Video 6.1b: The CS ➔ CA Theorem, Models of Neutral Geometry

produced by Mark Barsamian, 2021.04.02

for Ohio University MATH 3110/5110 College Geometry

Five Topics for this Video

- **Models of Neutral Geometry**
- **Facts of the Euclidean Plane**
- **The CS ➔ CA Theorem (Pons Asinorum)**
- **Using the CS ➔ CA Theorem to prove a fact about special rays in triangles**
- **Using the CS ➔ CA Theorem to prove a particular congruence**

Reading: Pages 128 – 129 of Section 6.1 The Side-Angle Side Axiom

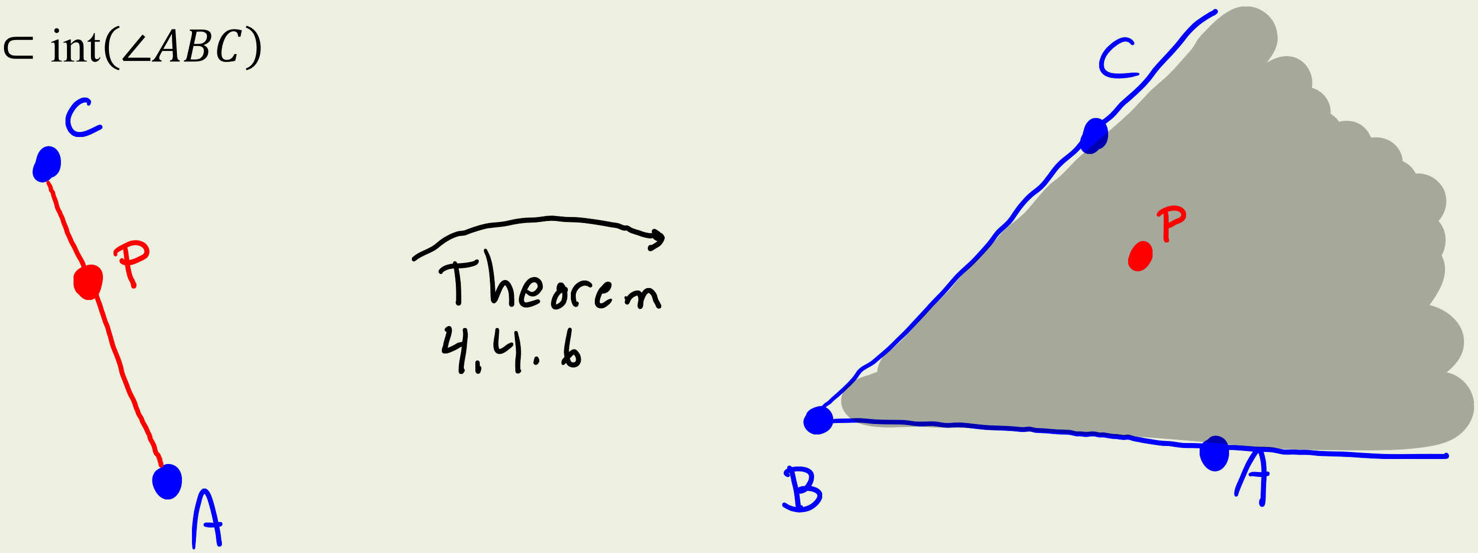
in *Geometry: A Metric Approach with Models, Second Edition* by Millman & Parker

Homework: Section 6.1 # 4, 5, 6, 7, 10, 12, 13

Stuff from Previous Sections that will be Needed in this Video

Theorem 4.4.6 Given ΔABC in a Pasch geometry, if $A - P - C$, then $P \in \text{int}(\angle ABC)$.

That is, int(~~$\overline{BC}$~~ $\overline{AC}$) $\subset$ int($\angle ABC$)

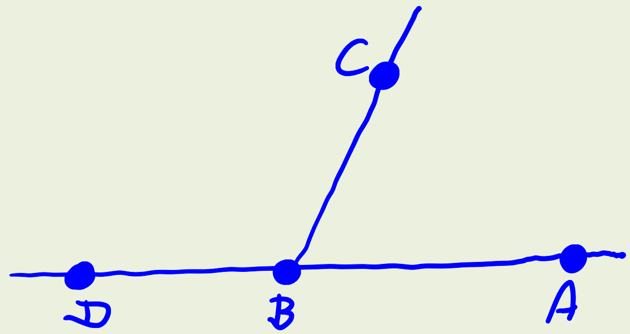


Definition of Linear Pair (from Section 5.3)

Words: *Two angles from a linear pair.*

Meaning: The two angles can be labeled $\angle ABC$ and $\angle CBD$ with $A - B - D$.

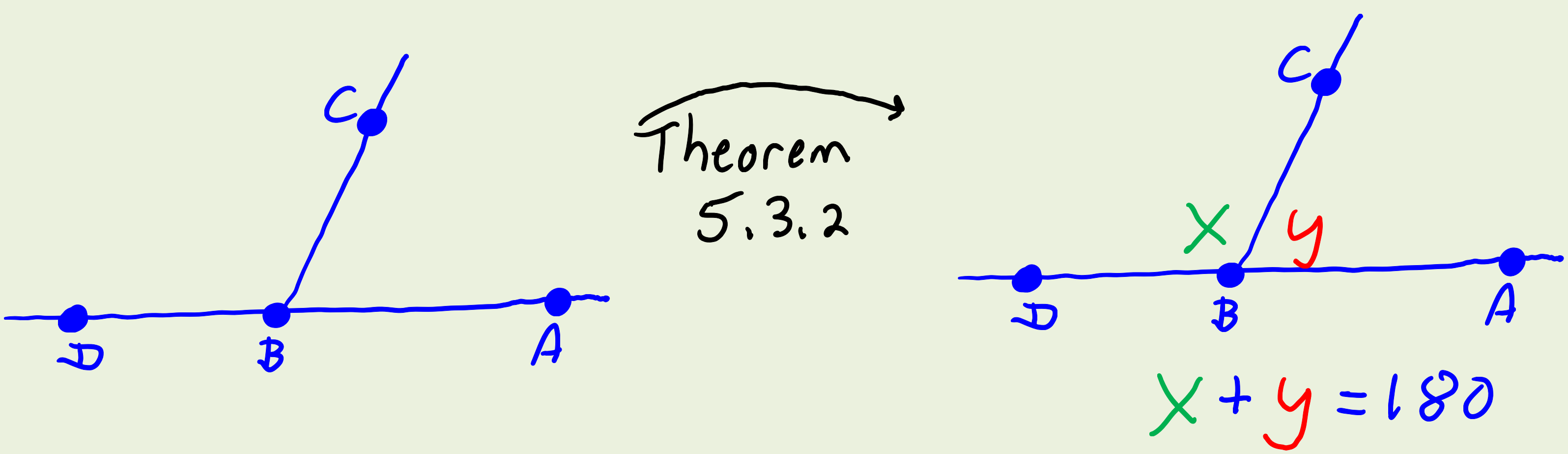
Illustration:



Theorem 5.3.2 The Linear Pair Theorem

In a protractor geometry,

if $\angle ABC$ and $\angle CBD$ form a linear pair, then $m(\angle ABC) + m(\angle ABC) = 180$

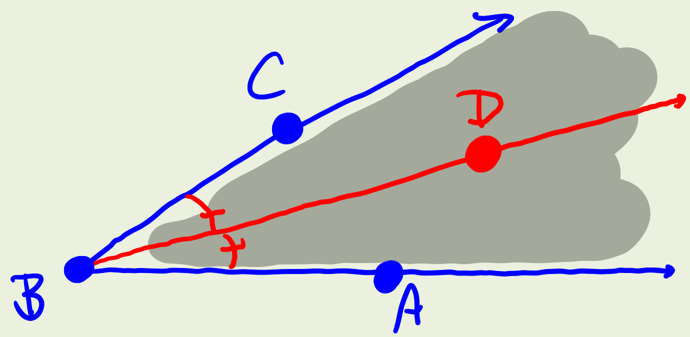


Definition of Angle Bisector (from Section 5.3)

Words: *a bisector of* $\angle ABC$

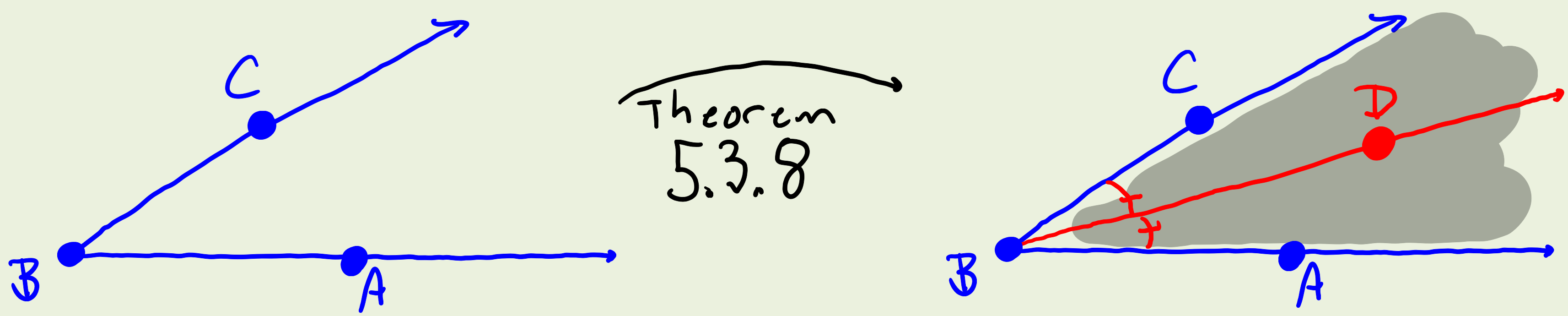
Usage: $\angle ABC$ is an angle in a protractor geometry

Meaning: a ray $\overrightarrow{BD}$ such that $D \in \text{int}(\angle ABC)$ and $m(\angle ABD) = m(\angle DBC)$.



Theorem 5.3.8 Existence of a Unique Angle Bisector

If $\angle ABC$ is an angle in a protractor geometry, then $\angle ABC$ has a *unique angle bisector*.

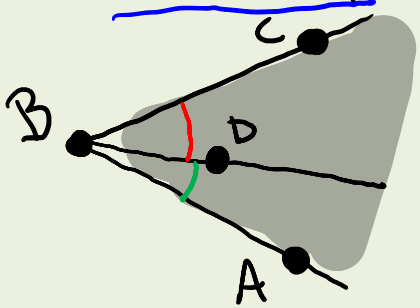


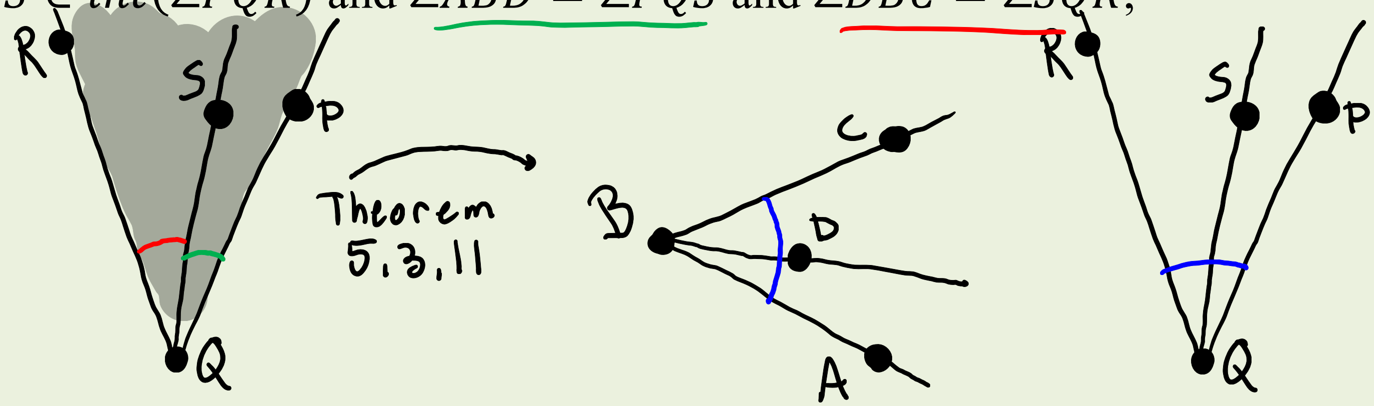
Theorem 5.3.11 (Congruent Angle Addition Theorem)

In a protractor geometry,

if $D \in int(\angle ABC)$ and $S \in int(\angle PQR)$ and $\angle ABD \simeq \angle PQS$ and $\angle DBC \simeq \angle SQR$,

then $\angle ABC \simeq \angle PQR$.



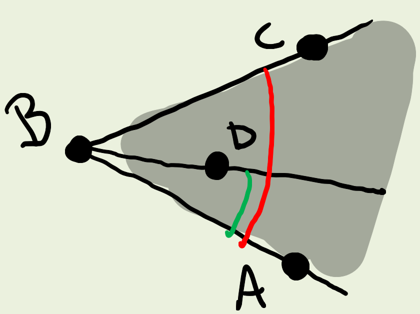


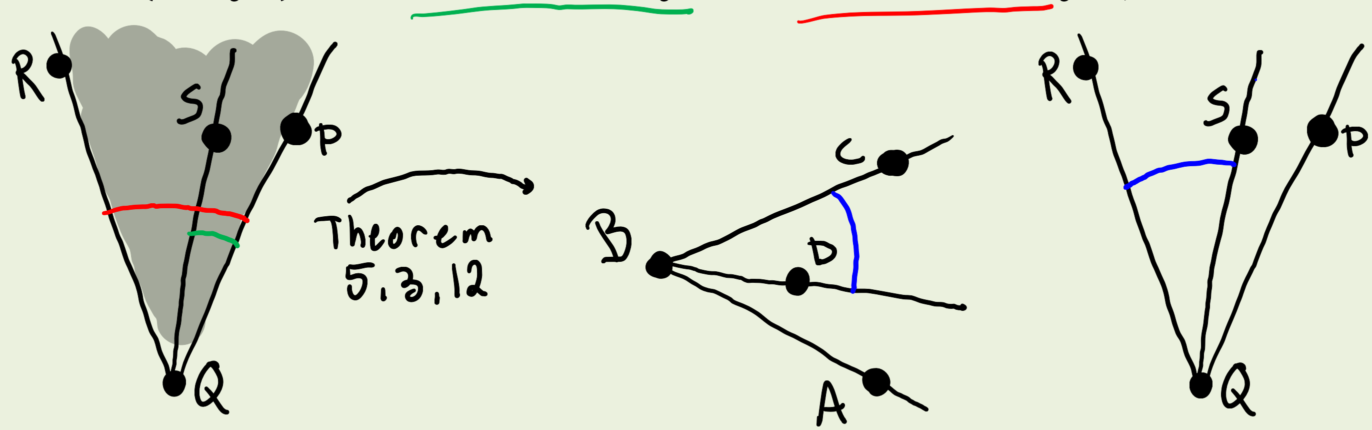
Theorem 5.3.12 (Congruent Angle Subtraction Theorem)

In a protractor geometry,

if $D \in int(\angle ABC)$ and $S \in int(\angle PQR)$ and $\angle ABD \simeq \angle PQS$ and $\angle ABC \simeq \angle PQR$,

then $\angle DBC \simeq \angle SQR$.





Remember that in our book Section 5.4, the cosine and sine functions are defined abstractly, using calculus. Their definitions do not involve right triangles or the unit circle.

The Trigonometric Functions (introduced in Section 5.4 on p.112)

The Cosine Function

$$\cos: [0,180] \to [-1,1]$$

is defined using calculus (NOT using right triangles or circles)

The Sine Function

$$\sin: [0,180] \to [0,1]$$

is defined using by the following equation (NOT using right triangles or circles)

$$\sin(\theta) = \sqrt{1 - (\cos(\theta))^2}$$

The Tangent Function (not introduced in the book)

$$\tan: [0,90) \to [0, \infty)$$

is defined using by the following equation (NOT using right triangles or circles)

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

Now Recall Content from Section 6.1 Discussed in Video 6.1a

We saw the Definition of Triangle Congruence

In order to formulate a definition of triangle congruence in an axiomatic geometry, it helps to have a notion of corresponding parts of triangles.

Definition of Corresponding Parts of Two Triangles

Let ΔABC and ΔDEF be two triangles in a protractor geometry, and let $f: \{A, B, C\} \rightarrow \{D, E, F\}$ be a bijection from the set of vertices of ΔABC to the set of vertices of ΔDEF. Then associated to the bijection f is an automatic correspondence of six pairs of parts of the two triangles.

Segment $\overline{AB}$ of ΔABC corresponds to segment $\overline{f(A)f(B)}$ of ΔDEF.

Segment $\overline{BC}$ of ΔABC corresponds to segment $\overline{f(B)f(C)}$ of ΔDEF.

Segment $\overline{CA}$ of ΔABC corresponds to segment $\overline{f(A)f(A)}$ of ΔDEF.

Angle $\angle ABC$ of ΔABC corresponds to angle $\angle f(A)f(B)f(C)$ of ΔDEF.

Angle $\angle BCA$ of ΔABC corresponds to angle $\angle f(B)f(C)f(A)$ of ΔDEF.

Angle $\angle CAB$ of ΔABC corresponds to angle $\angle f(C)f(A)f(B)$ of ΔDEF.

Definition of Congruence between Triangles

Words: *A congruence between* ΔABC *and* ΔDEF

Usage: ΔABC and ΔDEF are two triangles in a protractor geometry,

Meaning: A bijection $f: \{A, B, C\} \to \{D, E, F\}$ from the set of vertices of ΔABC to the set of vertices of ΔDEF such that each pair of corresponding parts is congruent. That is,

$$\overline{AB} \simeq \overline{f(A)f(B)}$$

$$\overline{BC} \simeq \overline{f(B)f(C)}$$

$$\overline{CA} \simeq \overline{f(C)f(A)}$$

$$\angle ABC \simeq \angle f(A)f(B)f(C)$$

$$\angle BCA \simeq \angle f(B)f(C)f(A)$$

$$\angle CAB \simeq \angle f(C)f(A)f(B)$$

Definition of Congruent Triangles

Words: ΔABC *and* ΔDEF *are congruent.*

Usage: ΔABC and ΔDEF are two triangles in a protractor geometry,

Meaning: There exists a congruence between ΔABC and ΔDEF.

Definition of Symbol to Indicate a Particular Congruence

Symbol: $\Delta ABC \simeq \Delta DEF$

Usage: ΔABC and ΔDEF are two triangles in a protractor geometry,

Meaning: The particular bijection $f: \{A, B, C\} \to \{D, E, F\}$ defined by

$$f(A) = (D), f(B) = (E), f(C) = (F)$$

is a congruence.

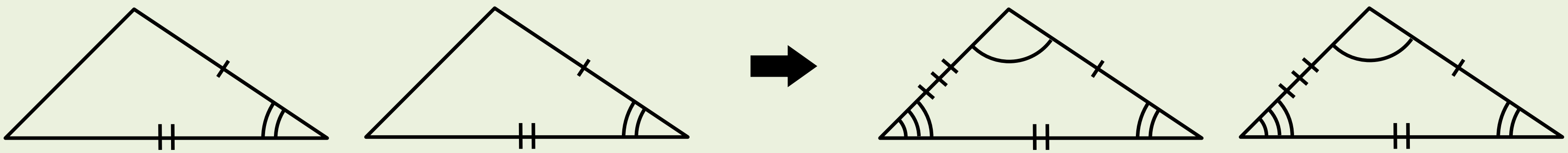
Theorem (Exercise 6.1#1)

In a protractor geometry, congruence is an equivalence relation on the set of all triangles

Definition of the Side-Angle-Side Axiom

Words: *A protractor geometry satisfies the Side-Angle-Side (SAS) Axiom.*

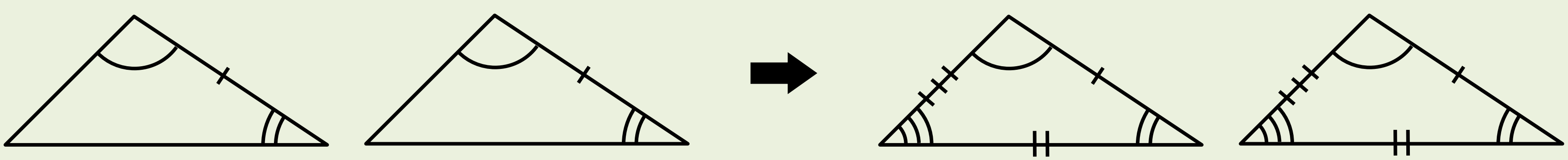
Meaning: If there is a bijection between the vertices of two triangles, and two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle, then all the remaining corresponding parts are congruent as well, so the bijection is a congruence and the triangles are congruent.



Definition of the Angle-Side-Angle Axiom

Words: *A protractor geometry satisfies the Angle-Side-Angle (ASA) Axiom.*

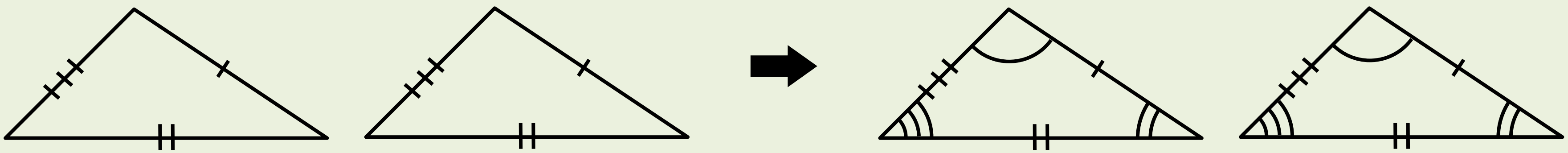
Meaning: If there is a bijection between the vertices of two triangles, and two angles and the included side of one triangle are congruent to the corresponding parts of the other triangle, then all the remaining corresponding parts are congruent as well, so the bijection is a congruence and the triangles are congruent.



Definition of the Side-Side-Side Axiom

Words: *A protractor geometry satisfies the Side-Side-Side (SSS) Axiom.*

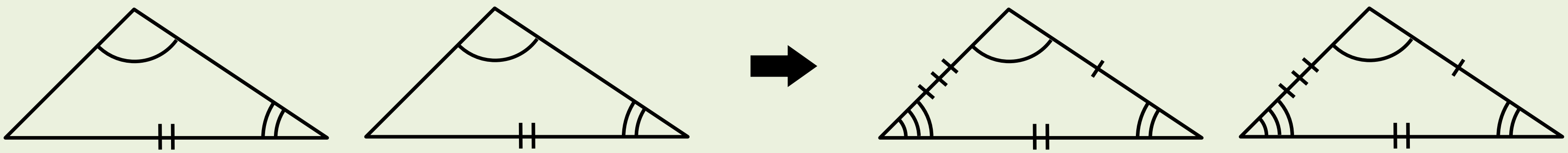
Meaning: If if there is a bijection between the vertices of two triangles, and the three sides of one triangle are congruent to the corresponding parts of the other triangle, then all the remaining corresponding parts are congruent as well, so the bijection is a congruence and the triangles are congruent.



Definition of the Angle-Angle-Side Axiom

Words: *A protractor geometry satisfies the Angle-Angle-Side (AAS) Axiom.*

Meaning: If if there is a bijection between the vertices of two triangles, and two angles and a non-included side of one triangle are congruent to the corresponding parts of the other triangle, then all the remaining corresponding parts are congruent as well, so the bijection is a congruence and the triangles are congruent.



Neutral Geometry

The four triangle congruence axioms articulate *desirable triangle congruence behavior*. We can require that just *one* axiom, about just *one particular kind* of *desirable behavior*, be satisfied. We can then prove *theorems* that show that triangles will *also* have the other three kinds of *desirable behavior*. That is the idea behind the definition of *neutral geometry*.

Definition of Neutral Geometry

A **neutral geometry** (or **absolute geometry**) is a protractor geometry that satisfies *SAS*.

In Chapter 6, we will prove three theorems about *desirable triangle congruence behavior*.

Theorem: Every neutral geometry satisfies *ASA*.

Theorem: Every neutral geometry satisfies *SSS*.

Theorem: Every neutral geometry satisfies *AAS*.

Note that the statements of the three theorems have been mentioned here just as an introduction to the coming material. The three theorems have not yet been proven and they do not yet have theorem numbers, so we may not yet use any of them in proofs. Soon, but not yet.

End of Review of previous Material

First Topic for Video 6.1b: Models of Euclidean Geometry

Which of our protractor geometries qualify to be called *neutral geometries*?

Recall from Video 6.1 [Example 3] that the Taxicab plane does not satisfy SAS.

In the Taxicab plane let

$A = (-1,1), B = (0,0), C = (1,1)$

and let

$D = (2,2), E = (2,0), F = (4,0)$

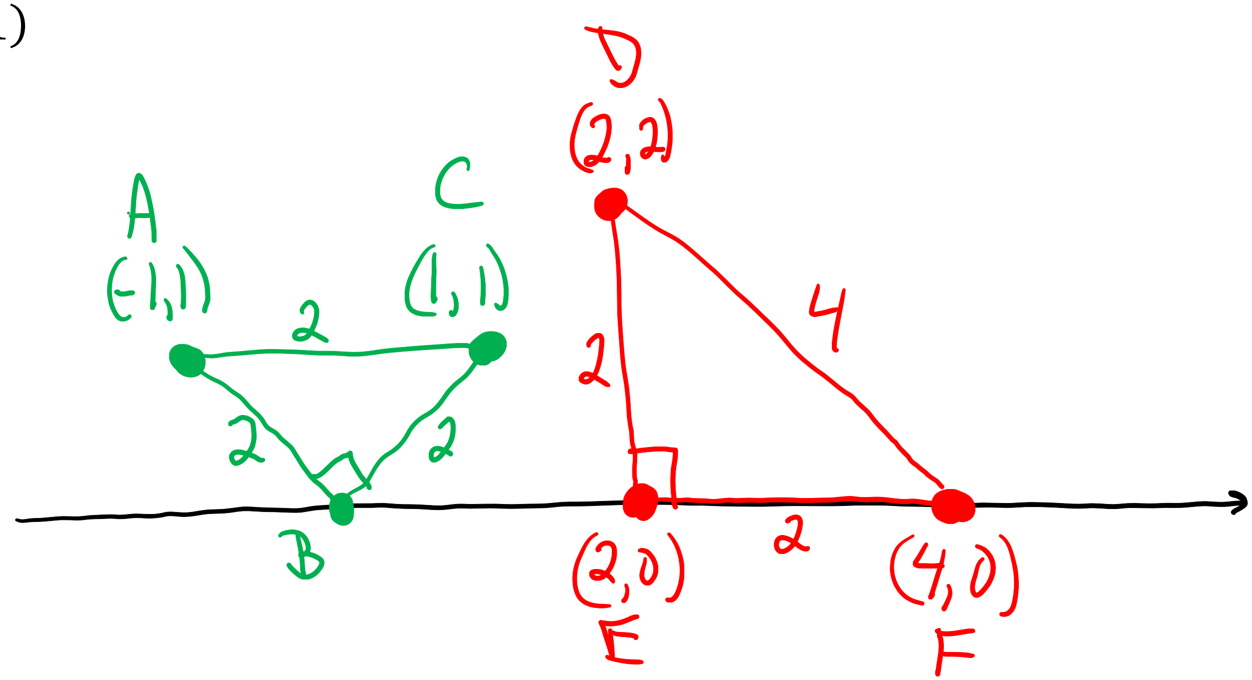
Observe that

$\overline{AB} \simeq \overline{DE}$

$\angle ABC \simeq \angle DEF$

$\overline{BC} \simeq \overline{EF}$

But $\overline{CA} \not\simeq \overline{FD}$



Therefore, the *Taxicab plane* is *not* a *neutral geometry*.

We could similarly show that the *Max plane* is *not* a *neutral geometry*.

But it is shown in the book that our two *most familiar models* of *protractor geometry do* qualify to be called models of *neutral geometry*.

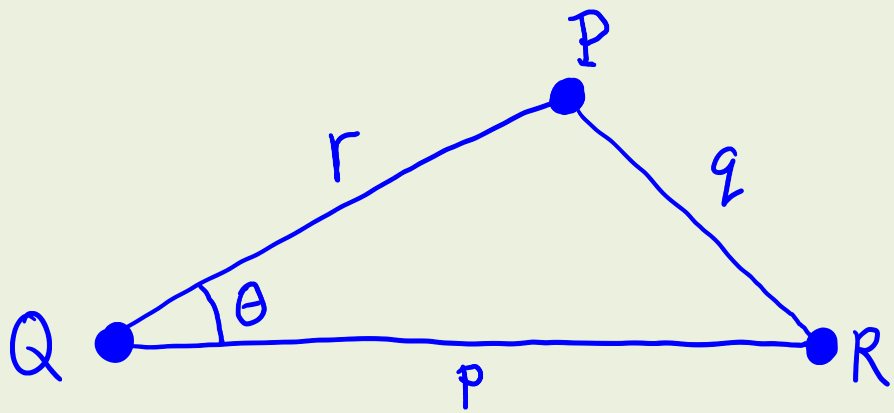
Proposition 6.1.2 (Euclidean Law of Cosines) (proven in exercise 5.4#3)

If P, Q, R are three non-collinear points in $\mathbb{R}^2$,

then $\left(d_E(P,R)\right)^2 = \left(d_E(Q,P)\right)^2 + \left(d_E(Q,R)\right)^2 - 2d_E(Q,P)d_E(Q,R)\cos\left(m_E(\angle PQR)\right)$.

In other words, for triangle ΔPQR, if p, q, r are defined to be the lengths of the sides opposite those vertices and $\theta = m_E(\angle PQR)$, then ~~$q^2 = p^2 + q^2 - 2pq\cos(\theta)$~~

$q^2 = p^2 + r^2 - 2pr\cos(\theta)$



It is straightforward to use the Euclidean Law of Cosines is used to prove the following.

Proposition 6.1.3 *The Euclidean plane satisfies SAS (and therefore is a neutral geometry).*

(See the book page 128 for a proof.)

The following proposition is presented in the book, although it is not proven until much later in the book. (We won't study its proof in our course.)

Proposition 6.1.4 *The Poincaré plane satisfies SAS (and therefore is a neutral geometry).*

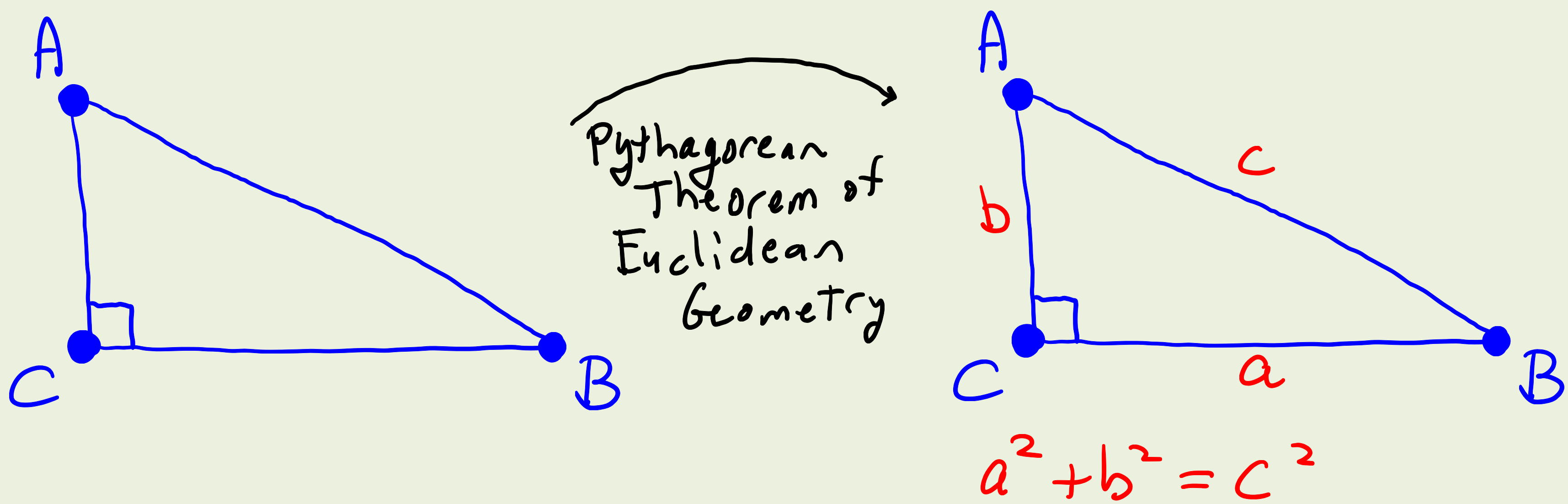
Second Topic for this Video: Facts of the Euclidean Plane

In your homework exercise 6.1#6, you'll use the Euclidean Law of Cosines to prove the following.

Corollary (The Pythagorean Theorem of Euclidean Geometry)(proven in exercise 6.1#6)
In the Euclidean plane, if triangle ΔABC has a right angle at C,

then $\left(d_E(A,B)\right)^2 = \left(d_E(C,B)\right)^2 + \left(d_E(C,A)\right)^2$

In other words, for triangle ΔABC, if a, b, c are the lengths of the sides opposite those vertices $\angle ACB$ is a right angle, then $c^2 = a^2 + b^2$



In your homework, you will use the *Euclidean Law of Cosines* and the *Pythagorean Theorem of Euclidean Geometry* to show that the trigonometric functions defined in Section 5.4 agree with the *SOHCAHTOA* definitions of sine and cosine that may have been your first introduction to those functions back in middle school. (Note that these results are valid only in the *Euclidean plane*.)

Corollary (*SOHCAHTOA* interpretation of Sine, Cosine, Tangent) (proven in 6.1#7)

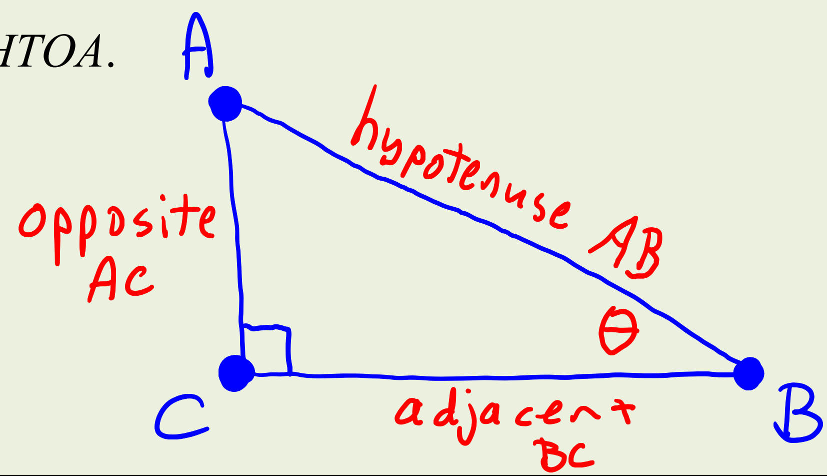
In the *Euclidean plane*, if triangle ΔABC has a right angle at C, and $m_E(\angle B) = \theta$, then

$$\sin(\theta) = \frac{AC}{AB} \quad \text{and} \quad \cos(\theta) = \frac{BC}{BA} \quad \text{and} \quad \tan(\theta) = \frac{AC}{BC}$$

in other words

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} \quad \text{and} \quad \cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} \quad \text{and} \quad \tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$

The acronym for this behavior is *SOHCAHTOA*.



Third Topic for this Video: the CS ➔ CA Theorem (Pons Asinorum)

Definition of Types of Triangles in a Protractor Geometry

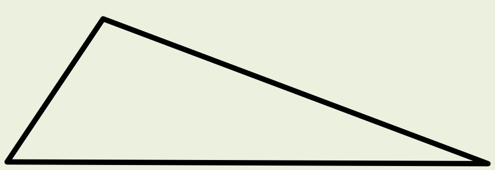
Given a triangle in a protractor geometry

The triangle is called **scalene** if no two sides are congruent. (In other words, all three sides have different lengths.)

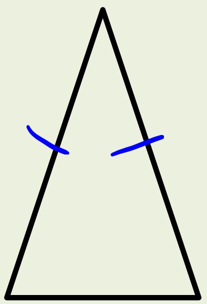
The triangle is called **isosceles** if at least two sides are congruent.

The triangle is called **equilateral** if all three sides are congruent.

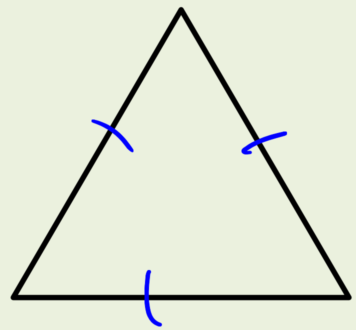
If a triangle is isosceles, then the **base angles** of the triangle are the angles opposite the congruent sides. (Note that an equilateral triangle will have three base angles, while a triangle that is isosceles but not equilateral will have two base angles.)



scalene



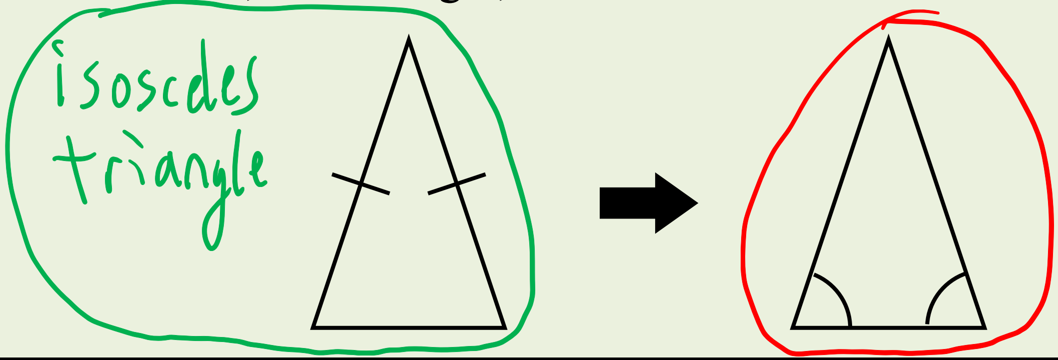
isosceles



isosceles and equilateral

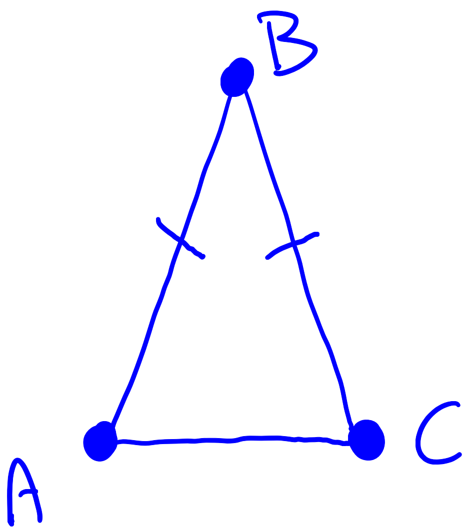
Theorem 6.1.5 (Pons Asinorum) (Isosceles Triangle Theorem) (CS ➔ CA Theorem)

In Neutral geometry, if two sides of a triangle are congruent, then the angles opposite those sides are also congruent. That is, in a triangle, if *CS* then *CA*.



Proof

(1) Suppose that ΔABC has $\overline{BA} \cong \overline{BC}$.



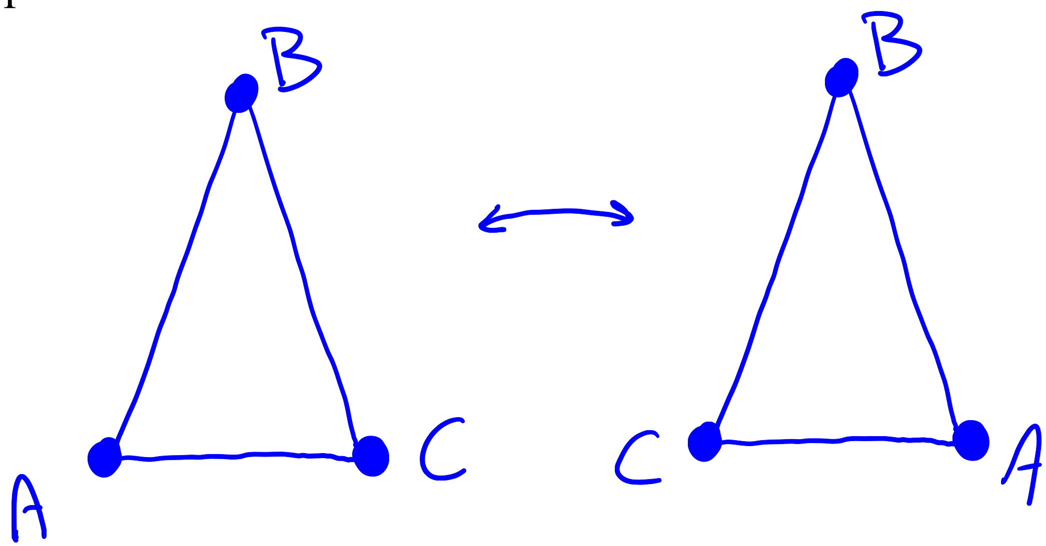
(2) Using the bijection $(A, B, C) \mapsto (C, B, A)$ between the vertices of ΔABC and ΔCBA, we have the following pairs of corresponding parts

parts of $\Delta ABC \leftrightarrow$ parts of ΔCBA

$$\overline{BA} \leftrightarrow \overline{BC}$$

$$\angle ABC \leftrightarrow \angle CBA$$

$$\overline{BC} \leftrightarrow \overline{BA}$$



(2) Observe that the corresponding parts in each of these three pairs are congruent

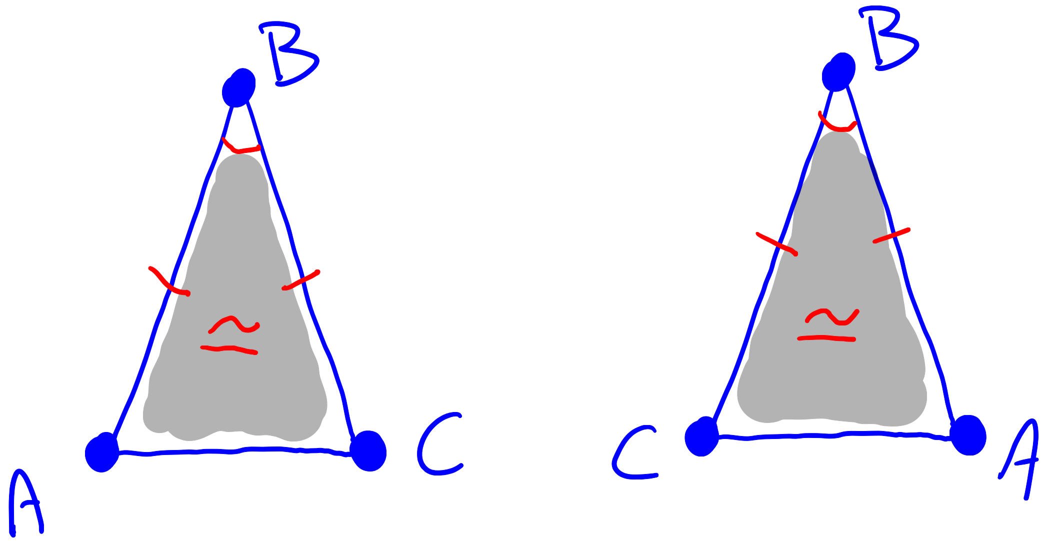
parts of $\Delta ABC \simeq$ parts of ΔCBA

$\overline{BA} \simeq \overline{BC}$ (given)

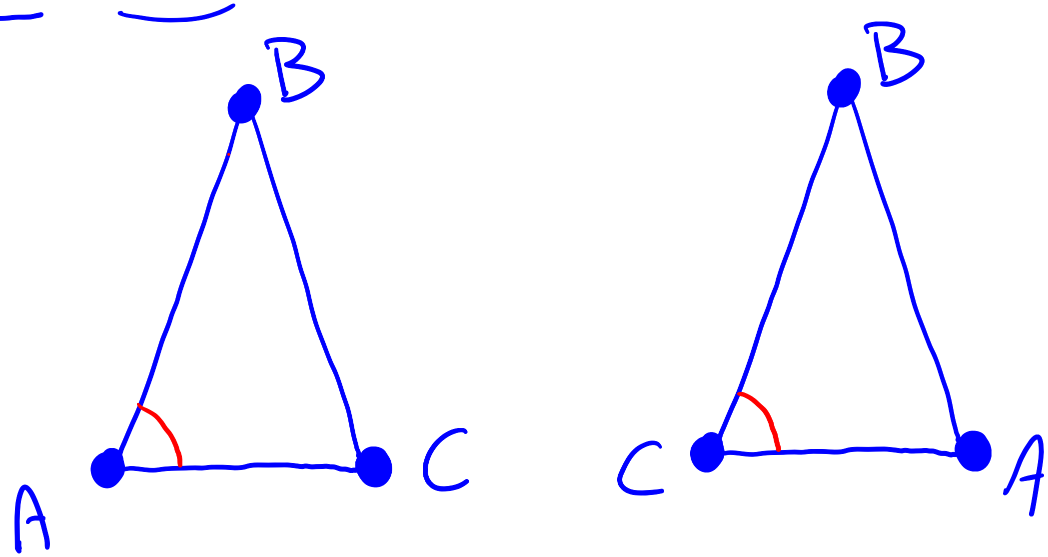
$\angle ABC \simeq \angle CBA$ (congruent to itself because it has the same measure as itself)

$\overline{BC} \simeq \overline{BA}$ (given)

(3) $\Delta ABC \cong \Delta CBA$ (by the SAS congruence axiom)



(4) Therefore, $\angle BAC \cong \angle BCA$. (by (3) and the definition of triangle congruence)



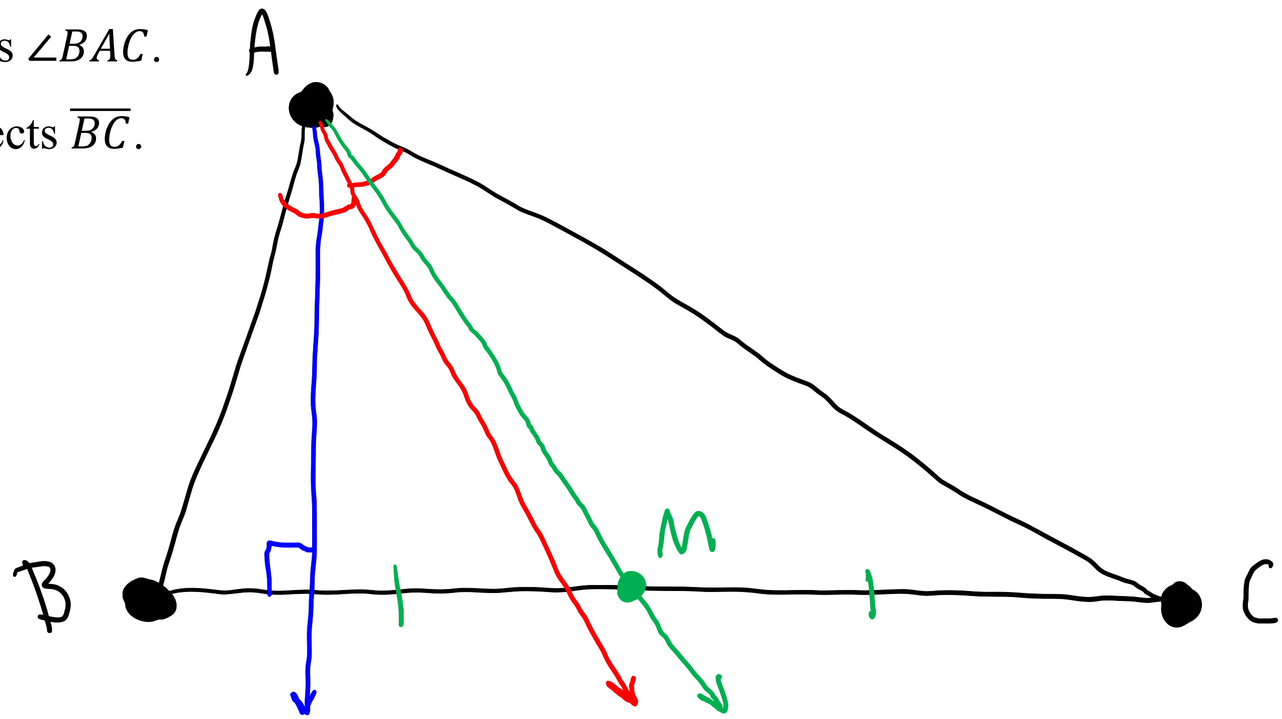
End of proof

Fourth Topic: Using the CS ➔ CA Theorem to prove a fact about special rays in triangles

Special Rays in Triangles

Given a triangle ΔABC in a protractor geometry, there exist three special rays from A.

- The ray from A that is perpendicular to $\overleftrightarrow{BC}$.
- The ray from A that bisects $\angle BAC$.
- The ray from A that is bisects $\overline{BC}$.



It is important to realize that these three rays are not in general the same ray.

But it can be shown that in neutral geometry, if ΔABC is an isosceles triangle, with $\overline{AB} \simeq \overline{AC}$, then the three rays are actually the same ray. Parts of this fact are proven in various places in our book and in my videos.

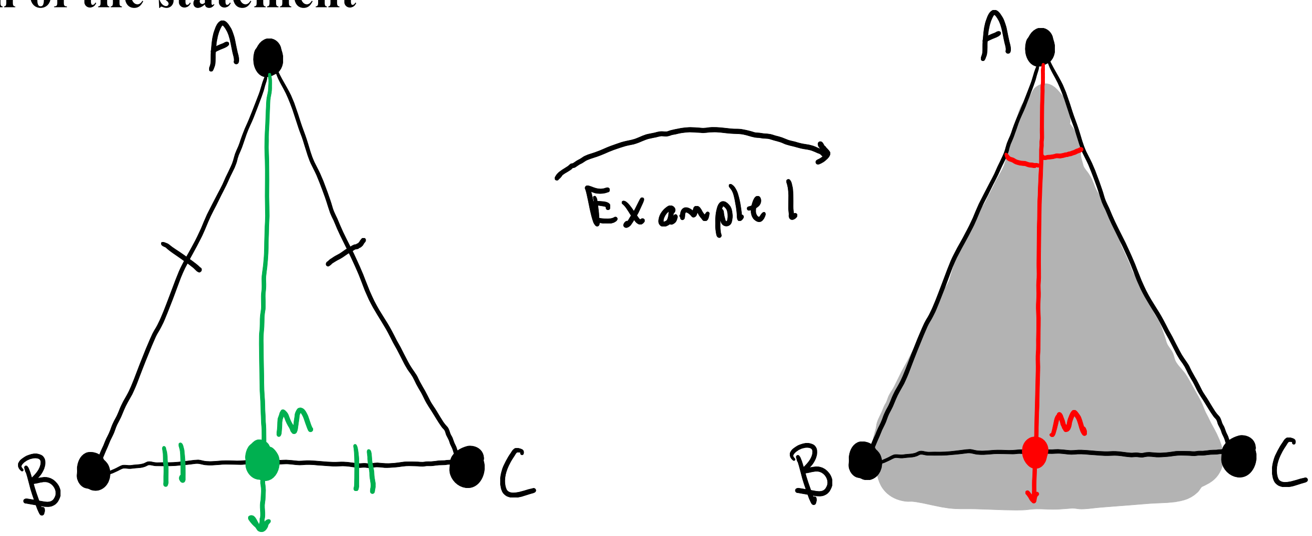
[Example 1] (presented below) Prove in neutral geometry, if ΔABC is an isosceles triangle, with $\overline{AB} \simeq \overline{AC}$, and if M is the midpoint of $\overline{BC}$, then $\overrightarrow{AM}$ bisects $\angle BAC$. (That is, the ray from A that bisects $\overline{BC}$ is also the ray from A that bisects $\angle BAC$.)

Exercise 6.1#8 Prove in neutral geometry, if ΔABC is an isosceles triangle, with $\overline{AB} \simeq \overline{AC}$, and if M is the midpoint of $\overline{BC}$, then $\overrightarrow{AM} \perp \overleftrightarrow{BC}$. (That is, the ray from A that bisects $\overline{BC}$ is also the ray from A that is perpendicular to $\overleftrightarrow{BC}$.)

[Example 1] In neutral geometry, suppose that ΔABC is isosceles, with $\overline{AB} \simeq \overline{AC}$.

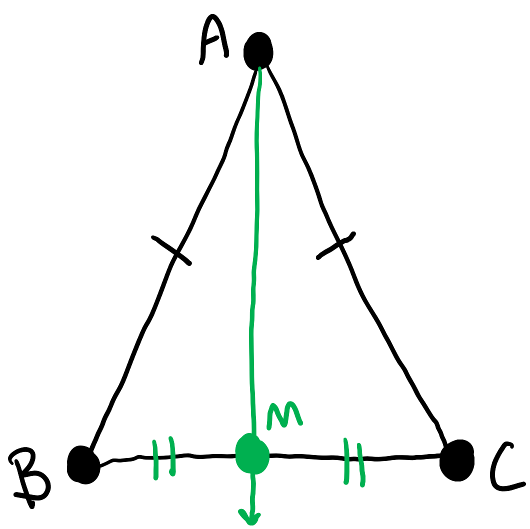
Show that if M is the midpoint of $\overline{BC}$, then $\overrightarrow{AM}$ bisects $\angle BAC$.

Illustration of the statement

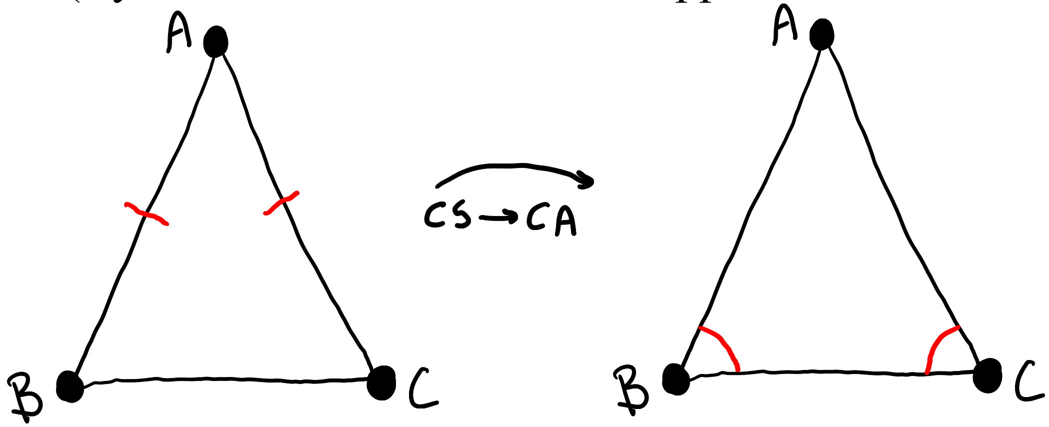


Proof

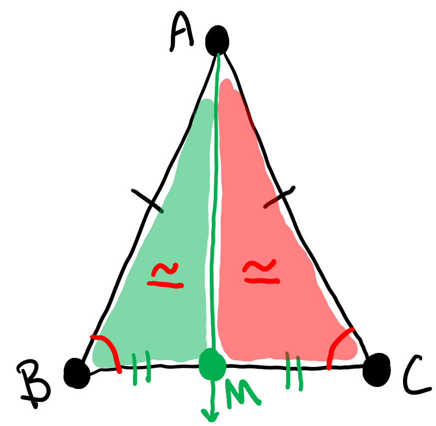
(1) In neutral geometry, suppose that ΔABC has $\overline{AB} \simeq \overline{AC}$ and that M is the midpoint of $\overline{BC}$.



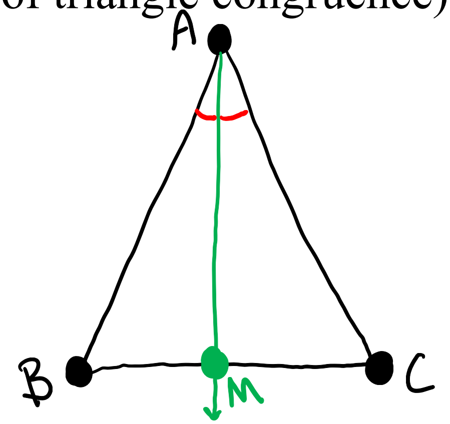
(2) $\angle ABC \simeq \angle ACB$ (By Theorem 6.1.5 CS ➔ CA applied to ΔABC with $\overline{AB} \simeq \overline{AC}$)



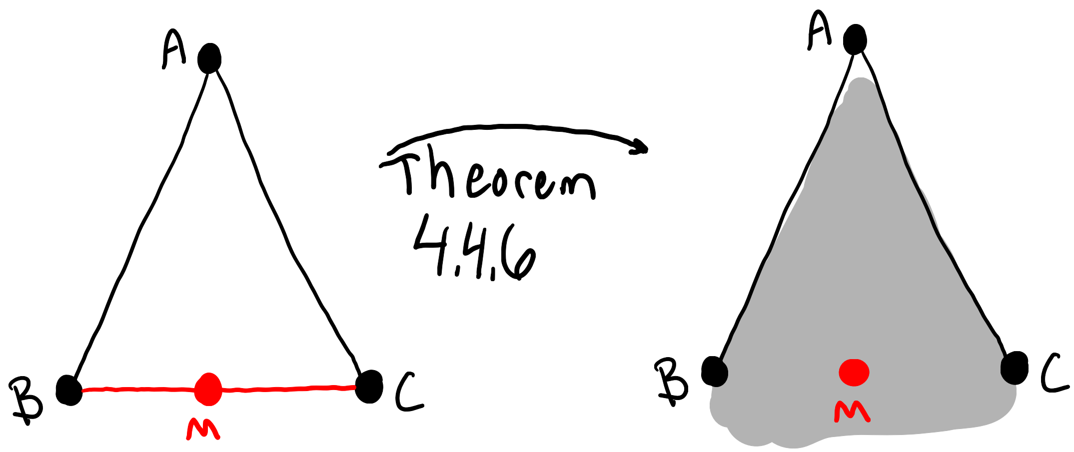
(3) $\Delta ABM \simeq \Delta ACM$ (By SAS axiom applied to $\overline{AB} \simeq \overline{AC}$ and $\angle ABM \simeq \angle ACM$ and $\overline{MB} \simeq \overline{MC}$)



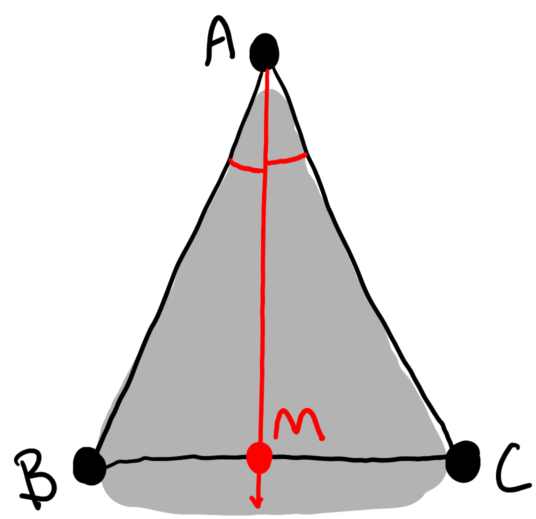
(4) $\angle MAB \simeq \angle MAC$ (By (3) and definition of triangle congruence)



(5) $M \in \text{int}(\angle ABC)$ (By Theorem 4.4.6 that tells us that $\text{int}(\overline{BC}) \subset \text{int}(\angle ABC)$)



(6) ray $\overrightarrow{AM}$ bisects $\angle BAC$. (by (4),(5) and definition of angle bisector.)



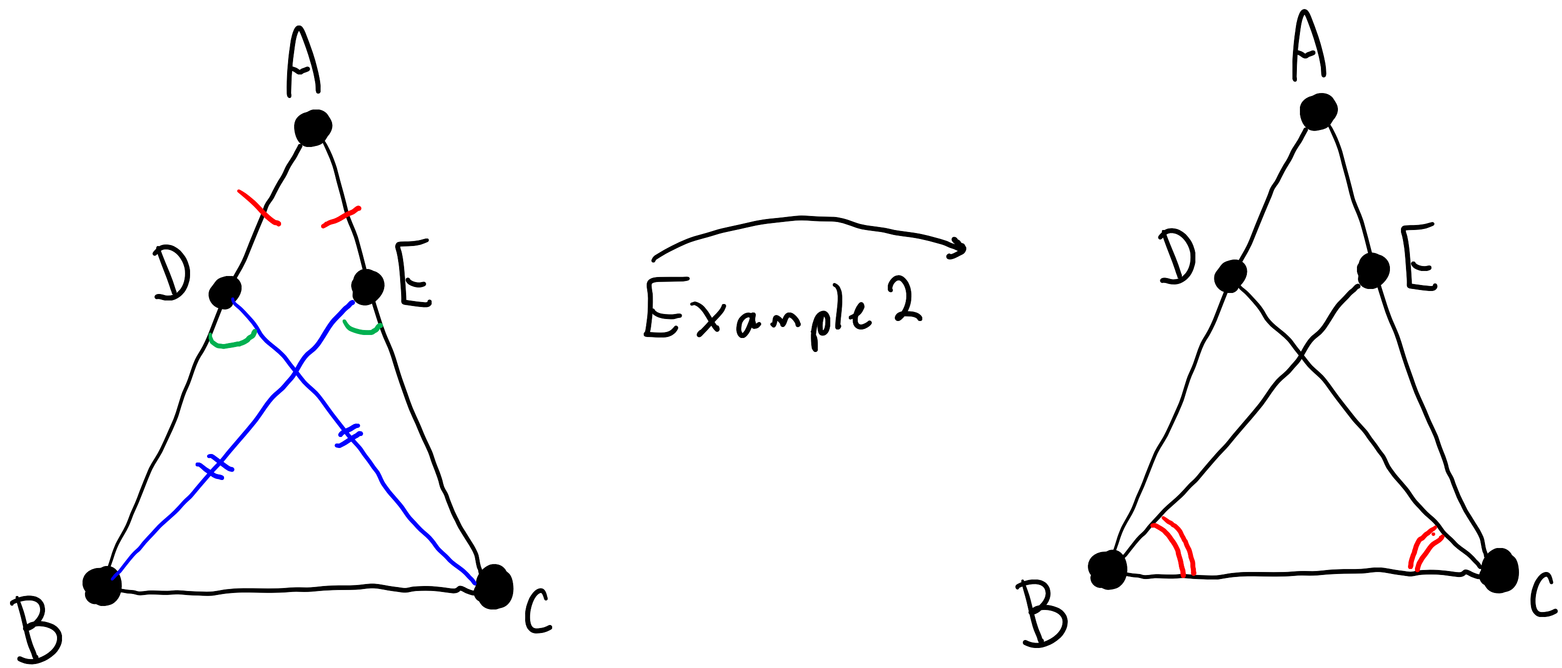
End of proof

End of [Example 1]

Fifth (and Final) Topic: Using the CS → CA Theorem to prove a particular congruence

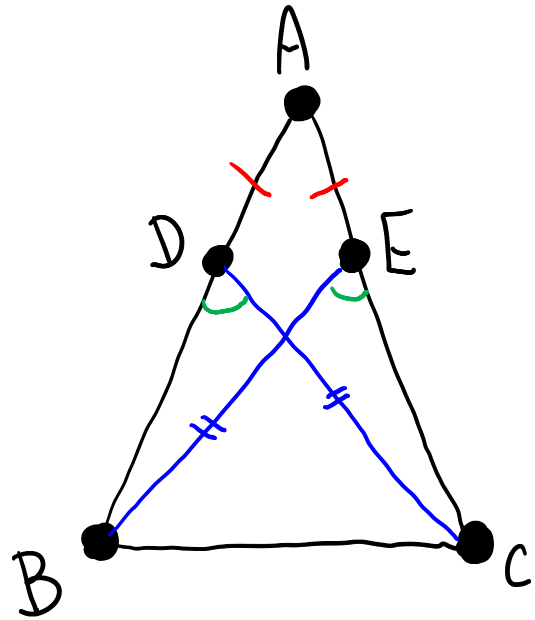
[Example 2](similar to 6.2#10) Prove the following: In a neutral geometry, if ΔABC and points D, E satisfy $A - D - B$ and $A - E - C$ and $\overline{AD} \simeq \overline{AE}$ and $\angle BDC \simeq \angle CEB$ and $\overline{DC} \simeq \overline{BE}$, then $\angle BCD \simeq \angle CBE$.

Illustration of the Statement:



Proof

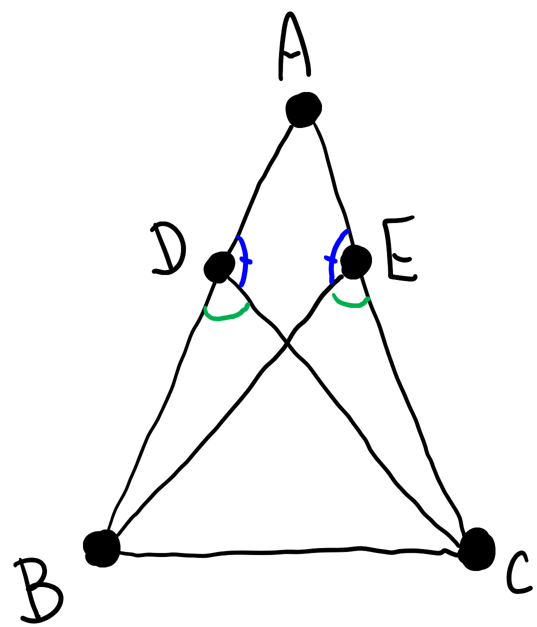
(1) Suppose that in a neutral geometry, ΔABC and points D, E satisfy $A - D - B$ and $A - E - C$ and $\overline{AD} \simeq \overline{AE}$ and $\angle BDC \simeq \angle CEB$ and $\overline{DC} \simeq \overline{BE}$.



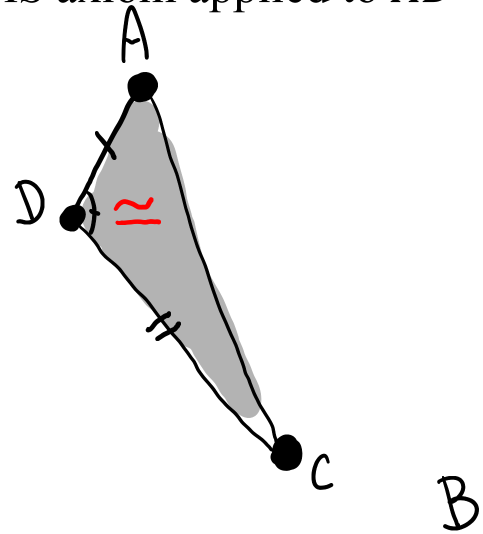
(2) $\angle ADC \simeq \angle AEB$

because

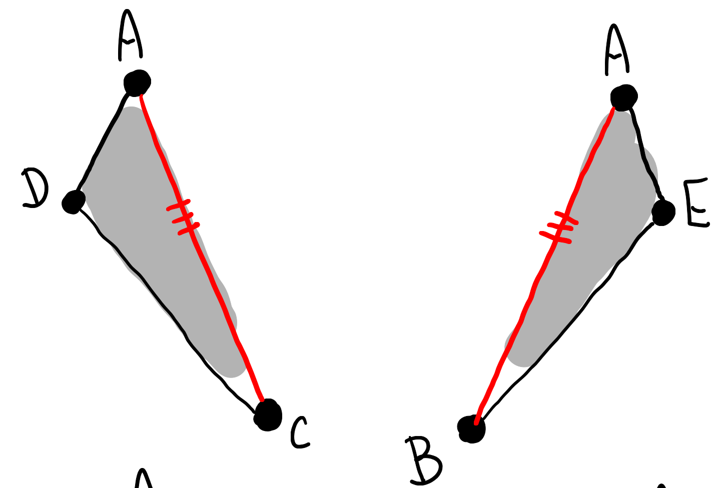
$$\begin{aligned} m(\angle ADC) &= 180 - m(\angle BDC) \text{ by linear pair theorem} \\ &= 180 - m(\angle BEC) \text{ because given } \angle BDC \simeq \angle CEB \\ &= m(\angle AEB) \text{ by linear pair theorem} \end{aligned}$$



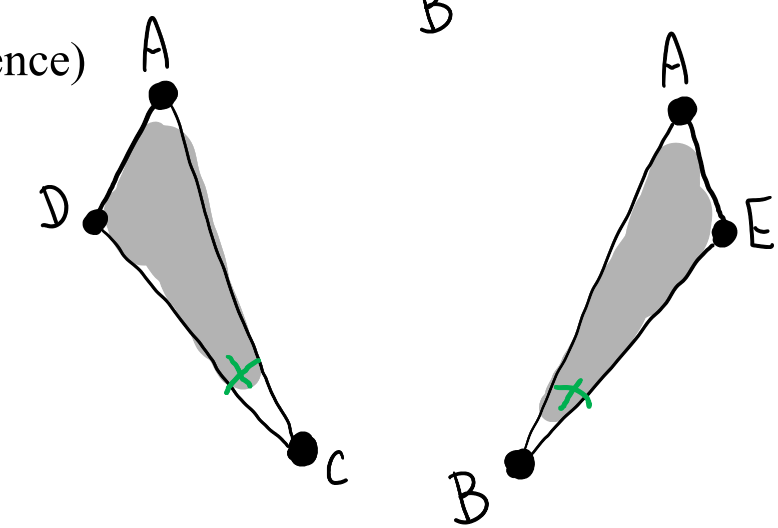
(3) $\Delta ADC \simeq \Delta AEB$ (By the *SAS* axiom applied to $\overline{AD} \simeq \overline{AE}$, $\angle ADC \simeq \angle AEB$, $\overline{DC} \simeq \overline{BE}$)



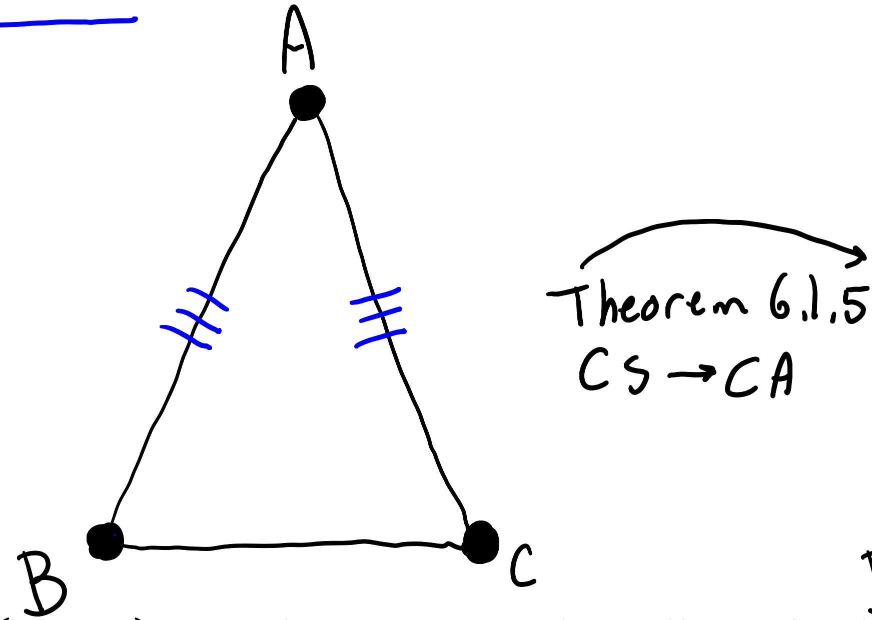
(4) $\overline{AC} \simeq \overline{AB}$ (by (2) and definition of triangle congruence)



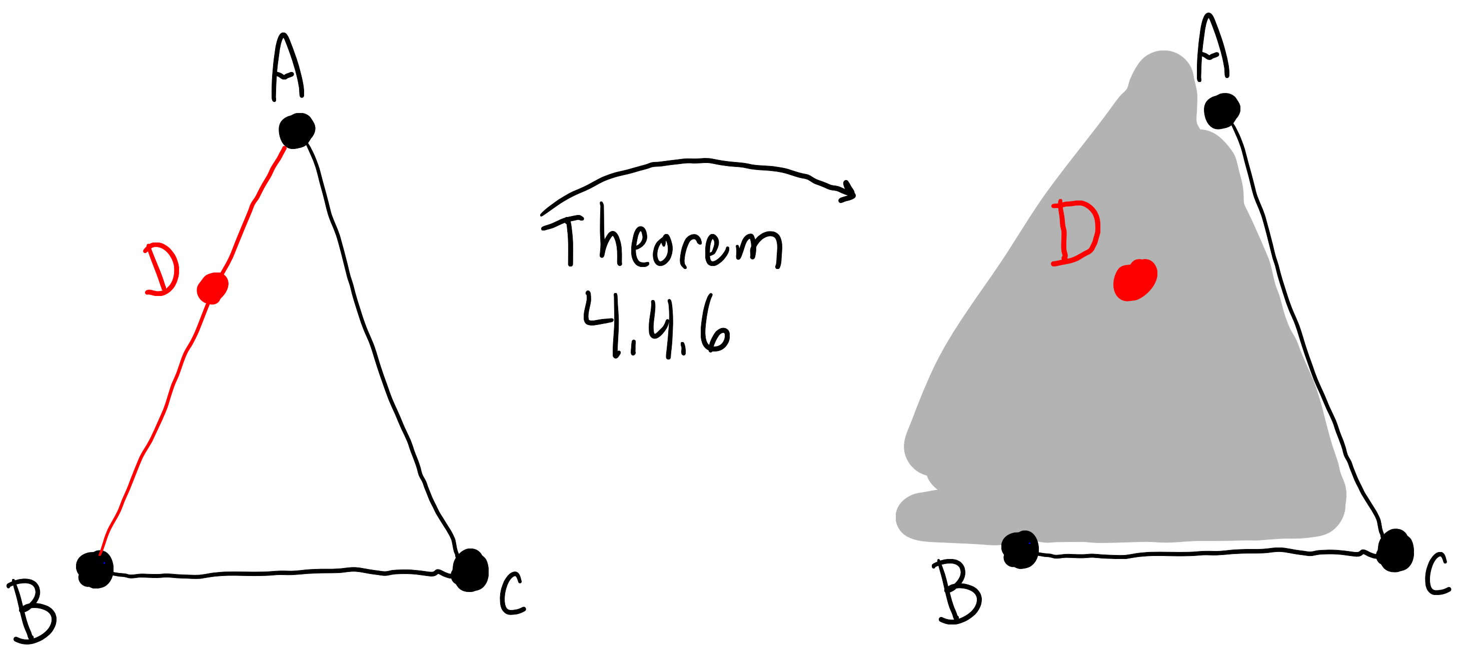
(5) $\angle ACD \simeq \angle ABE$ (by (2) and definition of triangle congruence)



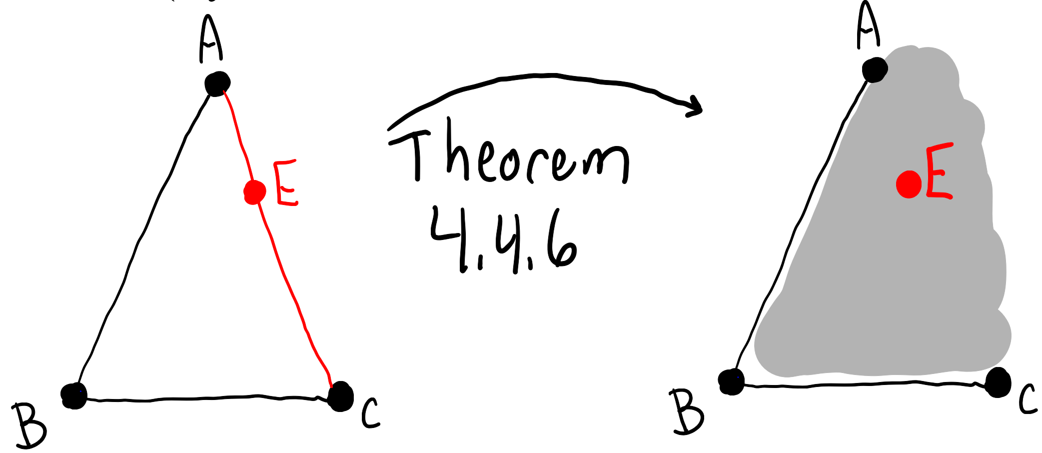
(6) $\angle ABC \simeq \angle ACB$ (By Theorem 6.1.5 CS ➔ CA applied to ΔABC with $\overline{AB} \simeq \overline{AC}$ by (4)



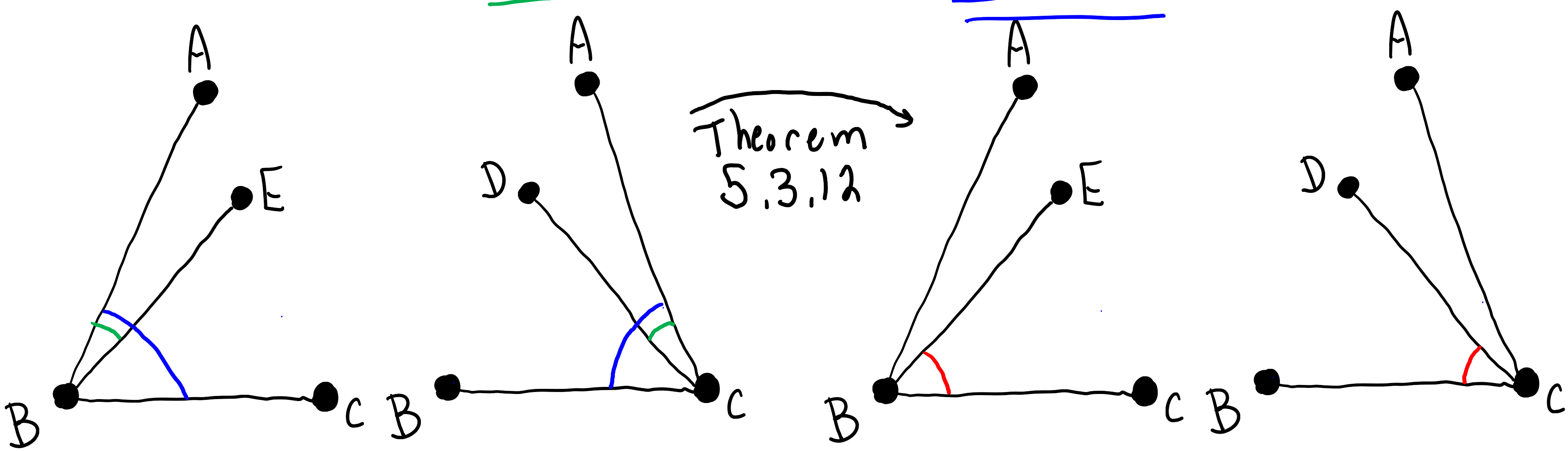
(7) $D \in \text{int}(\angle ACB)$ (By Theorem 4.4.6 that tells us that if $A - D - B$ then $D \in \text{int}(\angle ACB)$)



(8) $E \in \text{int}(\angle ABC)$ (By Theorem 4.4.6 that tells us that if $A - E - C$ then $D \in \text{int}(\angle ABC)$)



(9) $\angle BCD \simeq \angle CEB$ (By Theorem 5.3.12 (Congruent Angle Subtraction) applied to $D \in \text{int}(\angle ACB)$ and $E \in \text{int}(\angle ABC)$ such that $\angle ACD \simeq \angle ABE$ (by (4)) and $\angle ABC \simeq \angle ACB$ (by 5)



End of Proof

End of Video