Video 6.2: Basic Triangle Congruence Theorems

produced by Mark Barsamian, 2021.04.11

for Ohio University MATH 3110/5110 College Geometry

Six Topics for this Video

- **The Intermediate Triangle Strategy**
- **Theorem: Neutral Geometry Satisfies *ASA***
- **The *CA* ➔ *CS* Theorem (Converse of the Statement of Pons Asinorum, *CS* ➔ *CA*)**
- **Theorem: Neutral Geometry Satisfies *SSS***
- **Existence of a perpendicular to a line through a point *not* on the line**
- **Proving Particular Congruences**

Reading: Section 6.2 Basic Congruence Theorems (pages 131 – 134)

in *Geometry: A Metric Approach with Models, Second Edition* by Millman & Parker

Homework: Section 6.2 # 1, 2, 3, 5, 6, 7, 8, 9, 12, 13

First Topic: The Intermediate Triangle Strategy

In Section 6.1, we saw the definition of triangle congruence.

Definition of Congruence between Triangles

Words: *A congruence between* ΔABC *and* ΔDEF

Usage: ΔABC and ΔDEF are two triangles in a protractor geometry,

Meaning: A bijection $f:\{A,B,C\} \rightarrow \{D,E,F\}$ from the set of vertices of ΔABC to the set of vertices of ΔDEF such that each pair of corresponding parts is congruent. That is,

$$\overline{AB} \simeq \overline{f(A)f(B)}$$

$$\overline{BC} \simeq \overline{f(B)f(C)}$$

$$\overline{CA} \simeq \overline{f(C)f(A)}$$

$$\angle ABC \simeq \angle f(A)f(B)f(C)$$

$$\angle BCA \simeq \angle f(B)f(C)f(A)$$

$$\angle CAB \simeq \angle f(C)f(A)f(B)$$

Definition of Congruent Triangles

Words: ΔABC *and* ΔDEF *are congruent.*

Usage: ΔABC and ΔDEF are two triangles in a protractor geometry,

Meaning: There exists a congruence between ΔABC and ΔDEF.

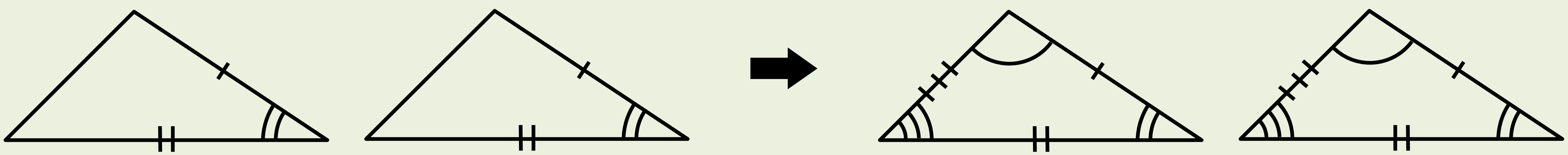
If the *definition of congruent triangles* is all one knows about congruent triangles, then if one wants to prove that two triangles are congruenct, one must produce a bijection between their sets of vertices and then show that all six pairs of corresponding parts are congruent. This is a very tedious job. Homework exercise 6.1#2 asked you to do this tedious job, and it is the only homework exercise of that sort.

Then, in Section 6.1, the Side-Angle-Side Axiom was introduced. It describes *desirable triangle congruence behavior*.

Definition of the Side-Angle-Side Axiom

Words: *A protractor geometry satisfies the Side-Angle-Side (SAS) Axiom.*

Meaning: If there is a bijection between the vertices of two triangles, and two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle, then all the remaining corresponding parts are congruent as well, so the bijection is a congruence and the triangles are congruent.



And a *neutral geometry* was defined to be protractor geometry that satisfies *SAS*.

Definition of Neutral Geometry

A **neutral geometry** (or **absolute geometry**) is a protractor geometry that satisfies *SAS*.

So far, the *definition of congruent triangles* and the Side-Angle-Side (*SAS*) axiom are all we know about congruent triangles. If one wants to prove that two triangles are congruenct, one must produce a bijection between their sets of vertices and then show that two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle, and finally use the Side-Angle-Side (*SAS*) axiom to say that the triangles are congruent.

In Sections 6.2 – 6.4, we will prove theorems that articulate other congruence conditions for neutral geometry. These theorems state that if certain other combinations of three corresponding parts of two triangles are congruent, then the triangles are congruent. A recurring strategy in proving such theorems will be to introduce an *intermediate triangle*:

Intermediate Triangle Strategy for Proving Two Triangles are Congruent.

Goal: Prove that Triangle $\Delta 1$ is congruent to Triangle $\Delta 2$

Strategy:

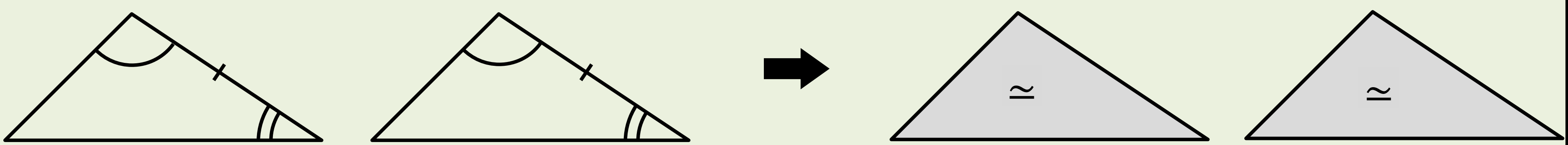
- Introduce a Triangle $\Delta 3$ (*the intermediate triangle*).
- Prove that $\Delta 3$ is congruent to $\Delta 1$ using *SAS* Axiom or a known Congruence Theorem.
- Prove that $\Delta 3$ is congruent to $\Delta 2$ using *SAS* Axiom or a known Congruence Theorem.
- Therefore, $\Delta 1$ is congruent to $\Delta 2$.

Second Topic: Theorem: Neutral geometry Satisfies *ASA*

Recall that *neutral geometry* was defined by adding a single axiom (*The Side-Angle-Side (SAS) axiom*) to the axioms for *protractor geometry*. It was mentioned in Section 6.1 that the other three axioms that articulate desirable triangle congruence behavior, (the *ASA*, *SSS*, and *SAA* axioms) did not need to be added as axioms of *neutral geometry*. It was mentioned that we would be proving theorems that say that those axioms would *automatically* be satisfied in *any neutral geometry*.

The first such theorem says that every neutral geometry satisfies *ASA*. (In other words, if a protractor geometry satisfies *SAS*, then it also satisfies *ASA*.)

Theorem 6.2.1 A Neutral Geometry Satisfies the Angle-Side-Angle (ASA) Axiom

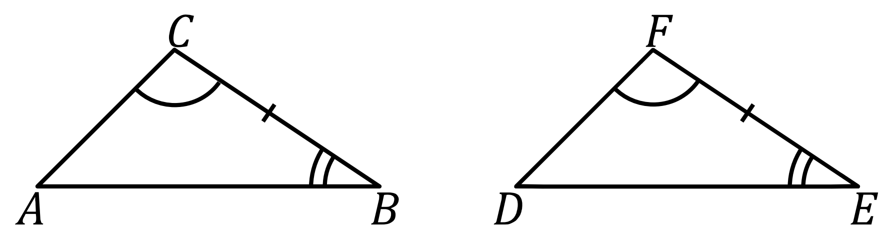


A proof of Theorem 6.2.1. is provided below (The proof roughly follows the book's proof, but with added statements and more drawings.) You will justify the statements in a homework exercise.

Proof

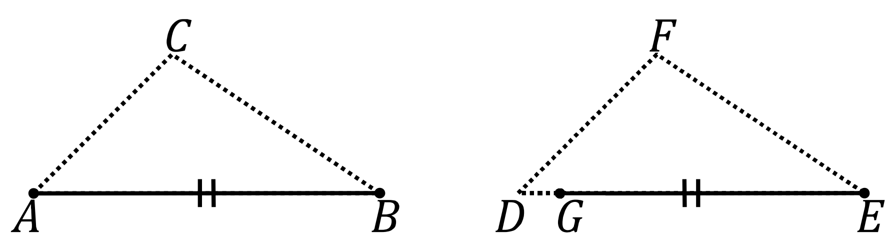
(1) In a neutral geometry, suppose that there is a bijection between the vertices of two triangles such that two angles and the included side of one triangle are congruent to the corresponding parts of the other triangle. Label the triangles ΔABC and ΔDEF so that $\angle ABC \cong \angle DEF$ and $\overline{BC} \cong \overline{EF}$ and $\angle BCA \cong \angle EFD$.

(**Remark**: In the *Intermediate Triangle Strategy*, triangle ΔABC is $\Delta 1$ and triangle ΔDEF is $\Delta 2$.)



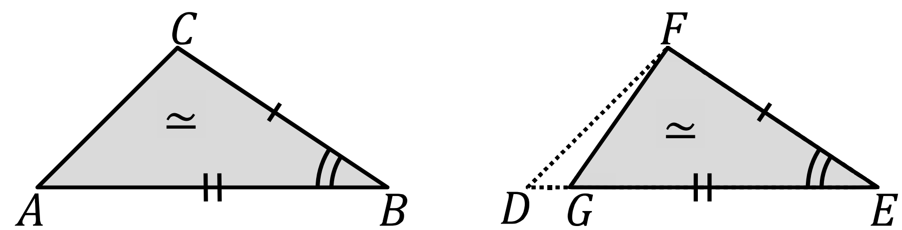
(2) There exists a point G on ray $\overrightarrow{ED}$ such that $\overline{EG} \cong \overline{BA}$. **(Justify)** (We suspect that G is the same point as D,but we have not yet proven that, so we should not draw it that way.)

(**Remark**: In the *Intermediate Triangle Strategy*, triangle ΔGEF is the *intermediate triangle* $\Delta 3$.

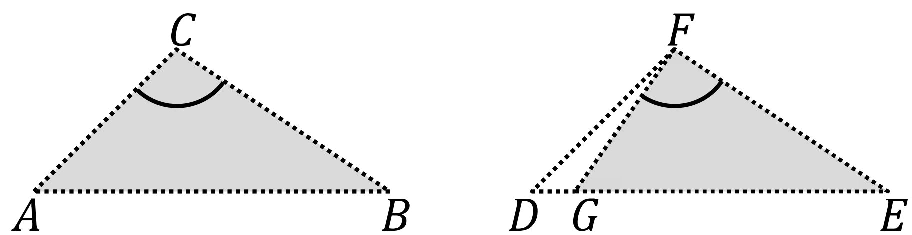


(3) $\Delta GEF \cong \Delta ABC$. **(Justify.)**

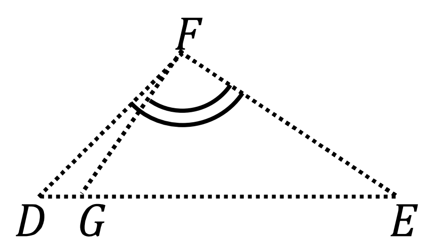
(**Remark**: In the *Intermediate Triangle Strategy*, we have proven that that $\Delta 3$ is congruent to $\Delta 1$.)



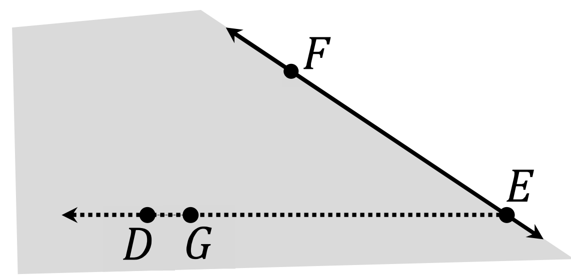
(4) $\angle EFG \cong \angle BCA$. **(Justify.)**



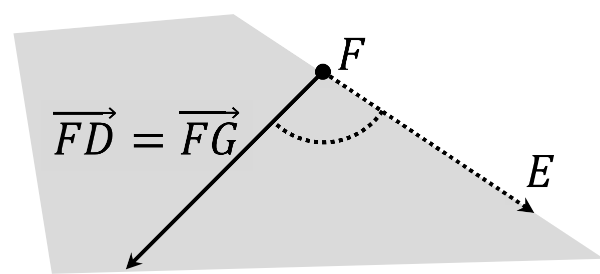
(5) $\angle EFG \cong \angle EFD$. **(Justify.)**



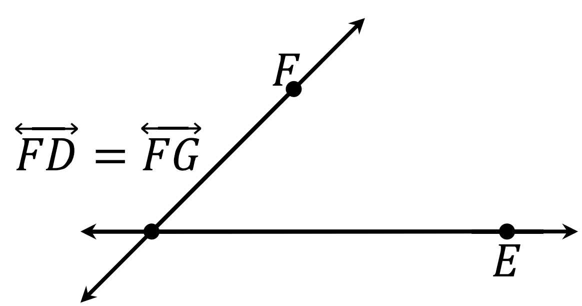
(6) Points D and G are on the same side of line $\overleftrightarrow{EF}$. **(Justify.)**



(7) Ray $\overrightarrow{FD}$ must be the same ray as $\overrightarrow{FG}$. **(Justify.)**

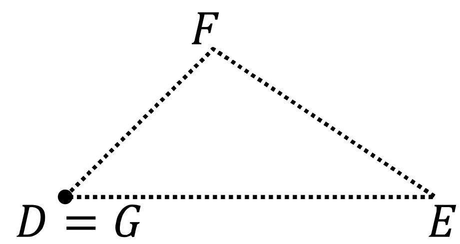


(8) Line $\overleftrightarrow{DF}$ can only intersect line $\overleftrightarrow{DE}$ at a single point. **(Justify.)**



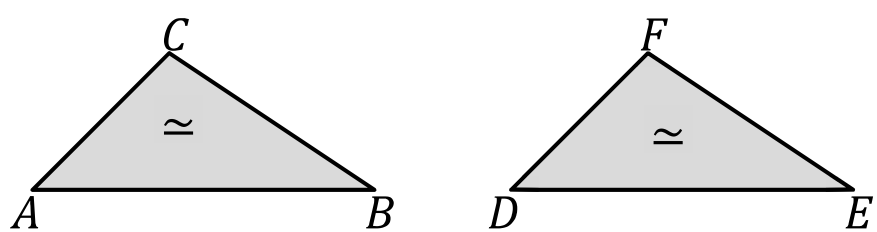
(9) Therefore, points D, G must be the same point. (This tells us that triangle $\cancel{\Delta DEG}$ ΔGEF and triangle ΔDEF are the same triangle.)

(**Remark**: In the *Intermediate Triangle Strategy*, we have proven that that $\Delta 3$ is congruent to $\Delta 2$.)



(10) $\Delta ABC \cong \Delta DEF$ **(Justify.)**

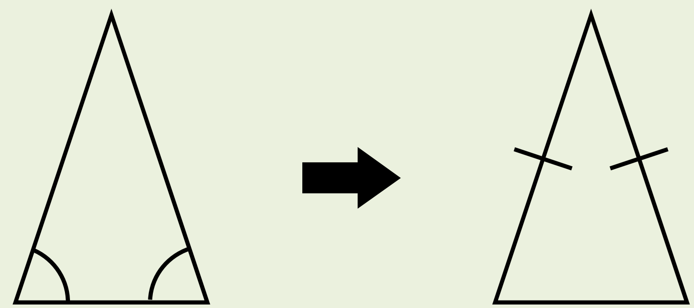
(**Remark:** In the *Intermediate Triangle Strategy*, this proves that $\Delta 1$ is congruent to $\Delta 2$.)



End of proof

Third Topic: The *CA* ➔ *CS* Theorem

Theorem 6.2.2 (Converse of the Statement of Pons Asinorum) (CA ➔ CS Theorem)
In Neutral geometry, if two angles of a triangle are congruent, then the sides opposite those angles are also congruent. That is, in a triangle, if *CA* then *CS*.



The Proof Mimics the proof of the CS ➔ CA Theorem that was presented in Video 6.1b.

Key difference:

- The proof of *CS* ➔ *CA* uses the *SAS axiom*.
- The proof of *CA* ➔ *CS* will use the *ASA theorem*.

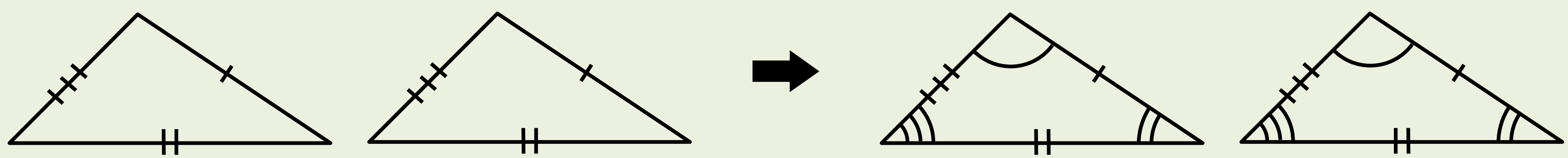
You'll prove Theorem 6.2.2 in a Homework Exercise.

Fourth Topic: Theorem: Neutral geometry Satisfies *SSS*

This video began with the first of three theorems about triangle congruence axioms that are automatically true in neutral geometry. Theorem 6.2.1 says that every neutral geometry satisfies *ASA*. (In other words, if a protractor geometry satisfies *SAS*, then it also satisfies *ASA*.)

The second theorem about the triangle congruence axioms says that every neutral geometry satisfies *SSS*. (In other words, if a protractor geometry satisfies *SAS*, then it also satisfies *SSS*.)

Theorem 6.2.3 A Neutral Geometry Satisfies the Side-Side-Side (SSS) axiom



The proof of this theorem is very long, involving many cases. The authors present a proof of one of the cases in the book on pages 132 – 133. We will study one of the other cases, and you will justify some of the statements in a homework exercise.

Proof of Theorem 6.2.3

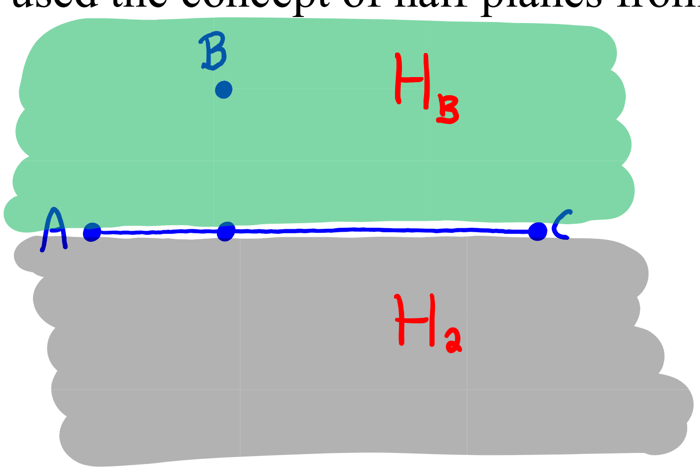
(1) In a neutral geometry, suppose that there is a bijection between the vertices of two triangles such that each pair of corresponding sides is congruent.

Label the triangles ΔABC and $\angle DEF$ so that $\overline{AB} \simeq \overline{DE}$ and $\overline{BC} \simeq \overline{EF}$ and $\overline{CA} \simeq \overline{FD}$.

(**Remark**: In the *Intermediate Triangle Strategy*, triangle ΔABC is $\Delta 1$ and triangle ΔDEF is $\Delta 2$.)

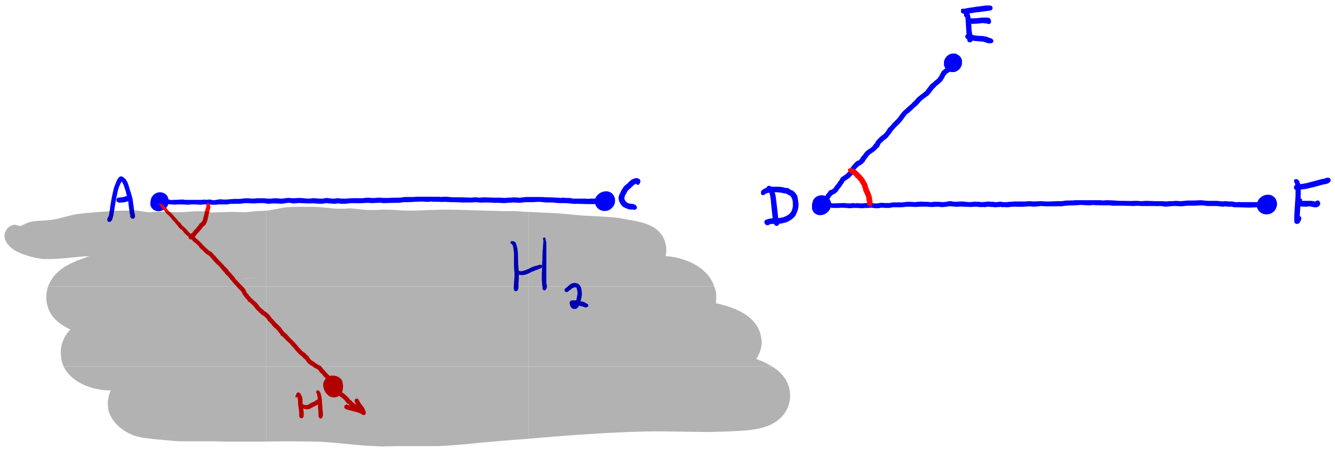
(There are lots of ways that these triangles can look. Here is just one possible illustration.)

(2) Line $\overleftrightarrow{AC}$ creates two half planes. Let H_B be the half plane containing B, and let H_2 be the other one. (Here, we have used the concept of half planes from the Plane Separation Axiom (PSA))



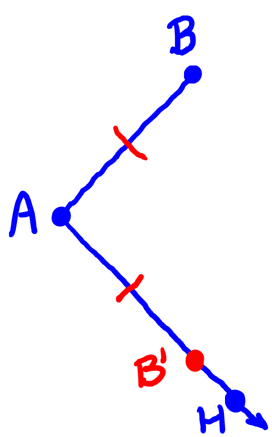
(3) There exists a ray $\overrightarrow{AH}$ such that $H \in H_2$ and such that $\angle CAH \cong \angle FDE$. **(Justify)**

(Again, this is just one possible illustration of many.)



(4) There exists a point B' on ray $\overrightarrow{AH}$ such that $\overline{AB'} \cong \overline{AB}$. **(Justify)** (Again, this is just one possible illustration of many.)

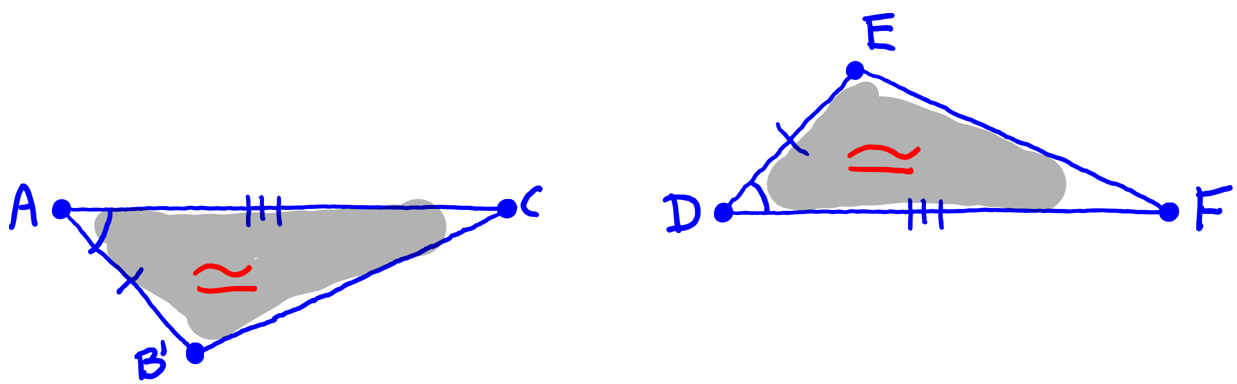
(**Remark**: In the *Intermediate Triangle Strategy*, triangle $\Delta AB'C$ is the *intermediate triangle* $\Delta 3$.)



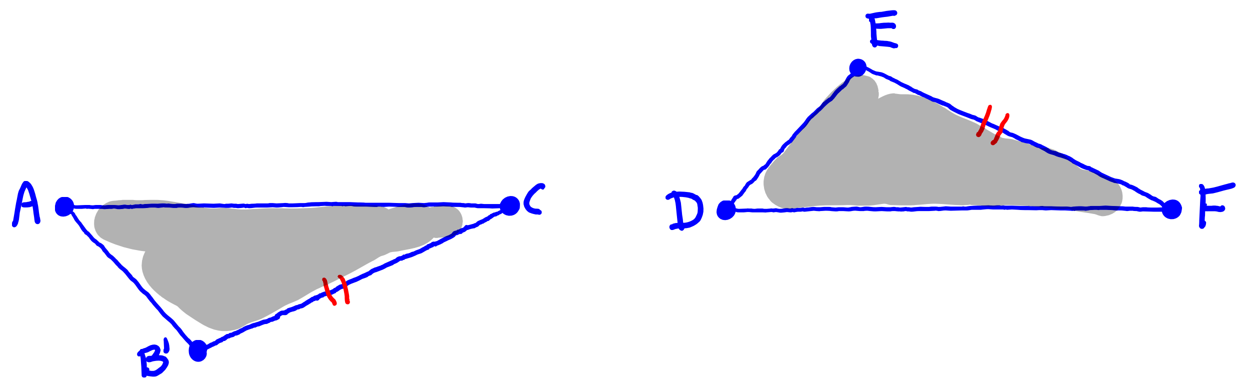
(5) $\overline{AB'} \cong \overline{DE}$. **(Justify)** (Again, this is just one possible illustration of many.)

(6) $\Delta AB'C \cong \Delta DEF$. **(Justify)** (Again, this is just one possible illustration of many.

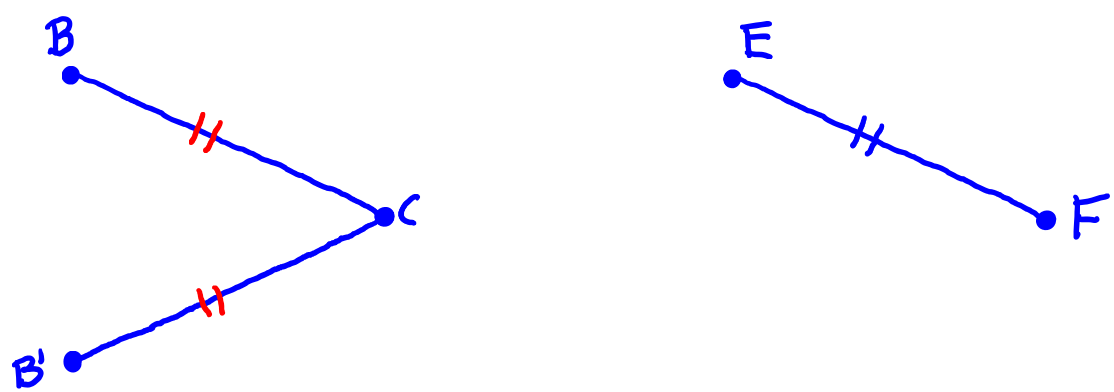
(**Remark**: In the *Intermediate Triangle Strategy*, this proves that that $\Delta 3$ is congruent to $\Delta 2$.)



(7) $\overline{B'C} \cong \overline{EF}$. **(Justify)**

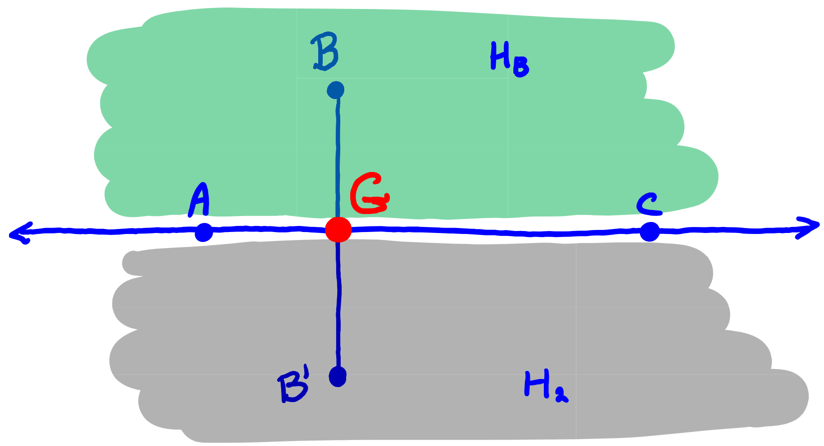


(8) $\overline{B'C} \cong \overline{BC}$. **(Justify)**



Introduce the point G and five possibilities for it.

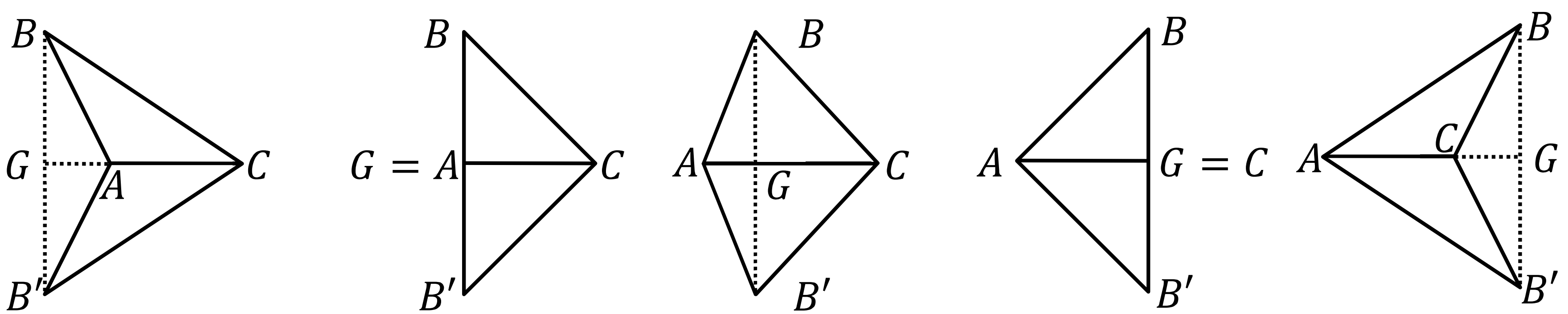
(9) Line $\overleftrightarrow{AC}$ must intersect segment $\overline{BB'}$ at a point G between B and B'. **(Justify.)**



(10) There are five possibilities for the betweenness relatinships among A, C, G on line $\overleftrightarrow{AC}$.

$$\text{(i) } G - A - C \quad \text{or} \quad \text{(ii) } G = A \quad \text{or} \quad \text{(iii) } A - G - C \quad \text{or} \quad \text{(iv) } G = C \quad \text{or} \quad \text{(v) } A - C - G$$

These possibilities are illustrated in the drawings below.



(i) $G - A - C$ (ii) $G = A$ (iii) $A - G - C$ (iv) $G = C$ (v) $A - C - G$

Case (i)

(11) In the book on pages 132 – 133, the authors present a proof that the claim $\Delta ABC \simeq \Delta DEF$ is true for Case (i).

Case (v)

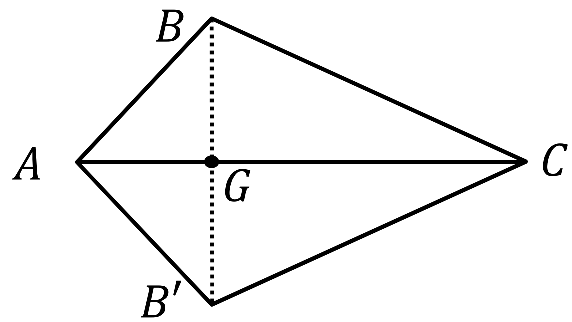
(12) A proof that the claim $\Delta ABC \simeq \Delta DEF$ is true for Case (v) would be similar to the book's proof for case (i).

Cases (ii) and (iv)

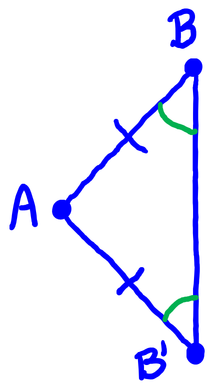
(13) The proof that the claim $\Delta ABC \simeq \Delta DEF$ is true for Cases (ii) & (iv) is straightforward.

Case (iii)

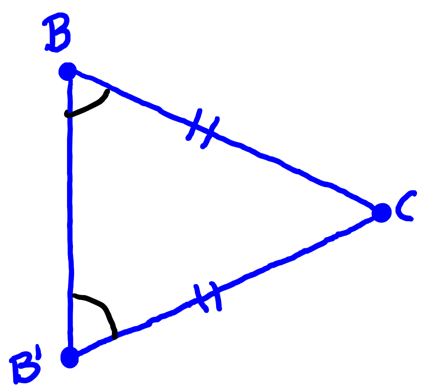
(14) Suppose $A - G - C$.



(15) $\angle ABB' \simeq \angle AB'B$. **(Justify.)**

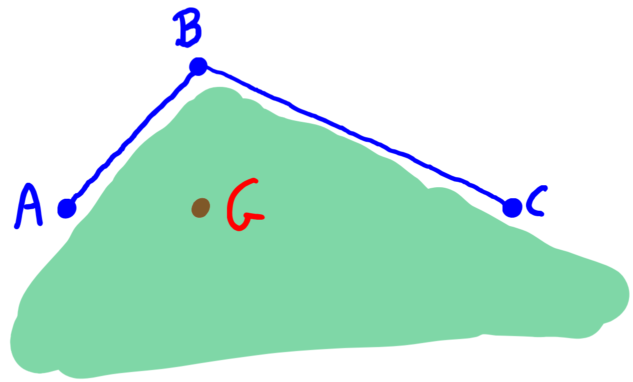


(16) $\angle CBB' \simeq \angle CB'B$. **(Justify.)**

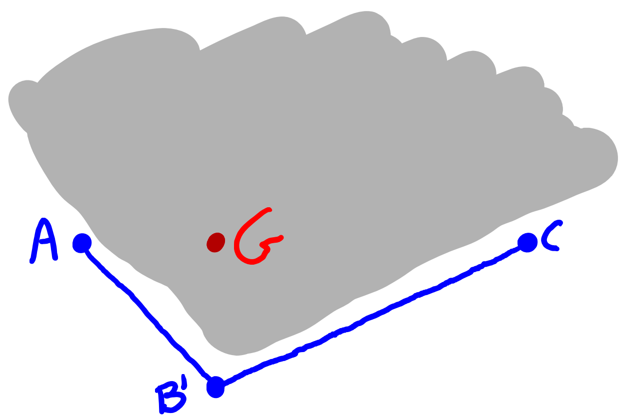


Establish that point G is in the interiors of two angles

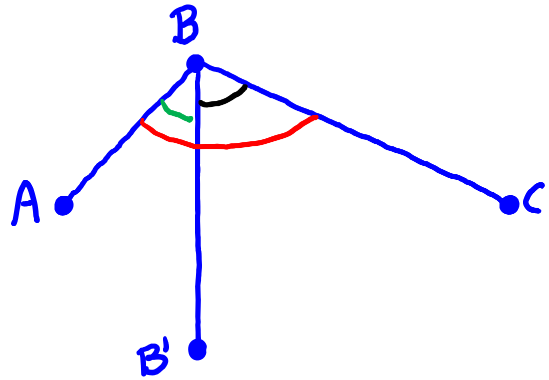
(17) Point G is in the interior of $\angle ABC$. **(Justify.)**

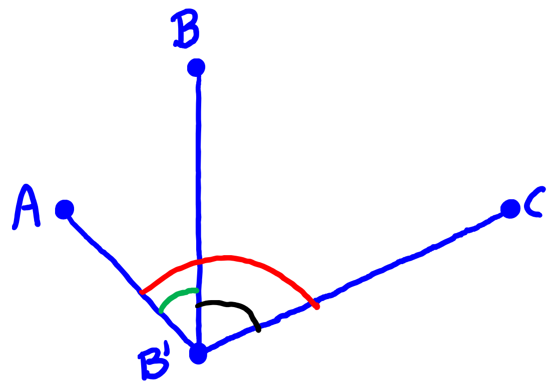


(18) Point G is in the interior of $\angle AB'C$. **(Justify.)**



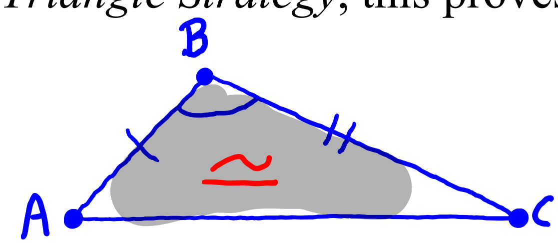
(19) $\angle ABC \simeq \angle AB'C$. **(Justify.)**

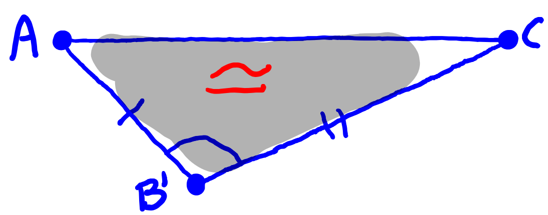




(~~120~~) (20) $\Delta ABC \simeq \Delta AB'C$. **(Justify.)**

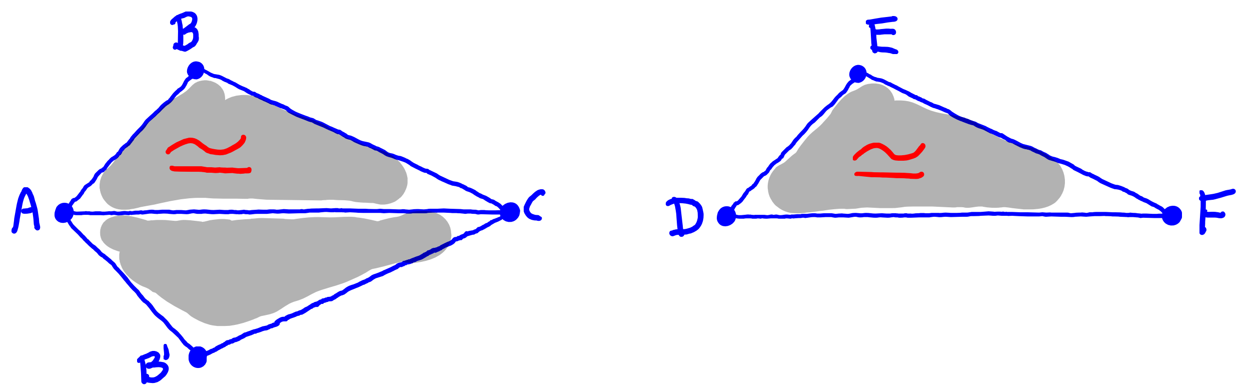
(**Remark**: In the *Intermediate Triangle Strategy*, this proves that that Δ3 is congruent to Δ1.)





(21) $\Delta ABC \simeq \Delta DEF$. **(Justify.)** So the claim is true in case (iii)

(**Remark:** In the *Intermediate Triangle Strategy*, this proves that $\Delta 1$ is congruent to $\Delta 2$.)



Conclusion of cases.

(22) Conclude that $\Delta ABC \simeq \Delta DEF$, because it is true in every case.

End of proof

~~**Remark:** In the above proof, notice that the Axiom or Thereom that was used to justify saying that $\Delta 3$ is congruent to $\Delta 2$ was not the same as the Axiom or Theorem that was used to justify saying that $\Delta 3$ is congruent to $\Delta 1$. This often happens when using the *Intermediate Triangle Strategy*.~~

Fifth Topic: Existence of a perpendicular to a line through a point not on the line

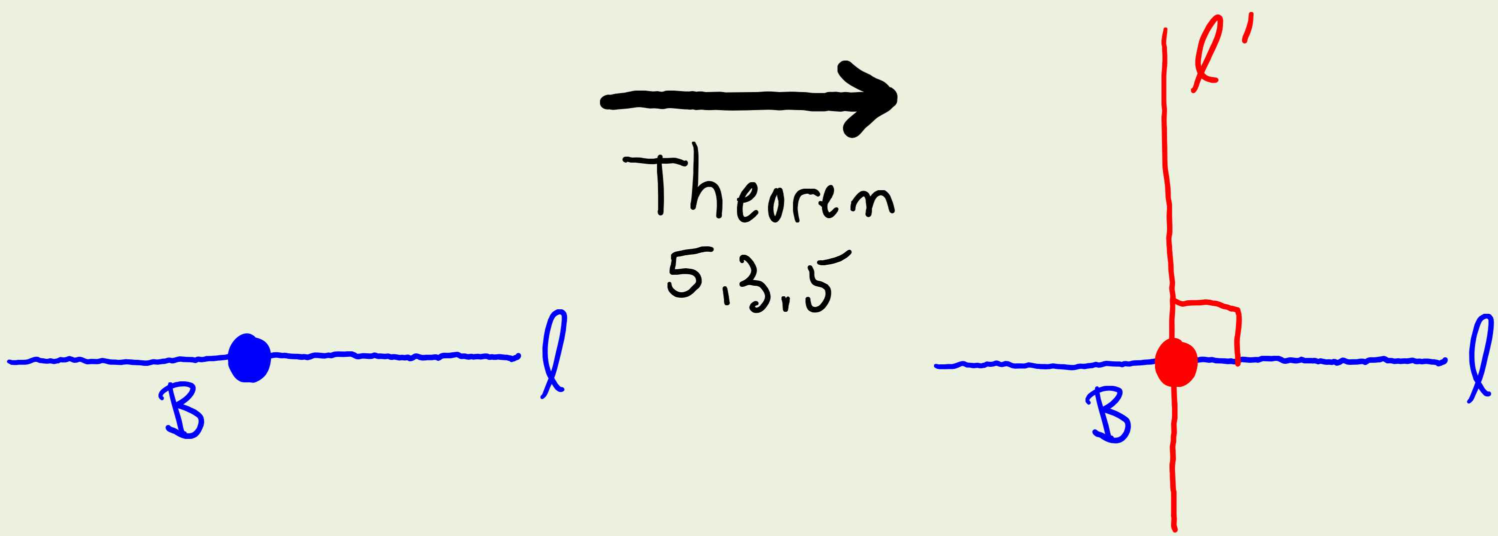
Recall Theorem 5.3.5

Theorem 5.3.5

Existence of a Unique Perpendicular to a Line through a Point *On the Line*

In a protractor geometry, if B is a point on a line l,

then there exists exactly one line l' such that l' contains B and $l' \perp l$.



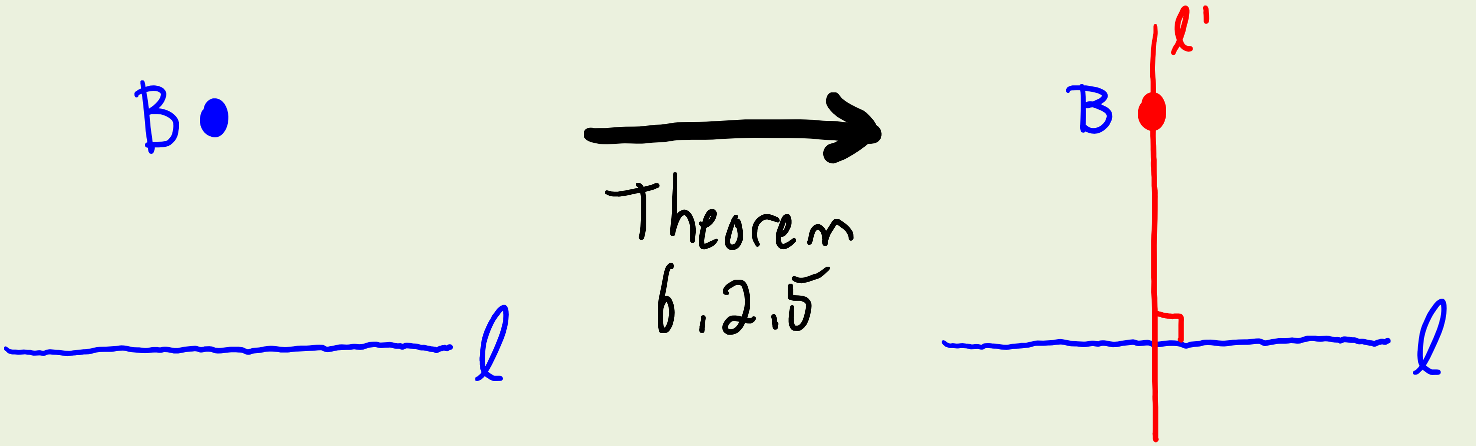
There is a related Theorem in Neutral Geometry but with point *B not* on line *l*.

Theorem 6.2.5

Existence of At Least One Perpendicular to a Line through a Point *Not On the Line*

In a ~~protractor~~ neutral geometry, if B is a point not on a line l,

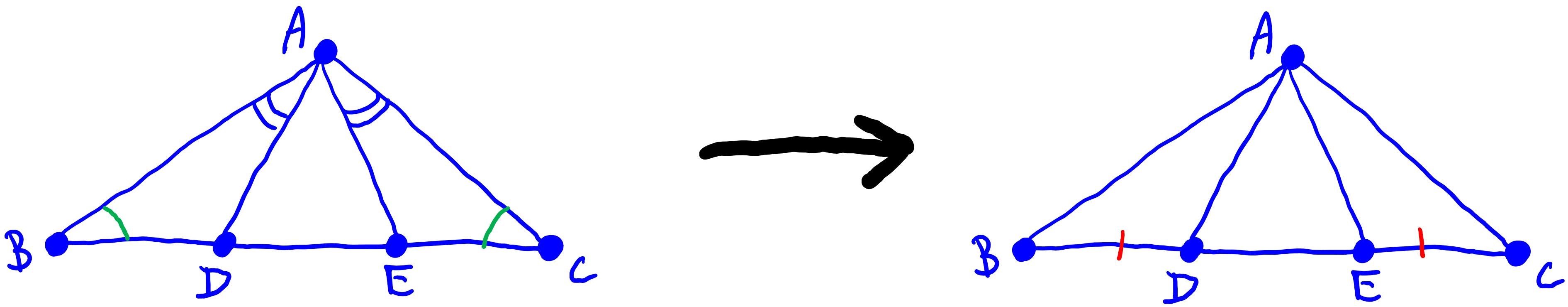
then there exists ***at least one*** line l' such that l' contains B and $l' \perp l$.



There is a proof of Theorem 6.2.5 in the book. It is very similar in style to the proof of Theorem 6.2.3 (*SSS*) that was just presented here. For that reason, I won't discuss the proof in this video.

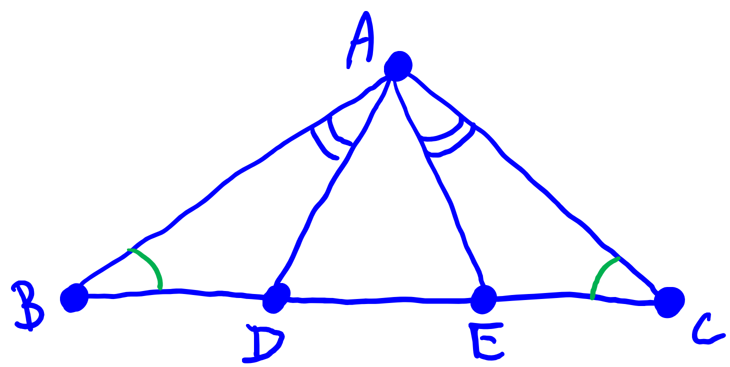
Sixth (and Final) Topic: Proving Particular Congruences

[Example 1](similar to 6.2#5,6) Prove the following: In a neutral geometry, if ΔABC and points D, E satisfy $B - D - E - C$ and $\angle ABC \simeq \angle ACB$, and $\angle BAD \simeq \angle CAE$, then $\overline{BD} \simeq \overline{CE}$.

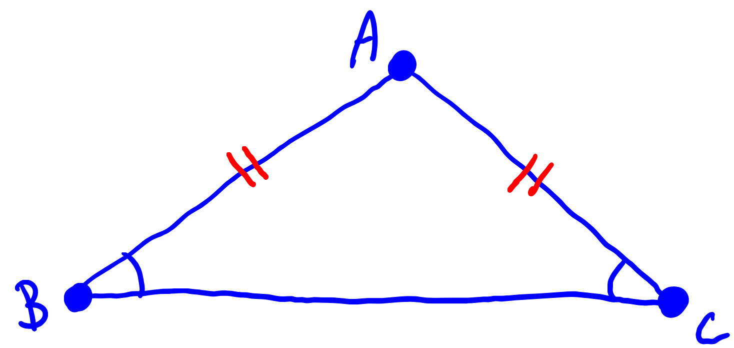


Proof

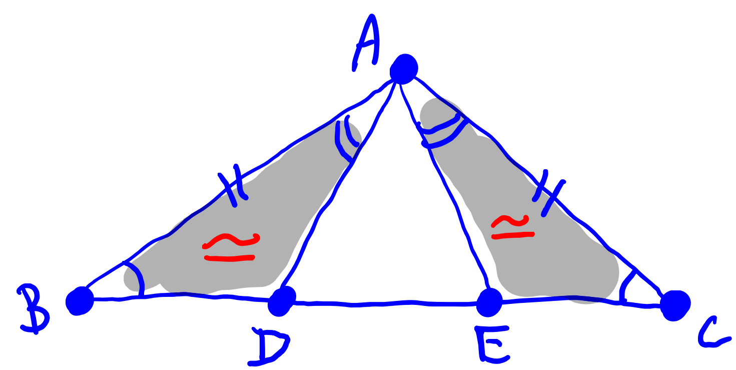
(1) Suppose that in a neutral geometry, triangle ΔABC and points D, E satisfy $B - D - E - C$ and $\angle ABC \simeq \angle ACB$, and $\angle BAD \simeq \angle CAE$.



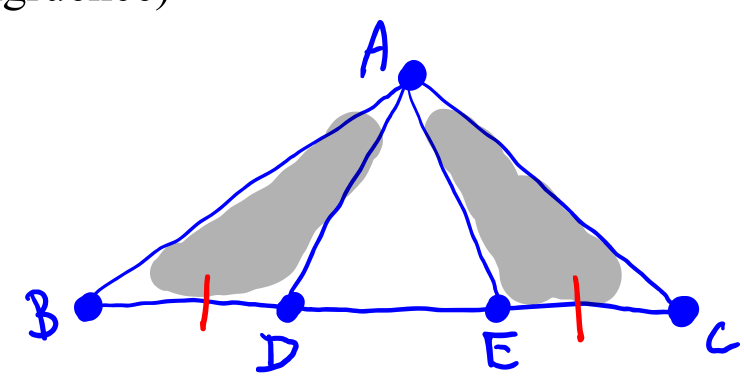
(2) $\overline{AB} \simeq \overline{AC}$ (by (1) and Theorem 6.2.2 CA ➔ CS)



(3) $\Delta ABD \simeq \Delta ACE$ (by Theorem 6.2.1 (ASA) applied to ΔABD and ΔACE with $\angle ABC \simeq \angle ACB$ and $\angle ADB \simeq \angle AEC$ and $\overline{AB} \simeq \overline{AC}$.)

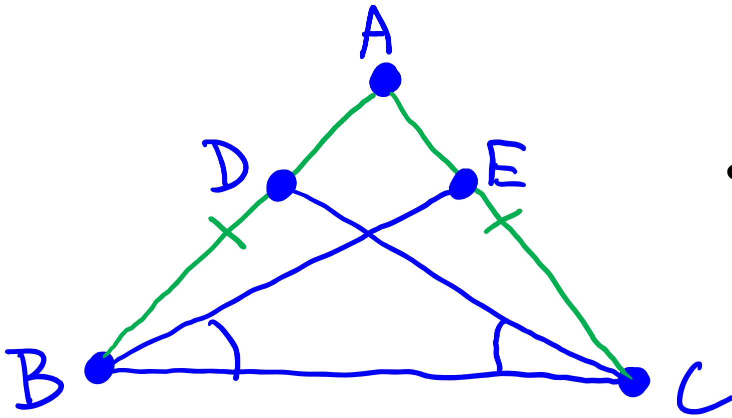


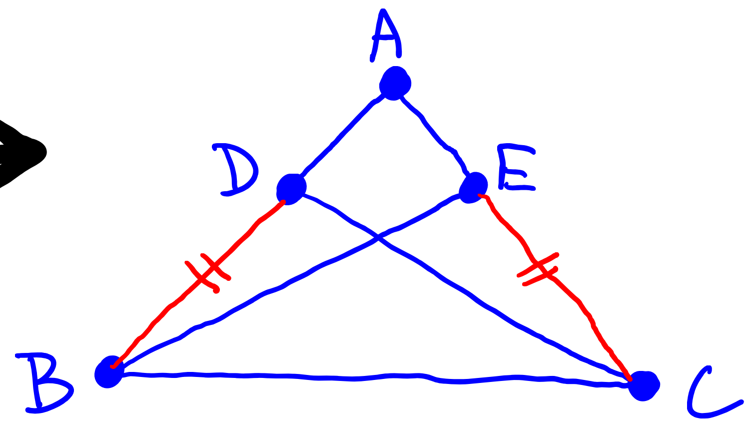
(4) $\overline{BD} \simeq \overline{CE}$ (by (3) and definition of triangle congruence)



End of Proof

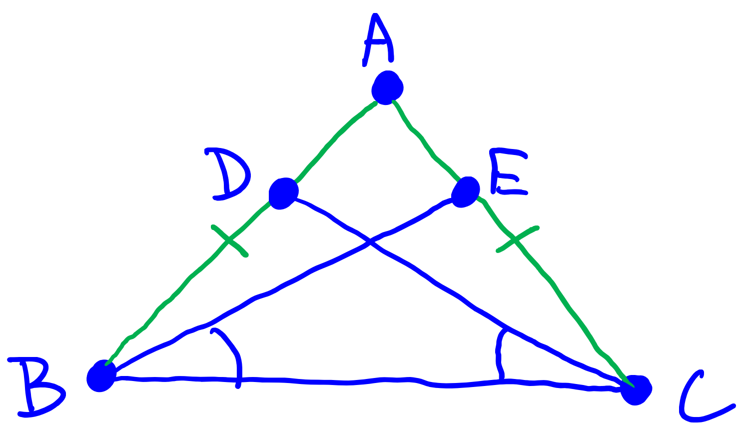
[Example 2](similar to 6.2#8) Prove the following: In a neutral geometry, if ΔABC and points D, E satisfy $A - D - B$ and $A - E - C$ and $\overline{AB} \simeq \overline{AC}$ and $\angle BCD \simeq \angle CBE$, then $\overline{BD} \simeq \overline{CE}$.



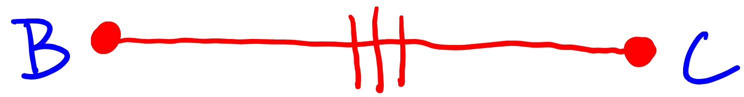


Proof

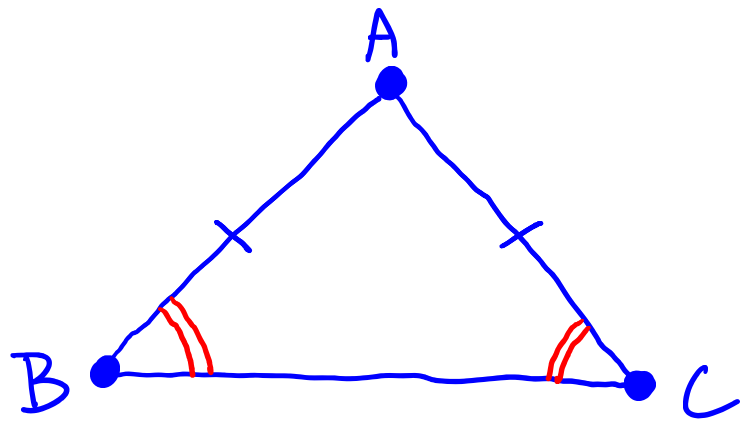
(1) Suppose that in a neutral geometry, ΔABC and points D, E satisfy $A - D - B$ and $A - E - C$ and $\overline{AB} \simeq \overline{AC}$ and $\angle BCD \simeq \angle CBE$.



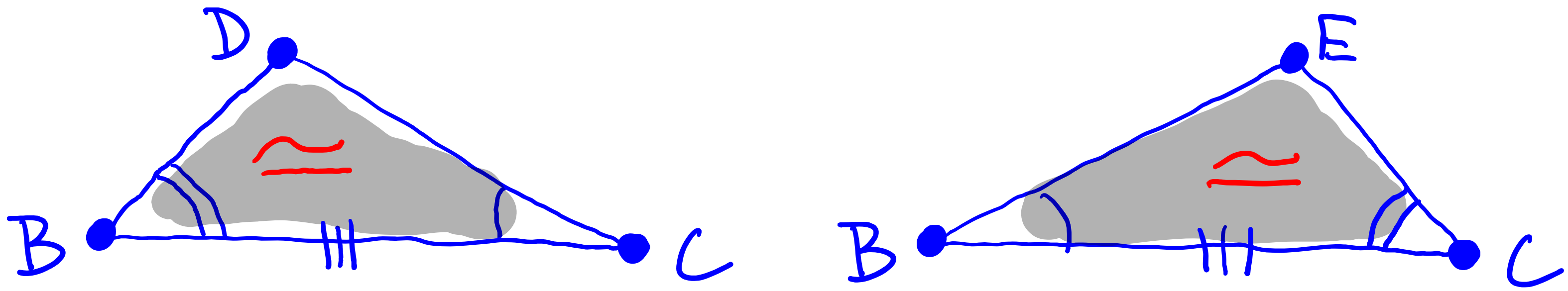
(2) $\overline{BC} \simeq \overline{CB}$ (a segment is congruent to itself because it has the same length as itself)



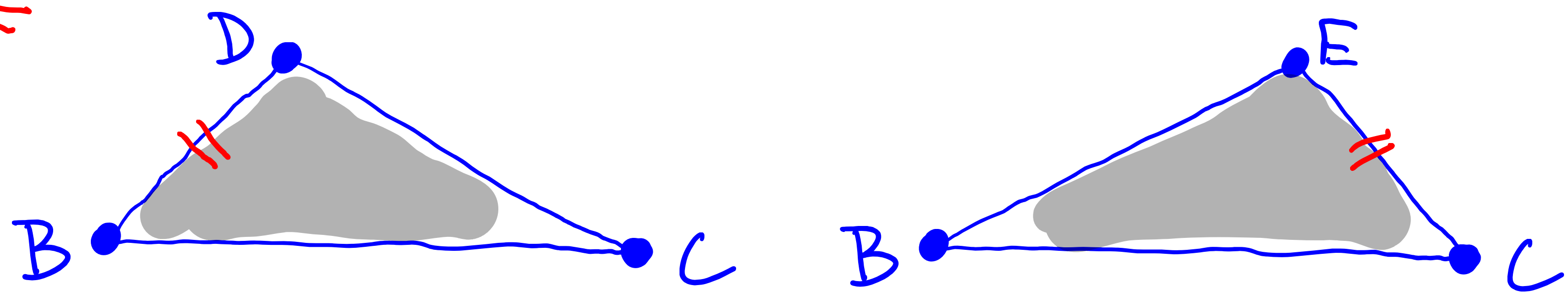
(3) $\angle ABC \simeq \angle ACB$ (by Theorem 6.1.5 CS ➔ CA)



(4) $\Delta BCD \simeq \Delta CBE$ (by Theorem 6.2.1 (ASA) applied to ΔBCD and ΔCBE with $\angle BCD \simeq \angle CBE$ and $\overline{BC} \simeq \overline{CB}$ and $\angle ABC \simeq \angle ACB$.)



(5) $\overline{BD} \simeq \overline{CE}$ (by (4) and definition of triangle congruence)



End of Proof

End of Video