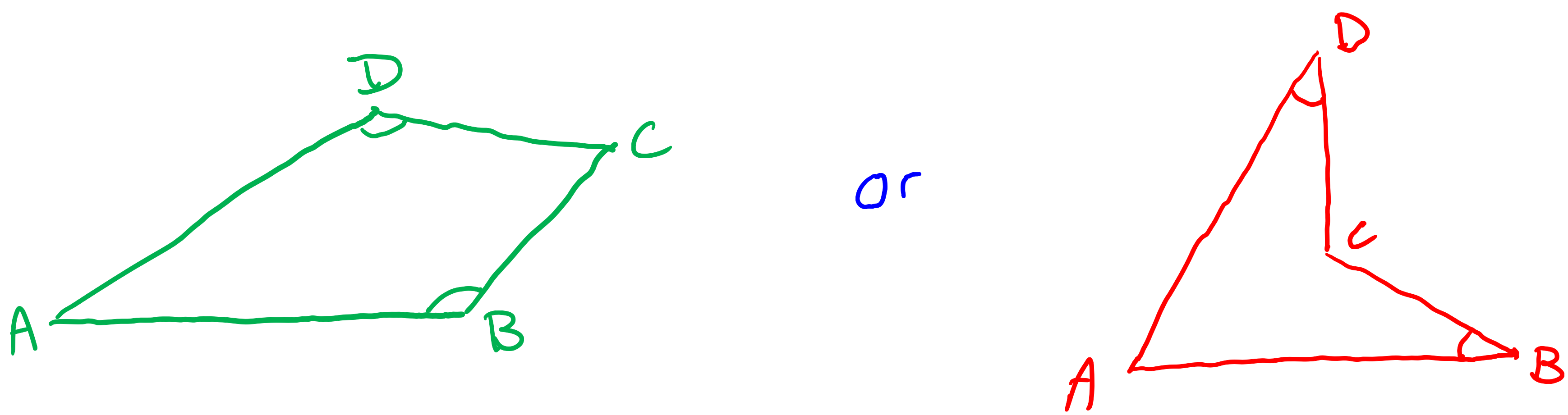
(3) $\overline{AC} \simeq \overline{AC}$ (by reflexivity of segment congruence. More simply, the segment is congruent to itself because it has the same length as itself.)



(4) $\Delta CAB \simeq \Delta CAD$ (by the Side-Angle-Side (SAS) axiom, using the fact that $\overline{AB} \simeq \overline{AD}$ and $\angle CAB \simeq \angle CAD$ and $\overline{AC} \simeq \overline{AC}$, from statements (1),(2),(3))



(5) $\angle ABC \simeq \angle ADC$ (by (4) and definition of triangle congruence)

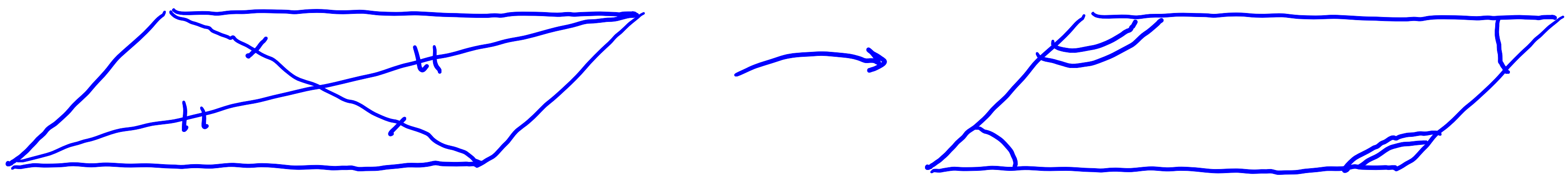


End of Proof

End of [Example 4]

[Example 5] Prove the following:

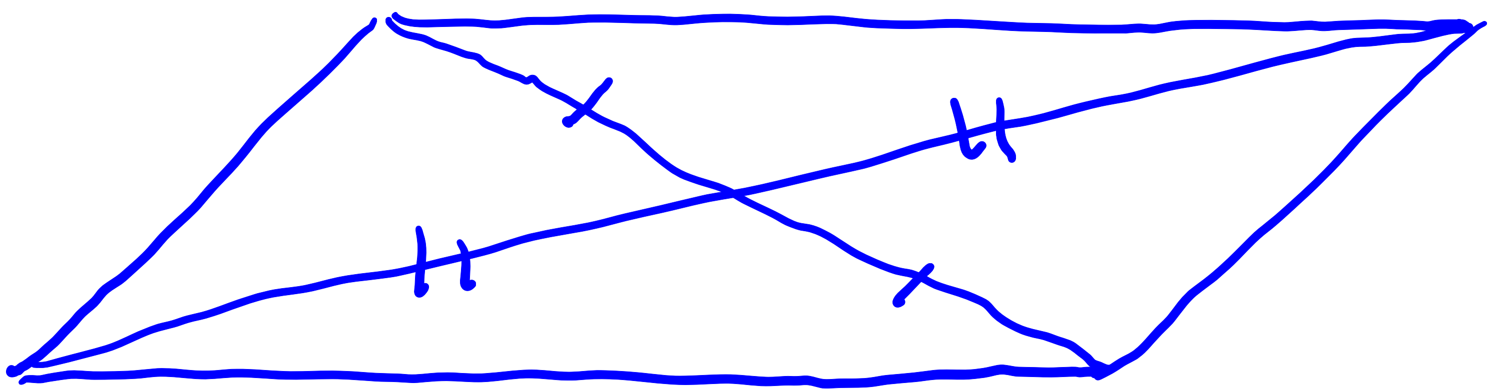
In Neutral Geometry, if the diagonals of a quadrilateral bisect each other, then the opposite angles of the quadrilateral are congruent.



Proof

(1) In Neutral Geometry, suppose that the diagonals of a quadrilateral bisect each other.

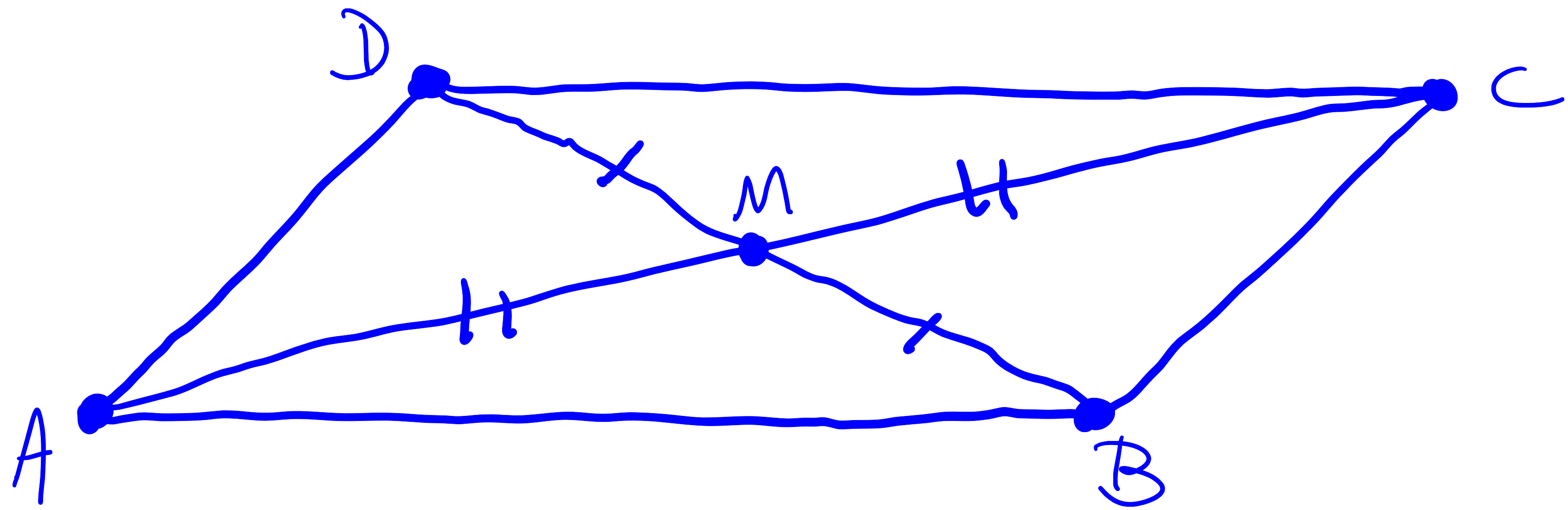
(2) The quadrilateral must be a convex quadrilateral. (In exercise 4.5#7, you proved that if the diagonals of a quadrilateral intersect, then the quadrilateral is convex.)



Part 1: Show that one pair of opposite angles is congruent

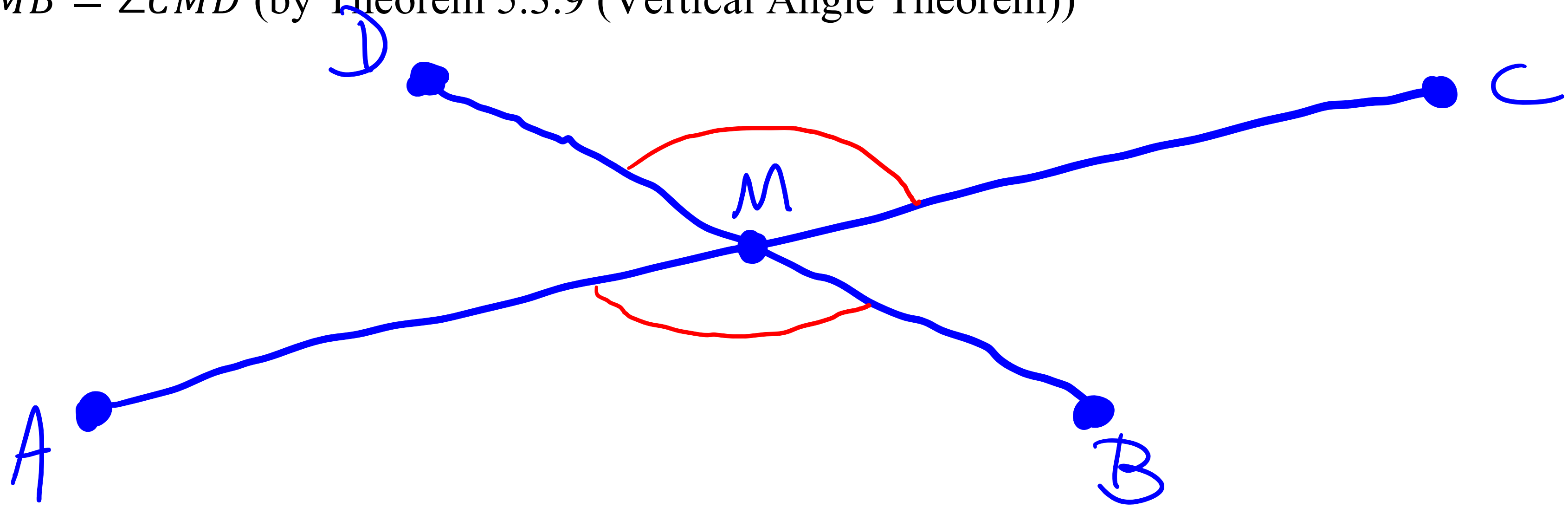
(3) The quadrilateral can be labeled $\square ABCD$. Then $\overline{AC}$ and $\overline{BD}$ intersect at a point M such that $\overline{AM} \simeq \overline{CM}$ and $\overline{BM} \simeq \overline{DM}$. (by (1) and definition of bisect)

Our goal will be to prove that $\angle BAD \simeq \angle DCB$.

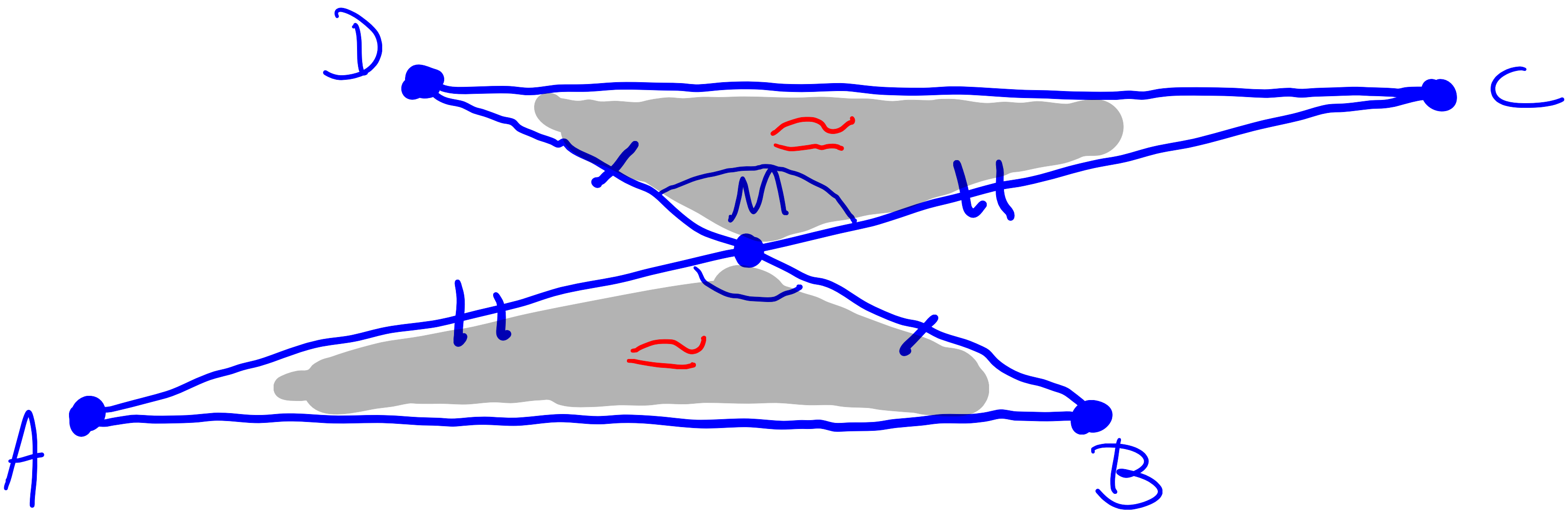


Show that $\angle CAB \simeq \angle ACD$

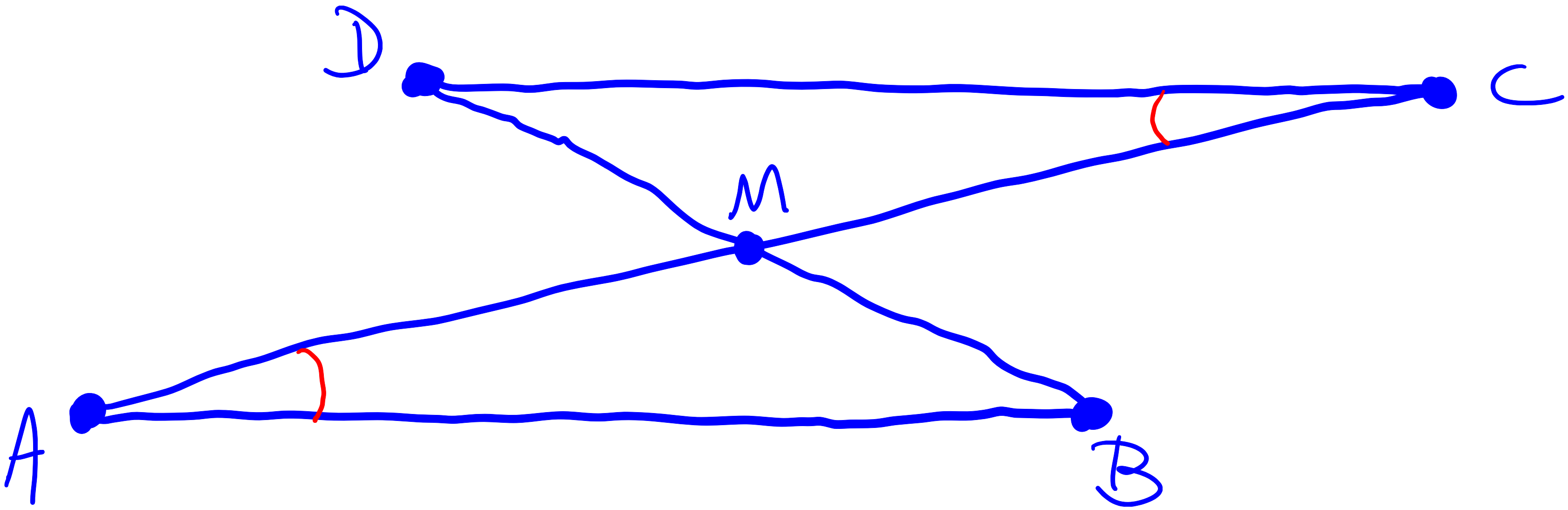
(4) $\angle AMB \simeq \angle CMD$ (by Theorem 5.3.9 (Vertical Angle Theorem))



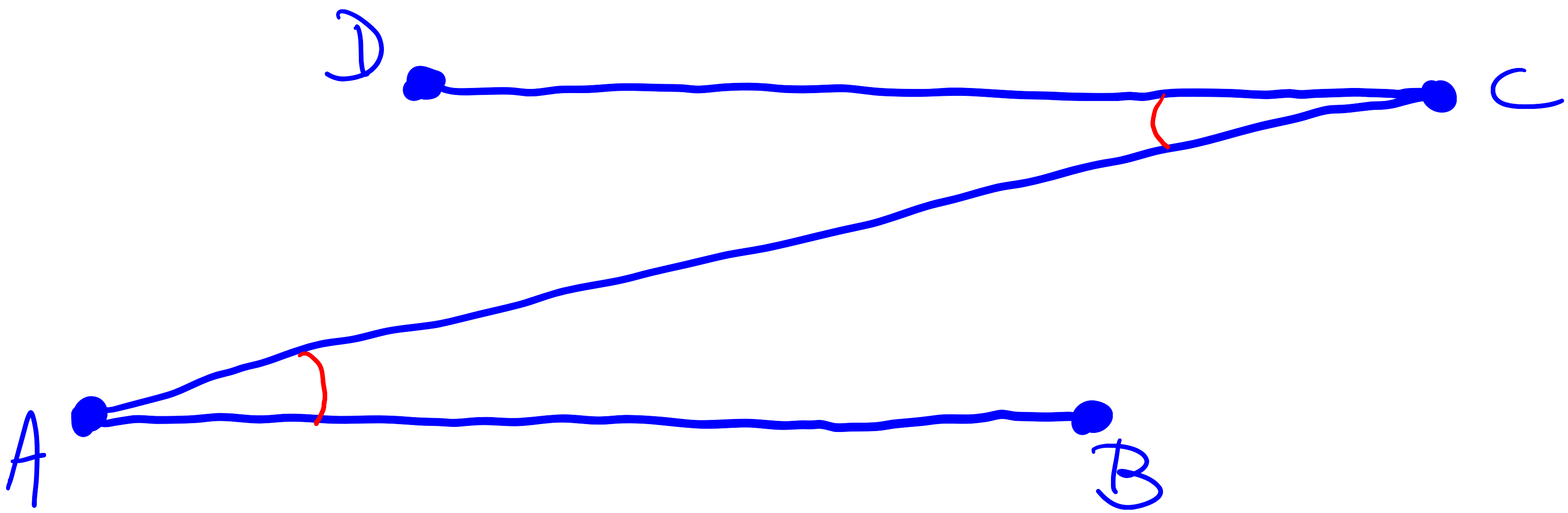
(5) $\Delta AMB \simeq \Delta CMD$ (by the SAS axiom, using the fact that $\overline{AM} \simeq \overline{CM}$ and $\angle AMB \simeq \angle CMD$ and $\overline{BM} \simeq \overline{DM}$, from statements (3),(4))



(6) $\angle MAB \simeq \angle MCD$ (by (5) and definition of triangle congruence)

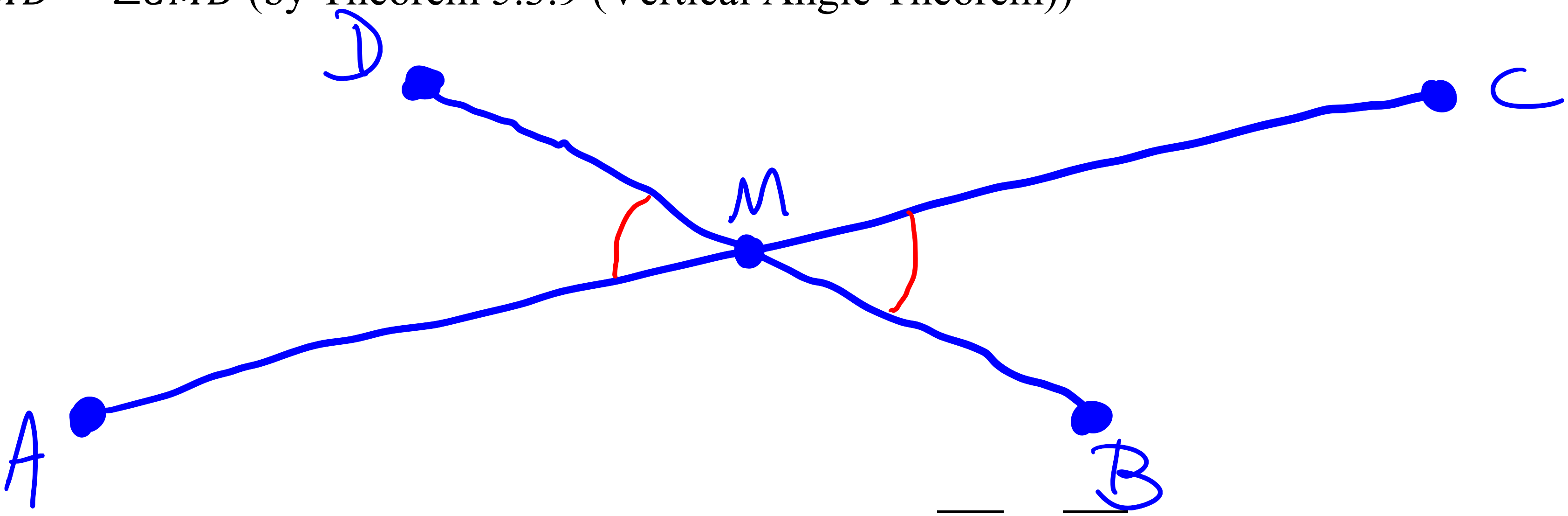


(7) $\angle CAB \simeq \angle ACD$ (because $\overrightarrow{AM}$ is the same ray as $\overrightarrow{AC}$, and $\overrightarrow{CM}$ is the same ray as $\overrightarrow{CA}$)

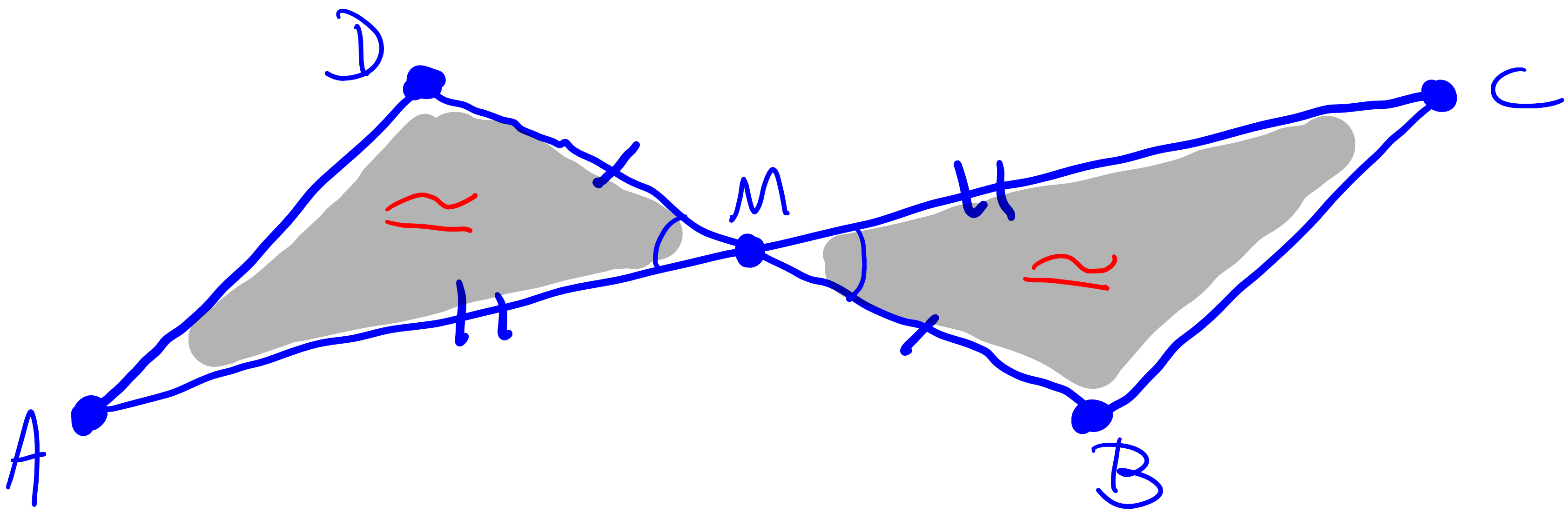


Show that $\angle CAD \simeq \angle ACB$

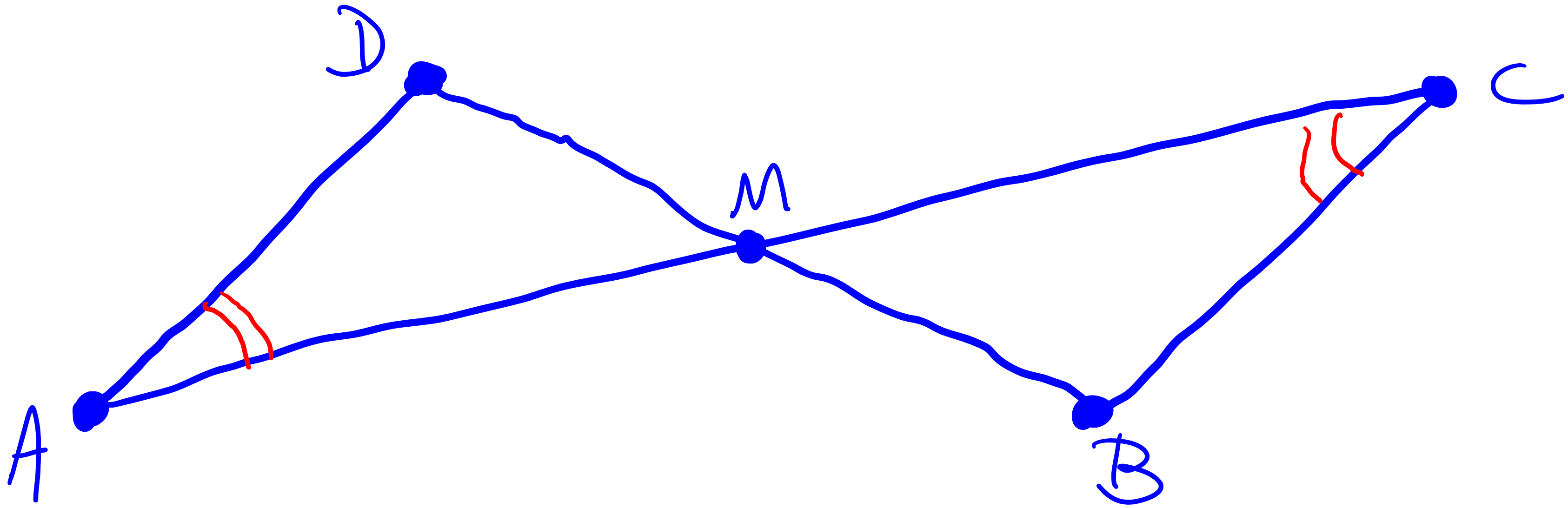
(8) $\angle AMD \simeq \angle CMB$ (by Theorem 5.3.9 (Vertical Angle Theorem))



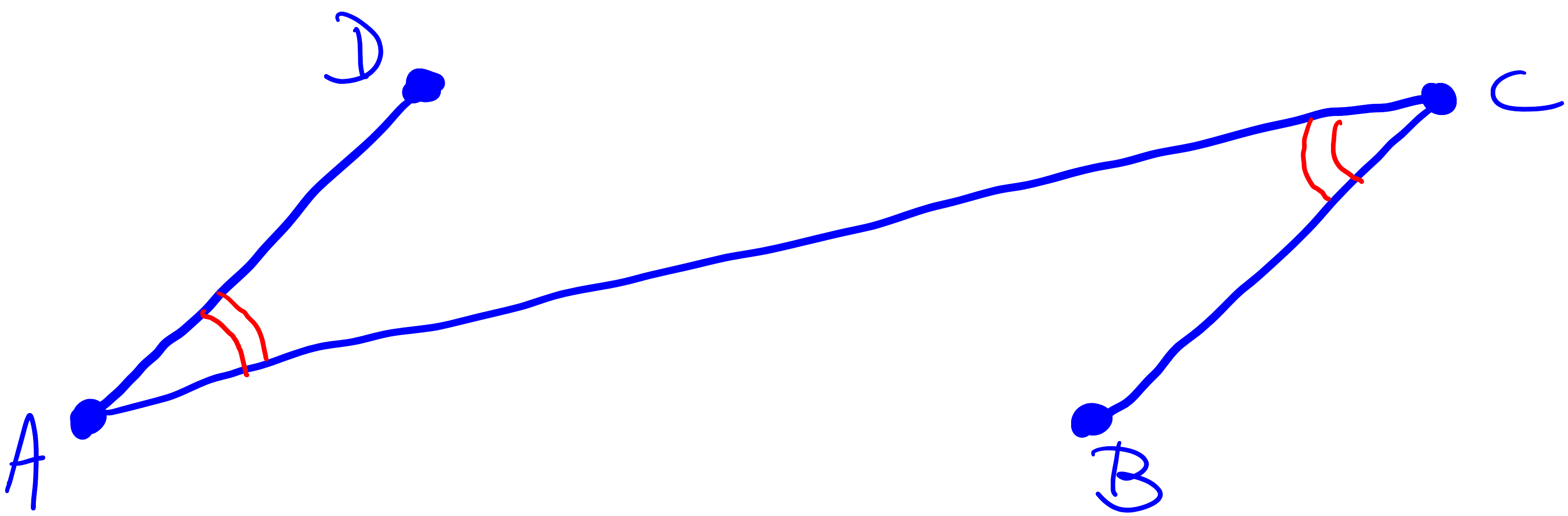
(9) $\Delta AMD \simeq \Delta CMB$ (by the SAS axiom, using the fact that $\overline{AM} \simeq \overline{CM}$ and $\angle AMD \simeq \angle CMB$ and $\overline{BM} \simeq \overline{DM}$, from statements (3),(8))



(10) $\angle MAD \simeq \angle MCB$ (by (9) and definition of triangle congruence)

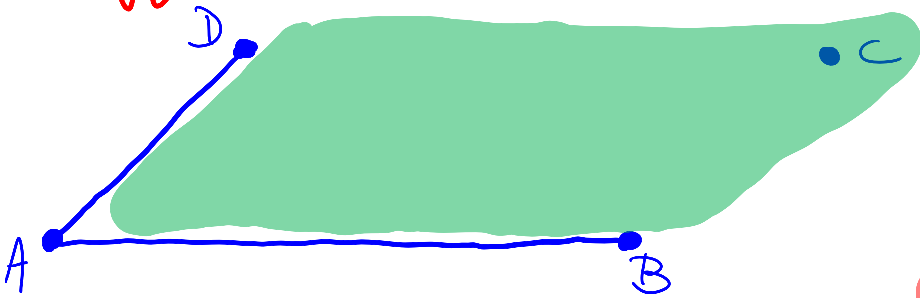


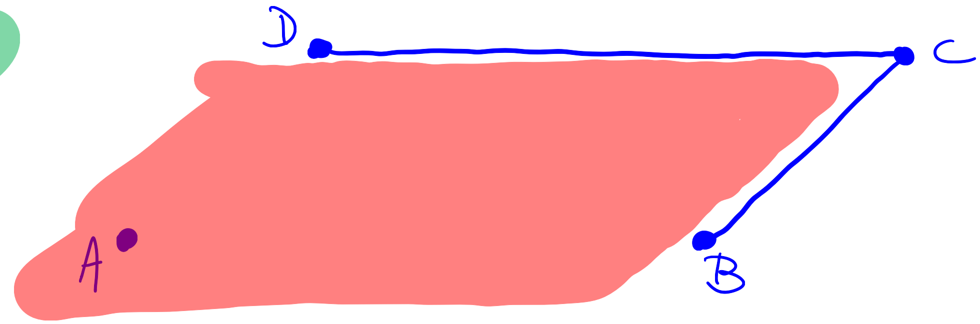
(11) $\angle CAD \simeq \angle ACB$ (because $\overrightarrow{AM}$ is the same ray as $\overrightarrow{AC}$, and $\overrightarrow{CM}$ is the same ray as $\overrightarrow{CA}$)



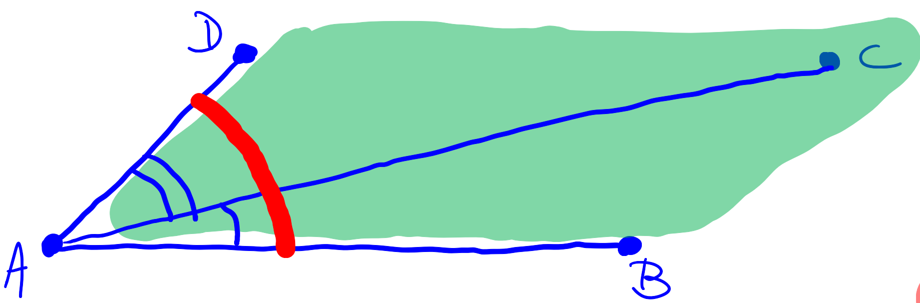
Show that $\angle BAD \simeq \angle DCB$

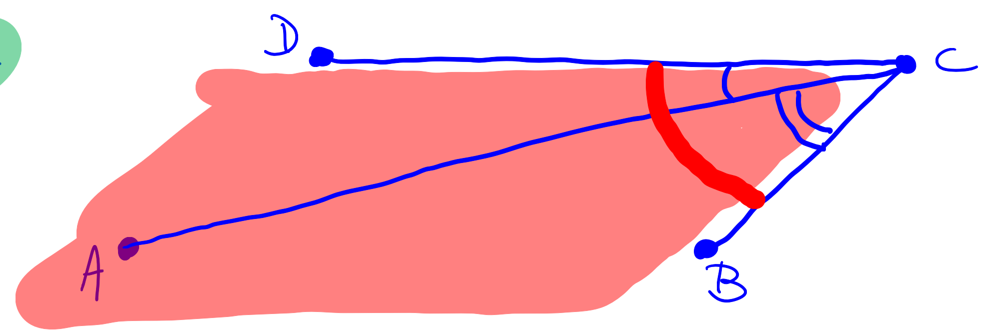
(12) $C \in \text{int}(\angle BAD)$ and $A \in \text{int}(\angle DCB)$. (by Theorem 4.5.4, a quadrilateral is convex if and only if each ~~angle~~ vertex is contained in the interior of the opposite angle.)





(13) $\angle BAD \simeq \angle DCB$ (by Theorem 5.3.11 the Congruent Angle Addition Theorem, applied to point $C \in \text{int}(\angle BAD)$ and $A \in \text{int}(\angle DCB)$ such that $\angle CAB \simeq \angle ACD$ and $\angle CAD \simeq \angle ACB$)

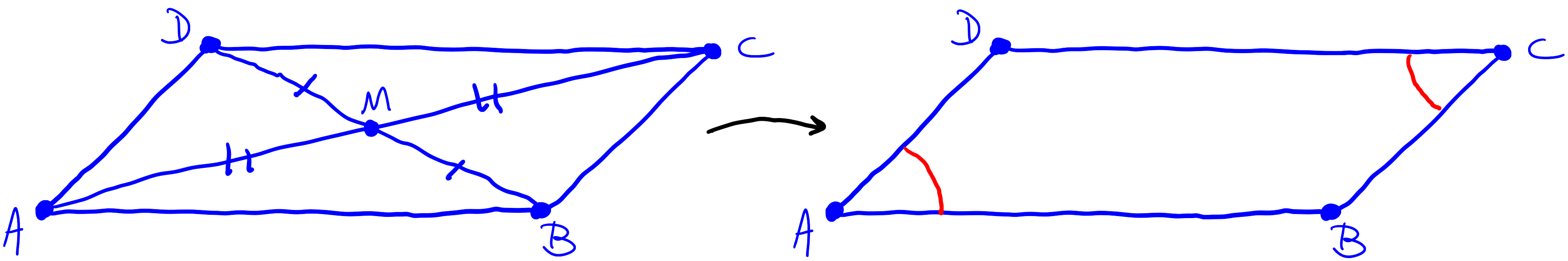




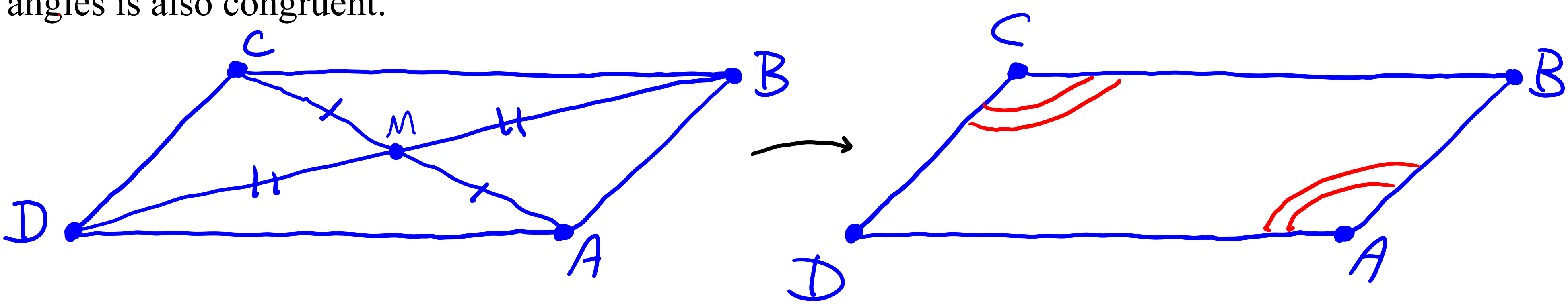
We have proven that one pair of opposite angles is congruent (End of Proof Part 1)

Proof Part 2: Generalize to the other pair of opposite angles

In Part 1, we showed that one pair of opposite angles is congruent. To do that, we chose letters A, B, C, D for the vertices of the quadrilateral, and showed that $\angle BAD \simeq \angle DCB$.



If we simply choose a different assignment of the letters A, B, C, D to the vertices of the quadrilateral, then our proof that $\angle BAD \simeq \angle DCB$ will serve to show that the other pair of opposite angles is also congruent.



End of [Example 5]

End of Video