<u>Video 6.3a The Neutral Exterior Angle Theorem and an Immediate Corollary</u>

produced by Mark Barsamian, 2022.04.03

for Ohio University MATH 3110/5110 College Geometry

Topics for this Video

- **The Neutral Exterior Angle Theorem**
- **Corollary about the number of lines through a given point perpendicular to a given line**

Reading: Pages 135 – 137 of Section 6.3 The Exterior Angle Theorem and Its Consequences in *Geometry: A Metric Approach with Models, Second Edition* by Millman & Parker

Relevant Book Exercises: Section 6.3 # 1, 2, 3, 4

Recall Old Stuff that Will be Useful

Recall Theorem 3.2.6

In a metric geometry, if A and B are distinct points, then

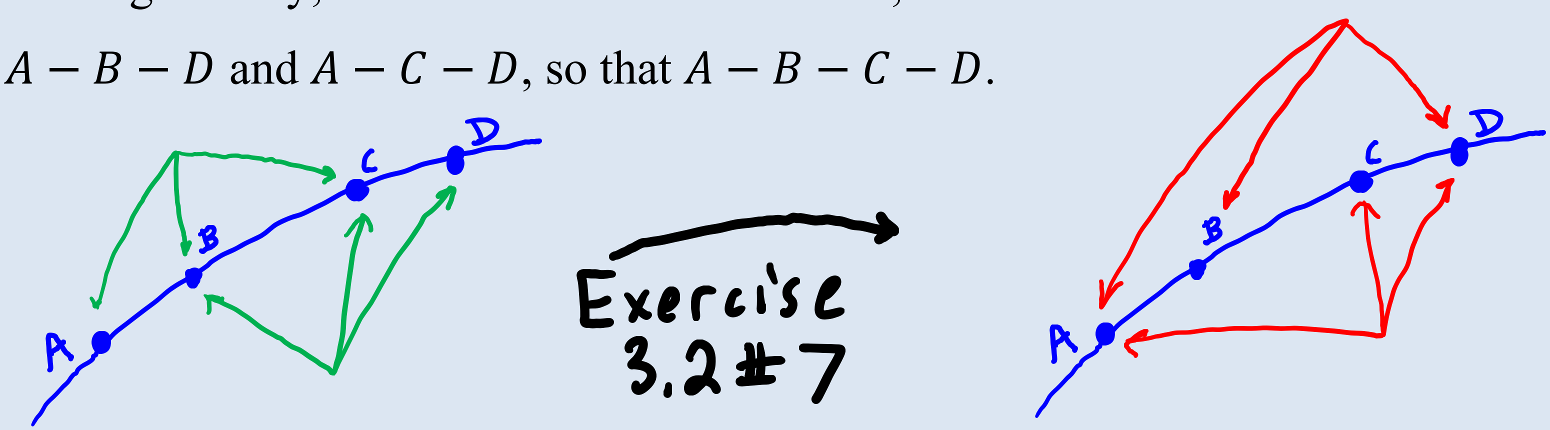
(i) there exists a point C such that $A - B - C$; and

(ii) there exists a point D such that $A - D - B$.

Recall Exercise 3.2#7

In a metric geometry, if $A - B - C$ and $B - C - D$,

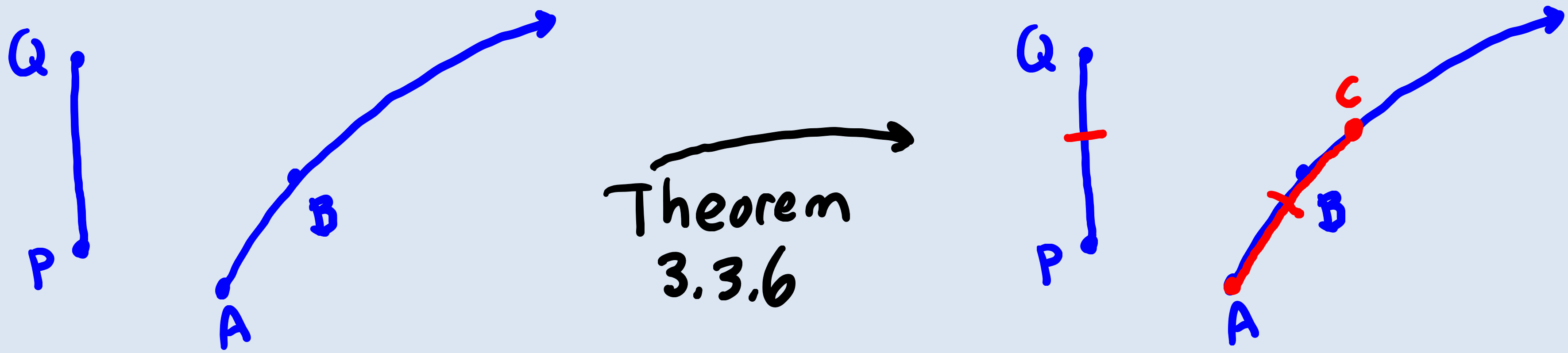
then $A - B - D$ and $A - C - D$, so that $A - B - C - D$.



Recall Theorem 3.3.6, the Congruent Segment Construction Theorem

In a metric geometry, given ray $\overrightarrow{AB}$ and segment $\overline{PQ}$,

there exists a unique point $C \in \overrightarrow{AB}$ such that $\overline{AC} \simeq \overline{PQ}$.

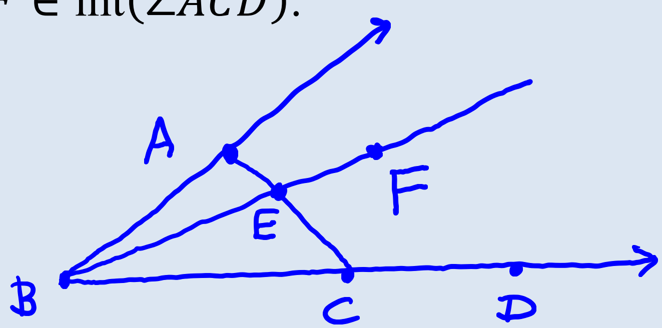


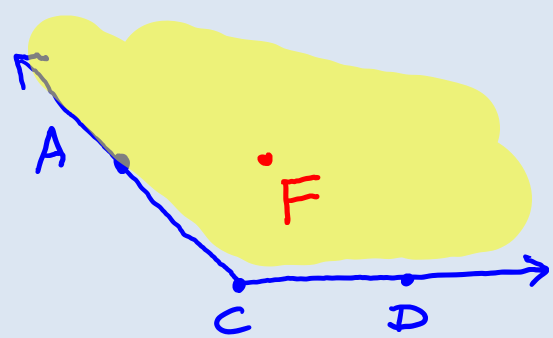
Recall book exercise 4.4#6

In a Pasch geometry,

if ΔABC and points D, E, F satisfy $B - C - D$ and $A - E - C$ and $B - E - F$,

then $F \in \text{int}(\angle ACD)$.

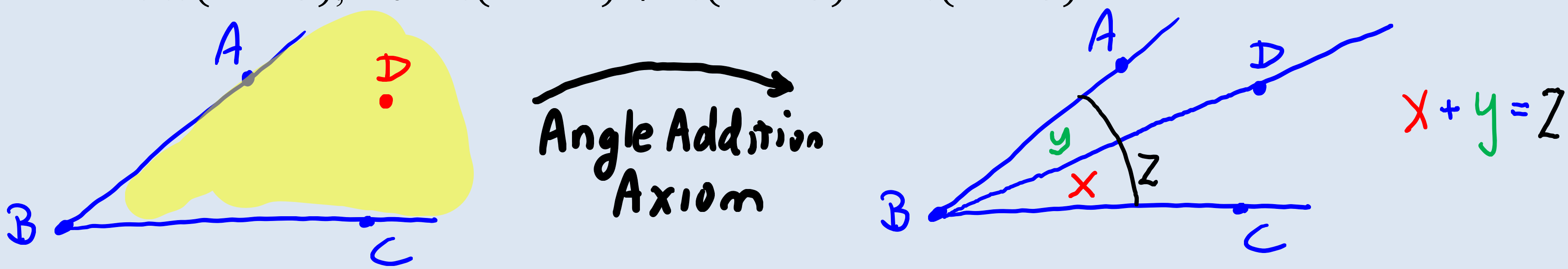




Recall that in Section 5.3, there was some discussion of *Bigger* and *Smaller Angles*.

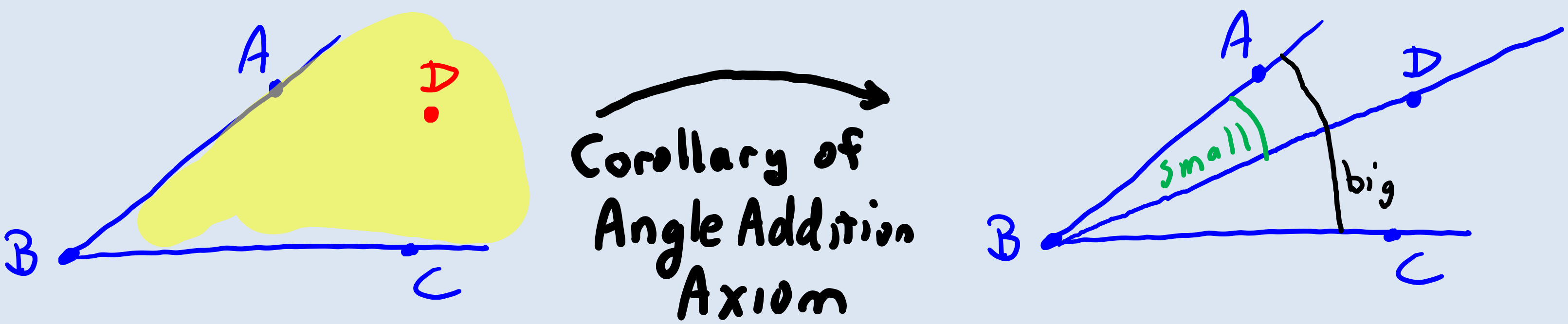
Recall the Angle Measure Axiom (iii) (the Angle Addition Axiom) from Section 5.1

If $D \in int(\angle ABC)$, then $m(\angle ABD) + m(\angle DBC) = m(\angle ABC)$



Unstated Corollary of the Angle Addition Axiom

If $D \in int(\angle ABC)$ then $m(\angle ABD) < m(\angle ABC)$.

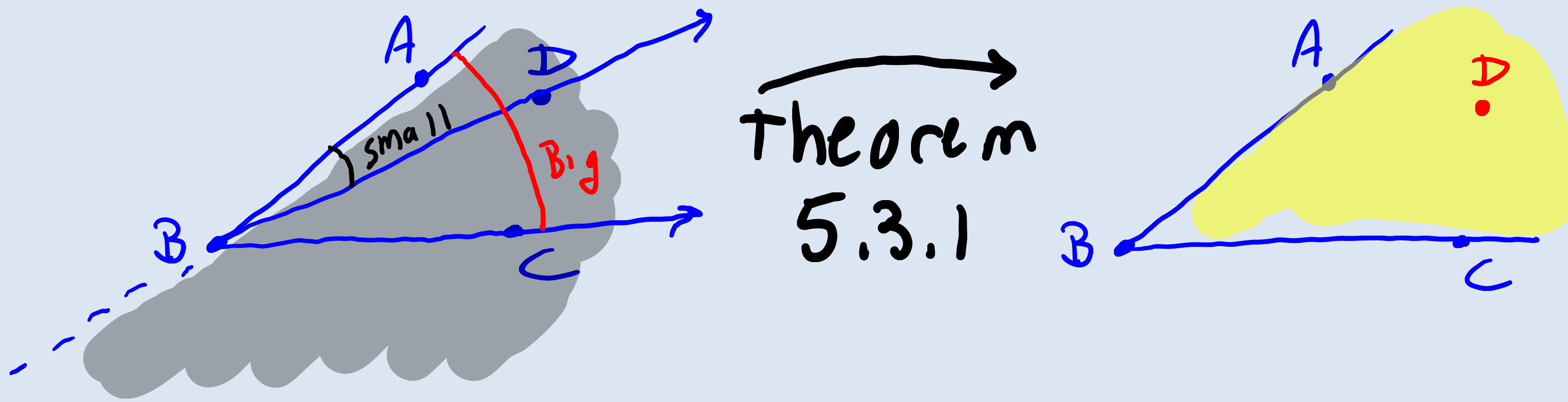


Contrapositive of Unstated Corollary: If $m(\angle ABD) \nless m(\angle ABC)$, then $D \notin int(\angle ABC)$.

Theorem 5.3.1

In a protractor geometry, given points C, D in the same half plane of $\overleftrightarrow{AB}$,

if $m(\angle ABD) < m(\angle ABC)$ then $D \in int(\angle ABC)$



Theorem 5.3.1 (Contrapositive Version)

In a protractor geometry, given points C, D in the same half plane of $\overleftrightarrow{AB}$,

if $D \notin int(\angle ABC)$ then $m(\angle ABD) \nless m(\angle ABC)$

Notice that since the introduction of the concept of Triangle Congruence in Section 6.1, the discussion has been about proving that certain objects are *congruent*. Initially, our only mathematical tool was the *Side-Angle-Side (SAS) Congruence Axiom*. Soon,,though, more theorems were proved that enlarged our mathematical toolbox. But those theorems were all about *congruent* objects.

In Section 6.3, we will study the *Neutral Exterior Angle Theorem*. This theorem is an extremely important new tool that will allow us to prove theorems about *bigger* and *smaller objects*.

Section 6.3 Starts with some definitions that are pretty self-explanatory.

Definition of *smaller* and *larger segments* and *angles* from the book

Symbol: $\overline{AB} < \overline{CD}$

Words: $\overline{AB}$ *is less than* $\overline{CD}$, or $\overline{AB}$ *is smaller than* $\overline{CD}$

Meaning: $AB < CD$

Symbol: $\overline{AB} > \overline{CD}$

Words: $\overline{AB}$ *is greater than* $\overline{CD}$, or $\overline{AB}$ *is larger than* $\overline{CD}$

Meaning: $AB > CD$

Symbol: $\angle ABC < \angle DEF$

Words: $\angle ABC$ *is less than* $\angle DEF$, or $\angle ABC$ *is smaller than* $\angle DEF$,

Meaning: $m(\angle ABC) < m(\angle DEF)$

Symbol: $\angle ABC > \angle DEF$

Words: $\angle ABC$ *is greater than* $\angle DEF$, or $\angle ABC$ *is larger than* $\angle DEF$,

Meaning: $m(\angle ABC) > m(\angle DEF)$.

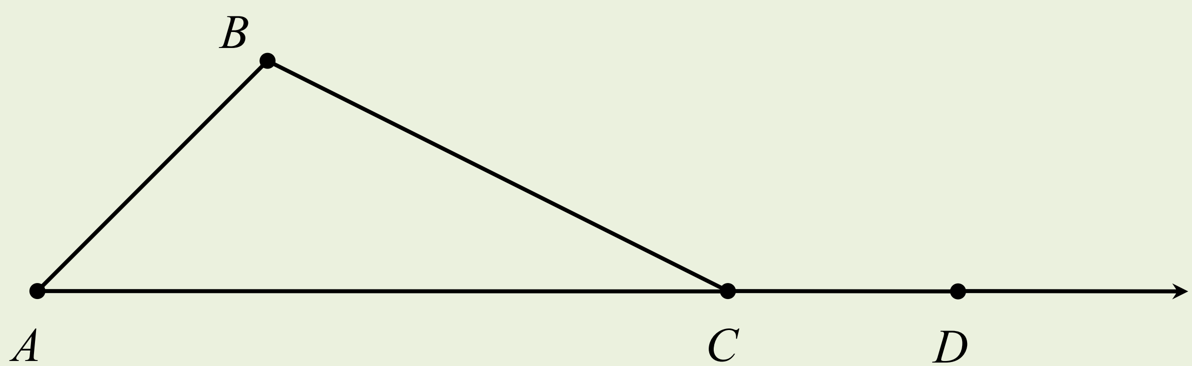
Definition of Exterior Angle and Remote Interior Angles

Words: *exterior angle of a triangle*

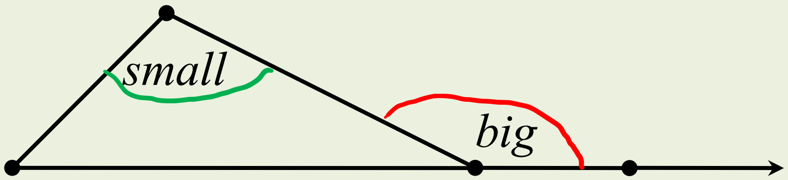
Usage: There is a protractor geometry and a triangle in the discussion.

Meaning: An angle that can be labeled $\angle BCD$ where ΔABC is the given triangle and $A - C - D$.

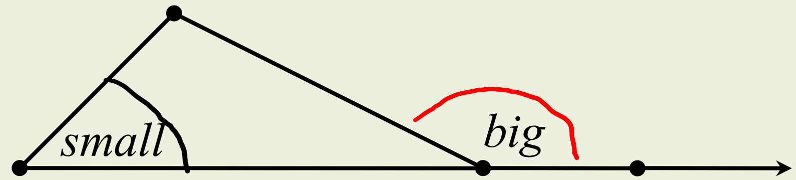
Related Terminology: for the exterior angle $\angle BCD$ of ΔABC, the two angles $\angle ABC$ and $\angle BAC$ are called *remote interior angles for the exterior angle* $\angle BCD$.



Theorem 6.3.3 (Neutral Exterior Angle Theorem) In a neutral geometry, the measure of an exterior angle of a triangle is greater than the measure of either of its remote interior angles.

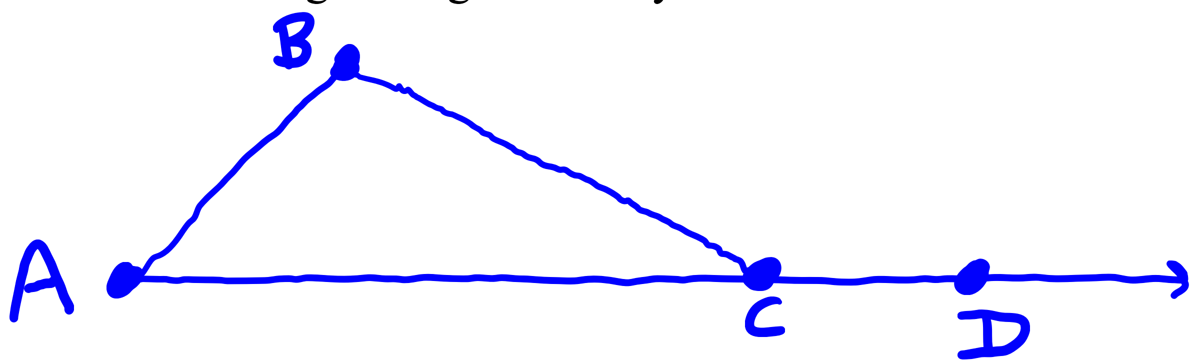


and



Proof

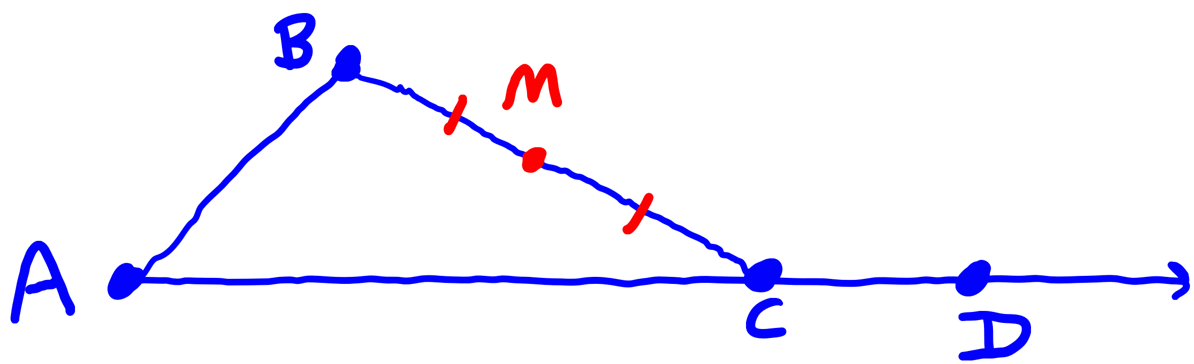
(1) Suppose a triangle and an exterior angle are given. They can be labeled ΔABC and $\angle BCD$, with $A - C - D$.



Part 1: Prove that $\boldsymbol{m(\angle BCD) > m(\angle ABC)}$

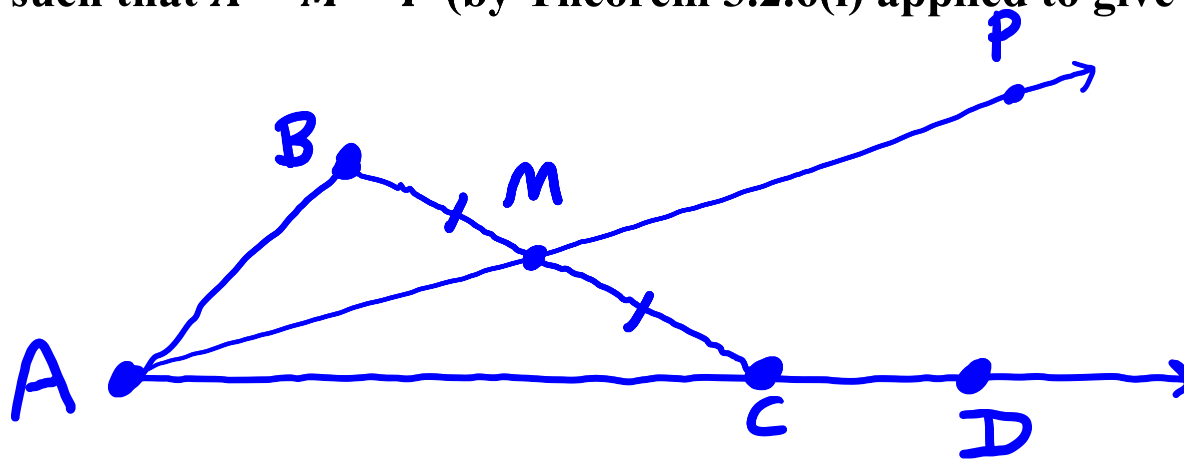
(2) There exists a point M that is the midpoint of $\overline{BC}$. **(Proven in Exercise 3.3#12)**

(3) $\overline{MB} \simeq \overline{MC}$. **(by (2) and definition of midpoint)**



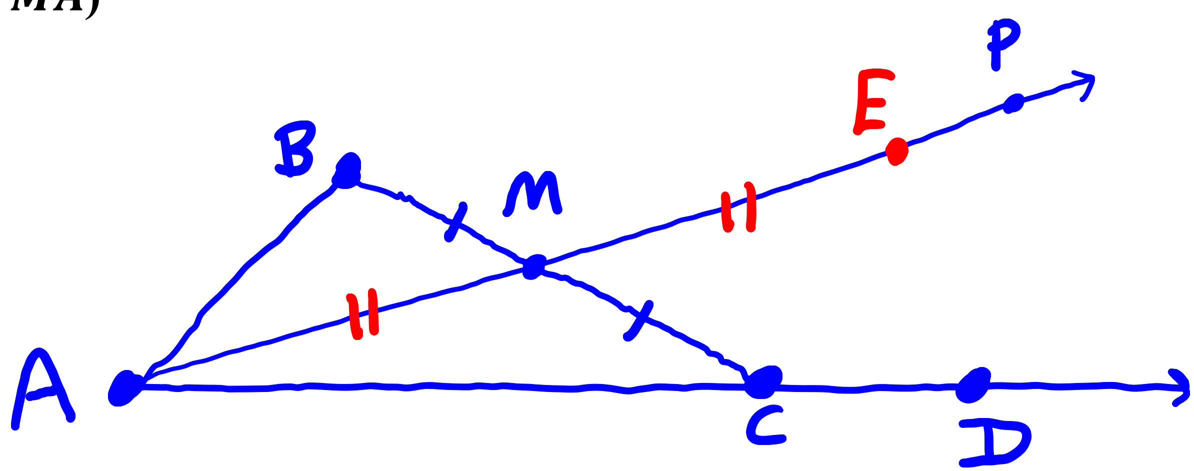
(4) There exists a point E such that $A - M - E$ and $\overline{ME} \simeq \overline{MA}$.

(There exists a point P such that $A - M - P$ (by Theorem 3.2.6(i) applied to give points A, M.)



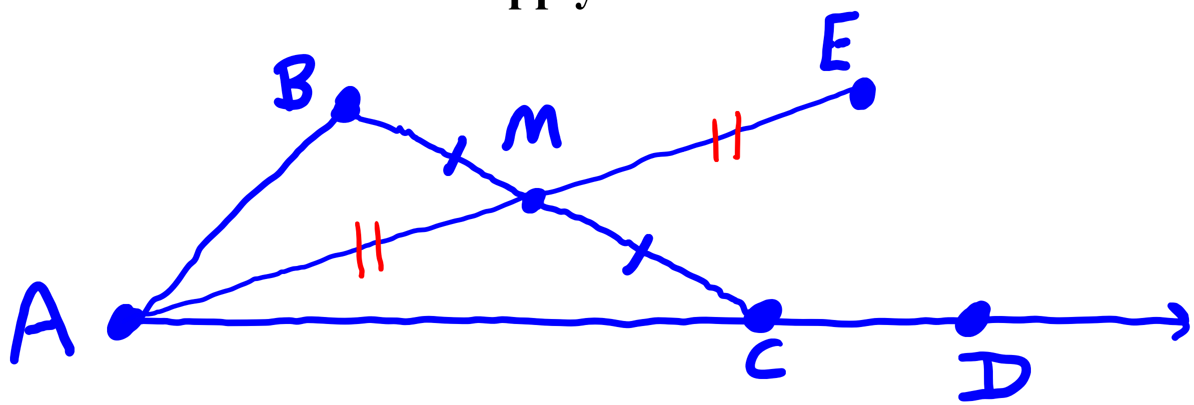
There exists a point E on ray $\overrightarrow{MP}$ such that $\overline{ME} \simeq \overline{MA}$.

(by Theorem 3.3.6, the Congruent Segment Construction Theorem, applied to given ray $\overrightarrow{MP}$ and given segement $\overline{MA}$)



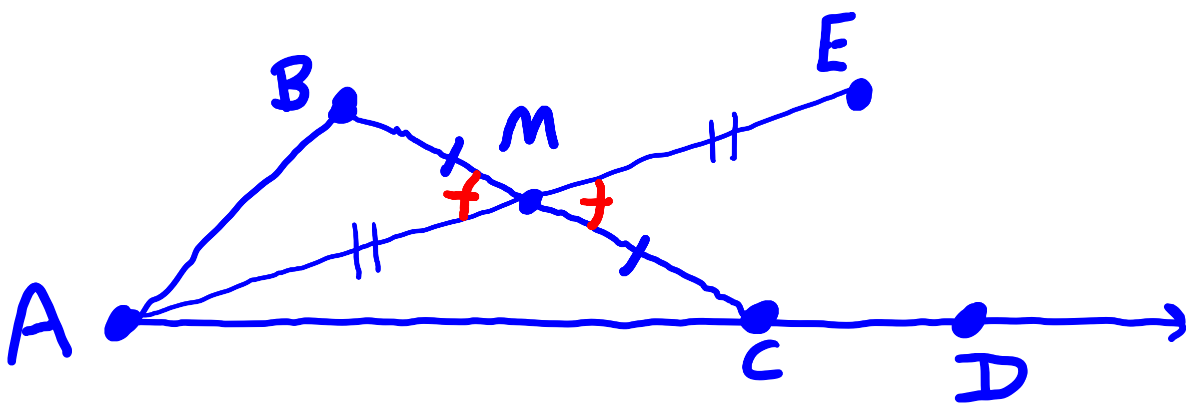
Points A, M, E will have the betweenness relationship $A - M - E$

(because $E \in \overrightarrow{MP}$, we know that either $E = M$ or $M - E - P$ or $E = P$ or $M - P - E$. We know that E cannot be M, because $ME = MA \neq 0$. So either $M - E - P$ or $E = P$ or $M - P - E$. But we also know that $A - M - P$. We can apply the result of Exercise 3.2#7 to say that $A - M - E$.)

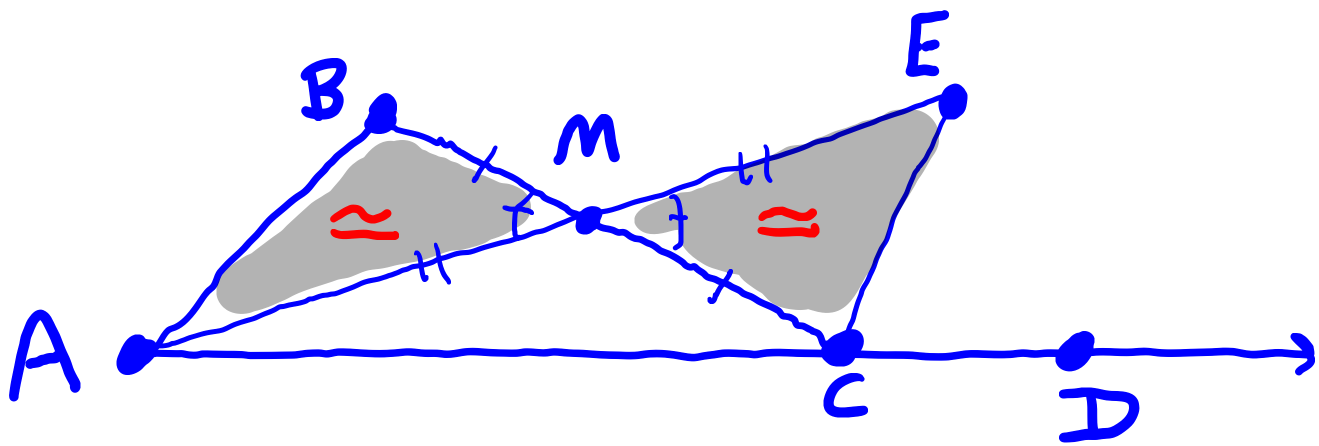


(5) $\angle AMB \simeq \angle EMC$.

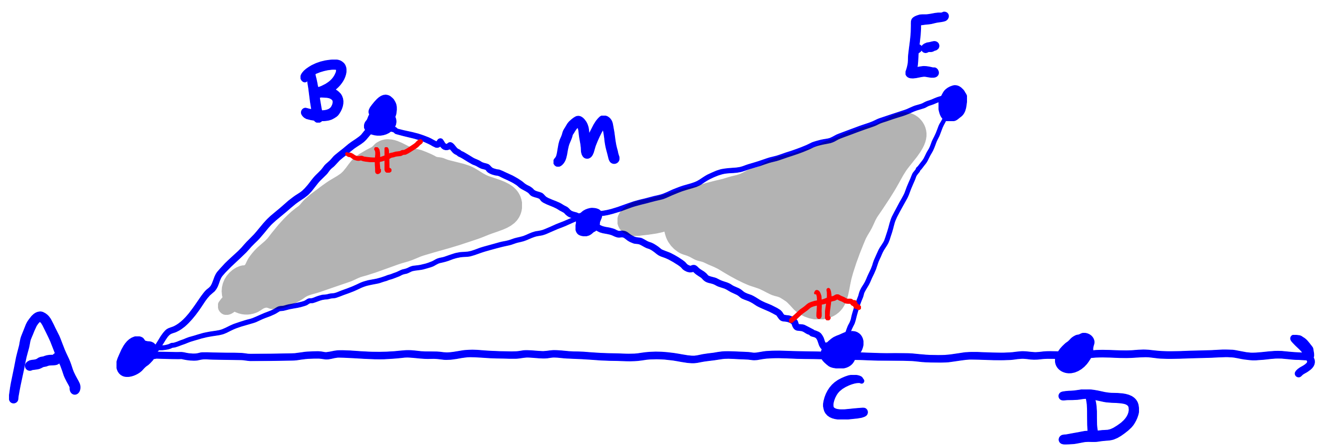
(Angles $\angle AMB$ and $\angle EMC$ are a vertical pair (because $B - M - C$ (proven in 3.3#11a) and $A - M - E$ (from step (4)). So Thm 5.3.9 Vertical Angle Theorem tells us they are congruent)



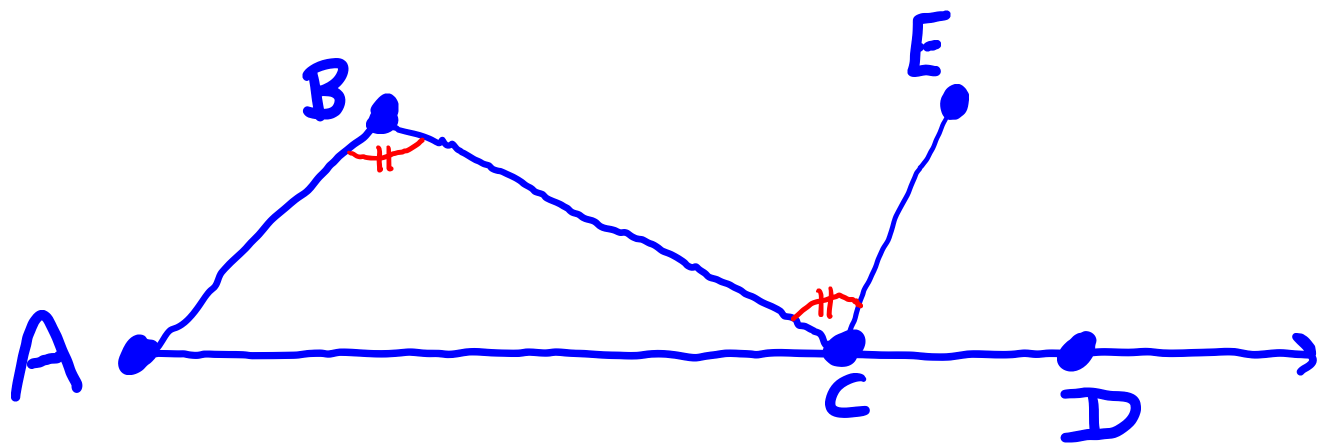
(6) $\Delta AMB \simeq \Delta EMC$. **(by SAS Axiom applied to $\boldsymbol{\Delta AMB}$ and $\boldsymbol{\Delta EMC}$ with $\overline{\boldsymbol{AM}} \simeq \overline{\boldsymbol{EM}}$ (by (4)) and $\boldsymbol{\angle AMB} \simeq \boldsymbol{\angle EMC}$ (by (5)) and $\overline{\boldsymbol{MB}} \simeq \overline{\boldsymbol{MC}}$ (by (3).)**



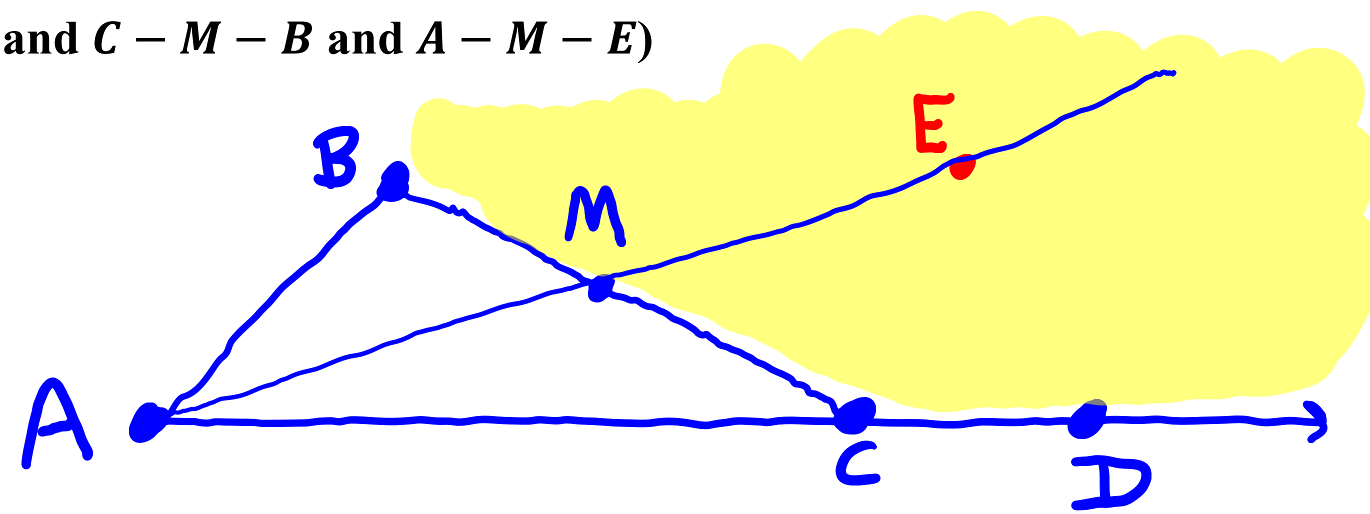
(7) $\angle ABM \simeq \angle ECM$. **(by (6) and definition of triangle congruence)**



(8) So $\angle ABC \simeq \angle ECB$. **(because $\overrightarrow{BM}$ and $\overrightarrow{BC}$ are the same ray, by Theorem 3.3.4(i), so $\angle ABM$ and $\angle ABC$ are the same angle. Similarly, $\angle ECM$ and $\angle ECB$ are the same angle.)**

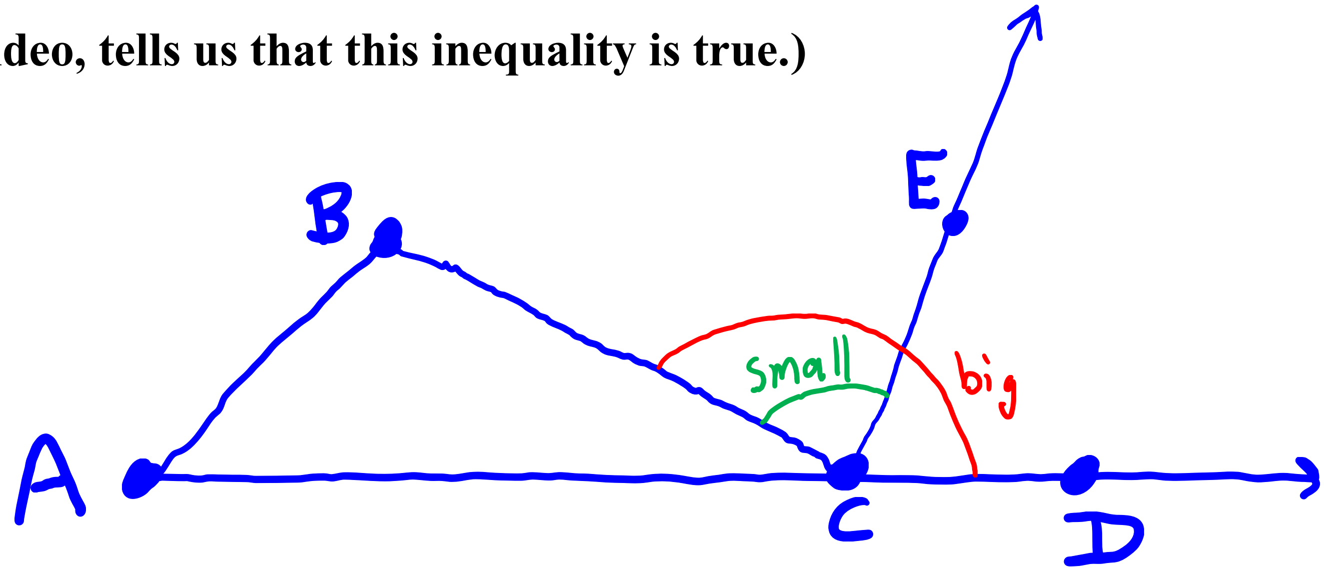


(9) $E \in \text{int}(\angle BCD)$. **(by the result of Exercise 4.4#6 applied to ΔABC and points D, M, E that satisfy $A - C - D$ and $C - M - B$ and $A - M - E$)**

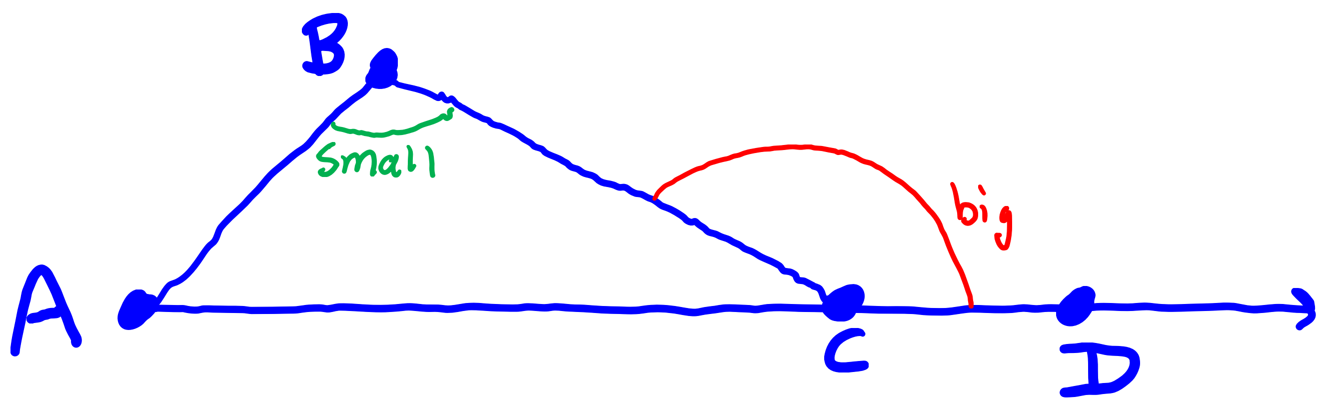


(10) $m(\angle BCD) > m(\angle ECB)$

(Because $\boldsymbol{E} \in \text{int}(\angle \boldsymbol{BCD})$, the Unstated Corollary of the Angle Addition Axiom, discussed at the start of the video, tells us that this inequality is true.)

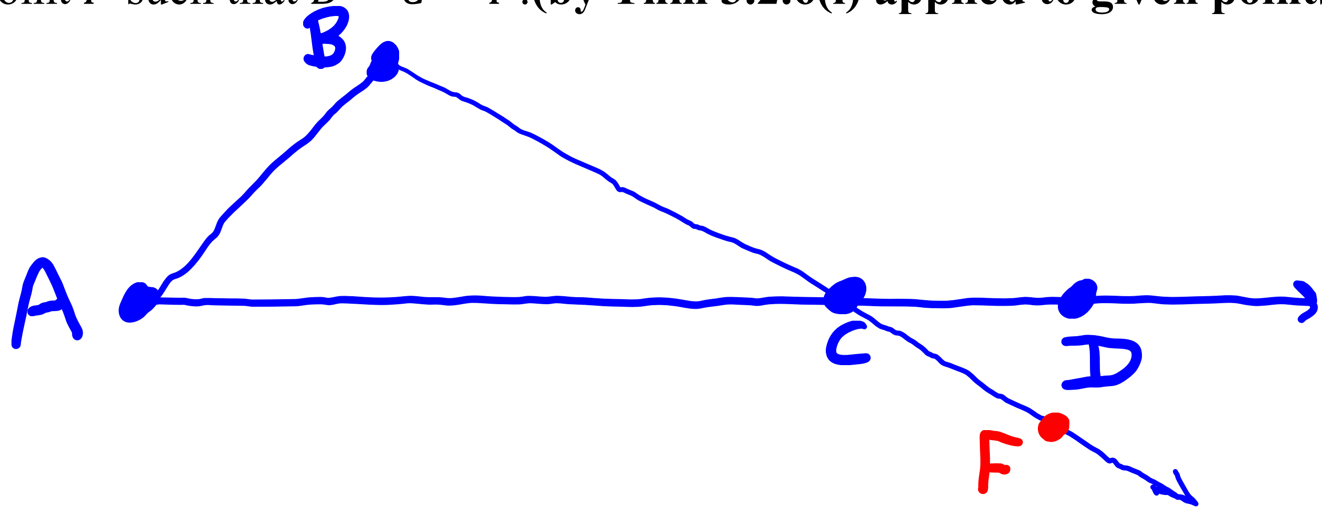


(11) $m(\angle BCD) > m(\angle ABC)$ **(by (10) and (8))**

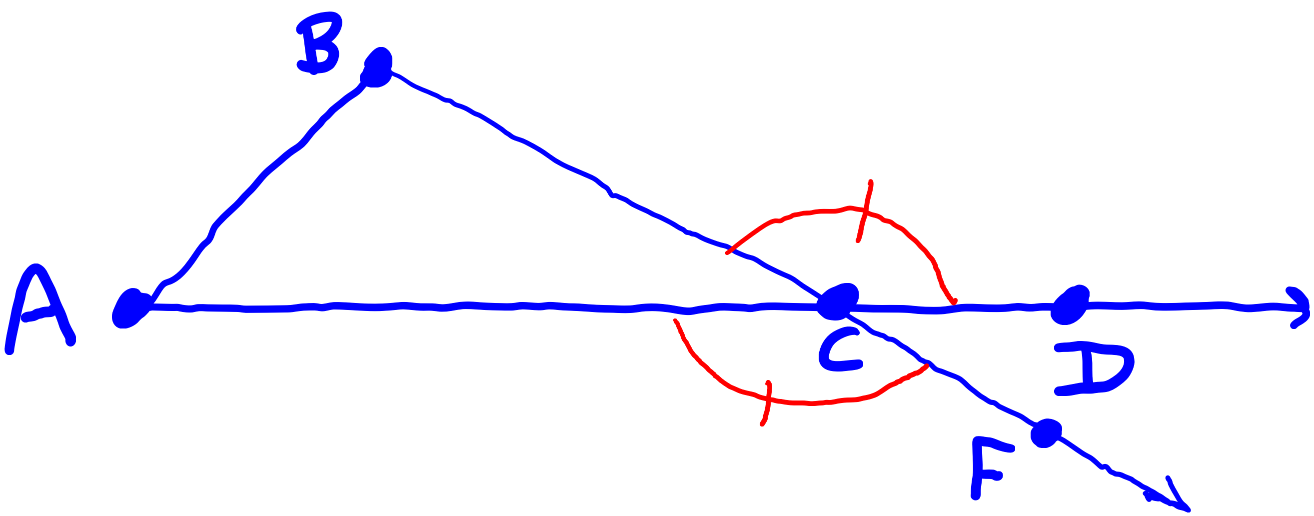


Part 2: Prove that $m(\angle BCD) > m(\angle BAC)$

(12) There exists a point F such that $B - C - F$.**(by Thm 3.2.6(i) applied to given points B, C)**

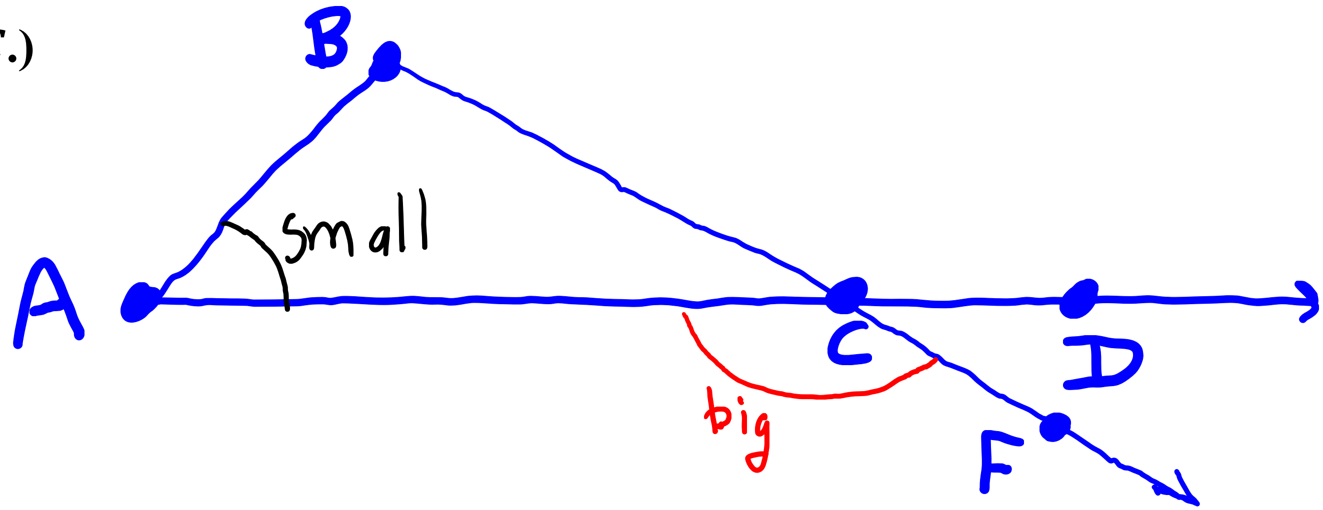


(13) $m(\angle BCD) = m(\angle FCA)$ **(The angles are a vertical pair (because $A - C - D$ from step (1) and $B - C - F$ from step (12), so the Thm 5.3.9 Vertical Angle Theorem tells us that they are congruent)**

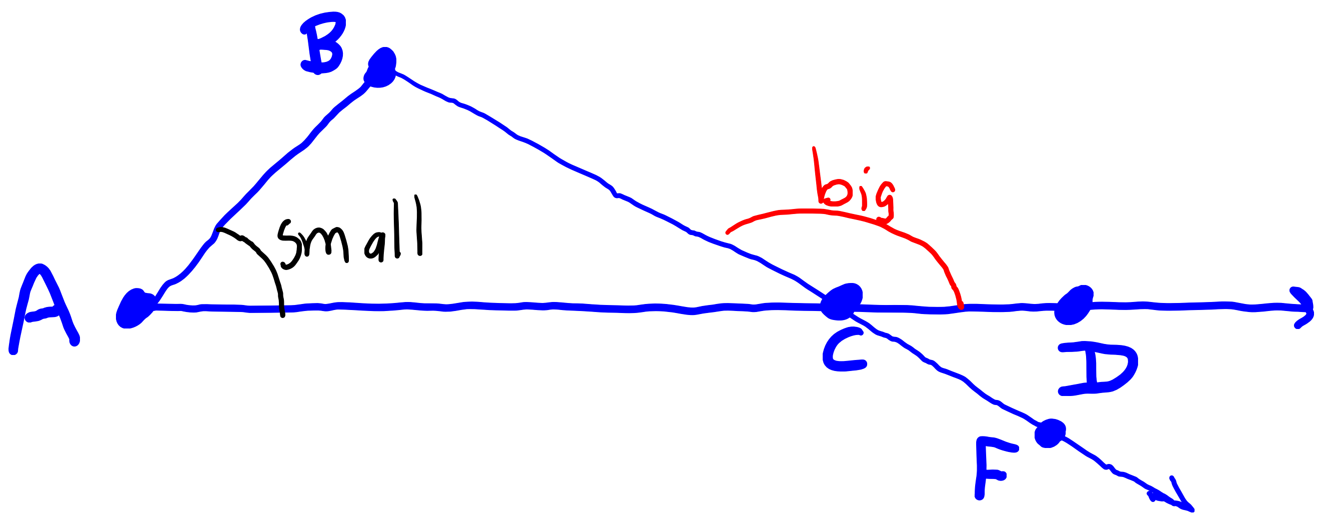


(14) $m(\angle FCA) > m(\angle ABC)$

(By result of Part 1 of this proof, applied to ΔABC and exterior angle $\angle FCA$ and remote interior angle $\angle ABC$.)



(15) $m(\angle BCD) > m(\angle BAC)$ **(by (14),(13))**



End of Proof

End of Proof

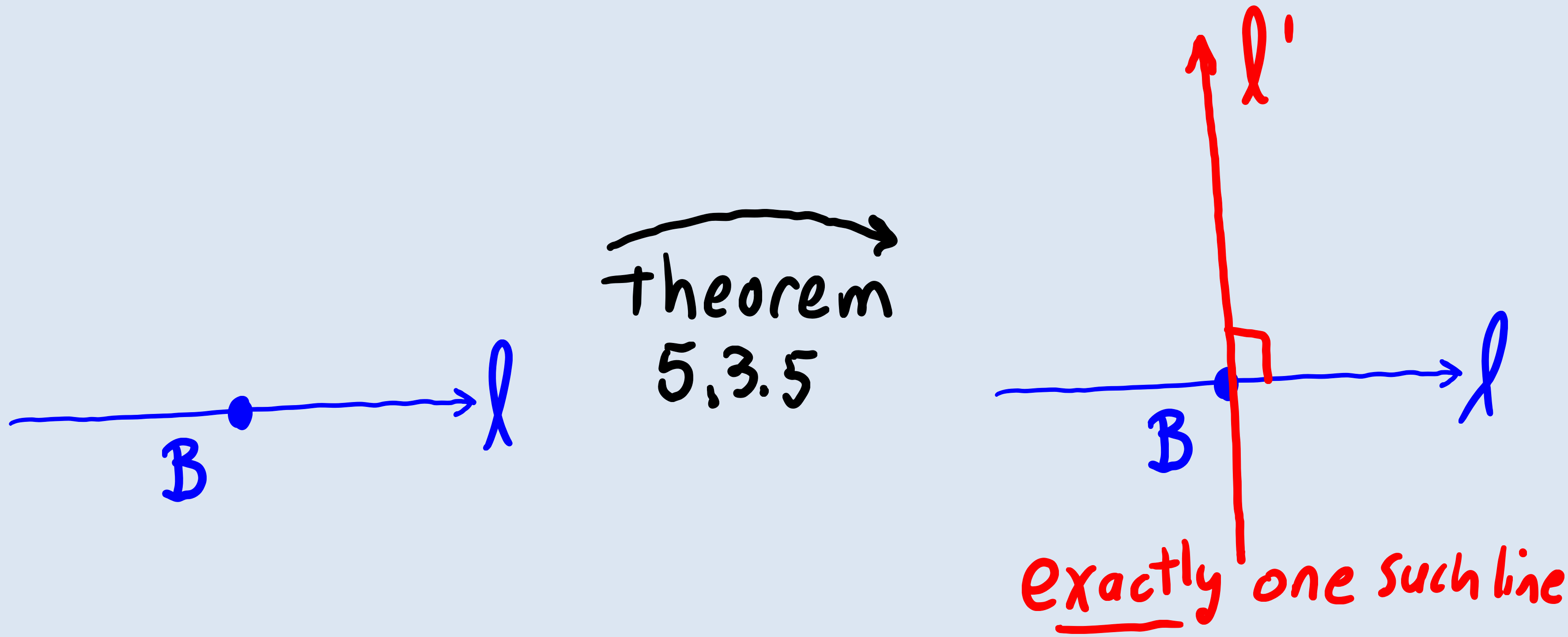
Recall Theorem 5.3.5

Theorem 5.3.5

Existence of a Unique Perpendicular to a Line through a Point *On the Line*

In a protractor geometry, if B is a point on a line l,

then there exists a ***unique*** line l' such that l' contains B and $l' \perp l$.



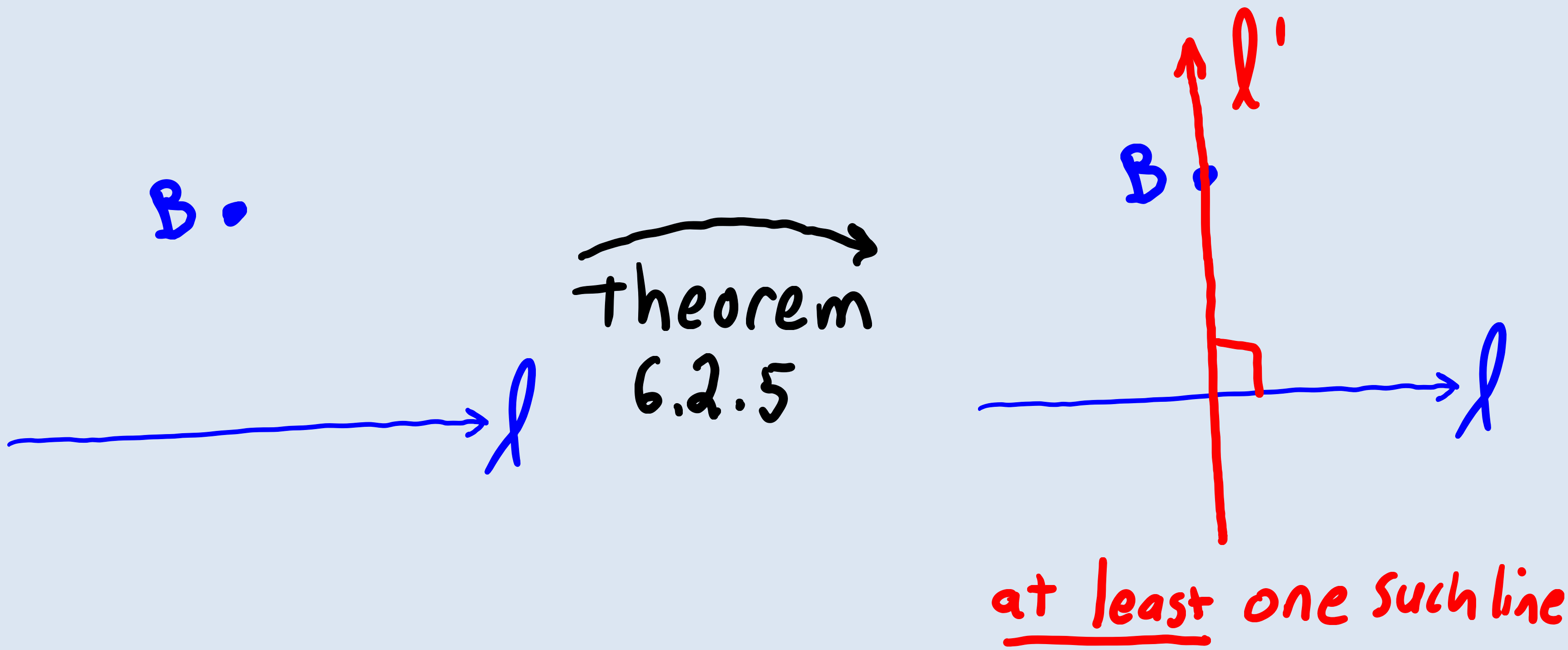
And recall that there is a related Theorem in Neutral Geometry but with point *B not on line l*.

Theorem 6.2.5

Existence of At Least One Perpendicular to a Line through a Point *Not On the Line*

In a neutral geometry, if B is a point not on a line l,

then there exists ***at least one*** line l' such that l' contains B and $l' \perp l$.



The Exterior Angle Theorem can be used to easily prove that if B is a point not on a line l, then there ***cannot be more than one*** line through B that is perpendicular to l. The proof is so easy that the statement aamounts to a *Corollary* of the Neutral Exterior Angle Theorem, although it is not presented in the book.

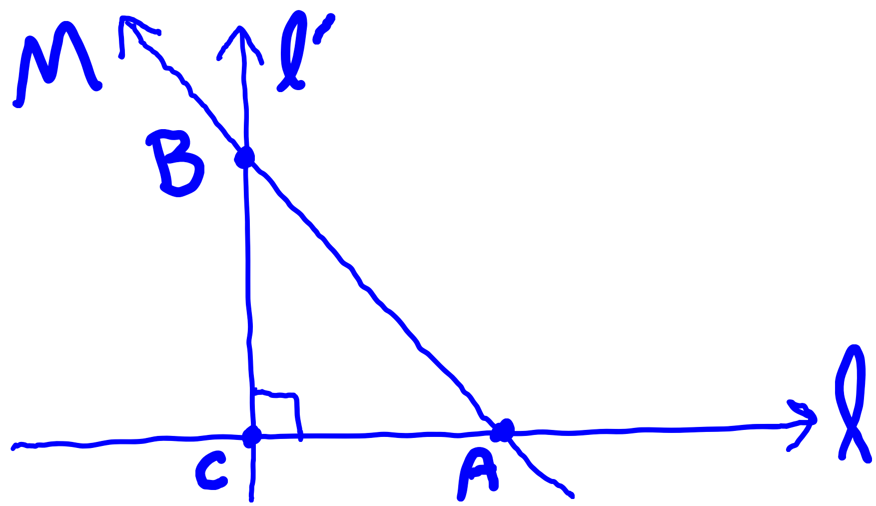
Unstated Corollary of the Neutral Exterior Angle Theorem

There Cannot Be More Than One Perpendicular to a Line through a Point *Not On the Line*

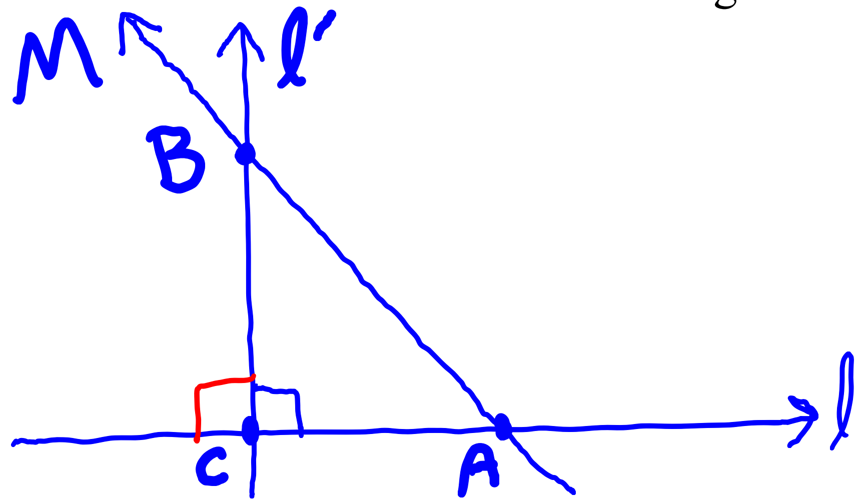
In a neutral geometry, if B is a point not on a line l,

then there ***cannot be more than one*** line that contains B and is perpendicular to l.

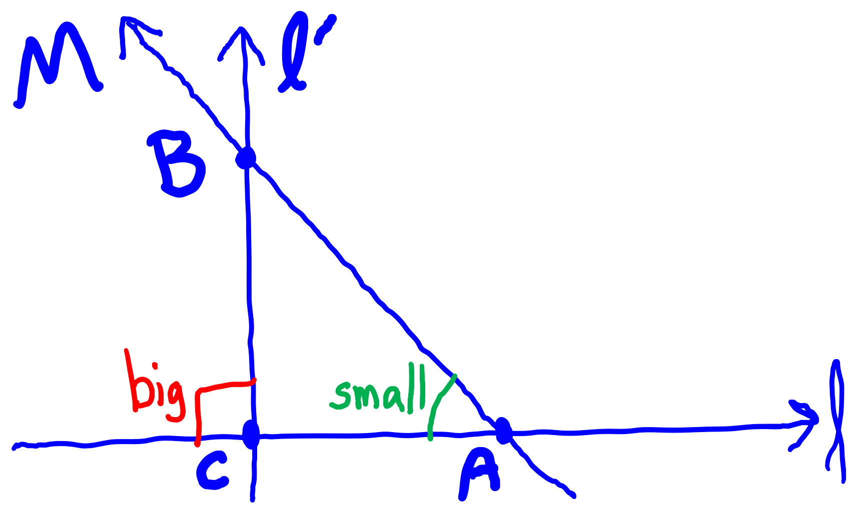
Proof: Suppose that line l is given, and B is a point not on l, and that line l' passes through B and is perpendicular to l at a point C. And suppose that there is another line M through B and that M intersects l at a point A.



This creates a right triangle ΔABC. There exists a point D such that $D - C - A$, creating an exterior angle $\angle BCD$. The Linear Pair Theorem tells us that this exterior angle will also be a right angle.



By the Neutral Exterior Angle Theorem, we know that $m(\angle BCD) > m(\angle BAC)$. Therefore, $\angle BAC$ is an acute angle, so line M is not perpendicular to line l.



End of Proof

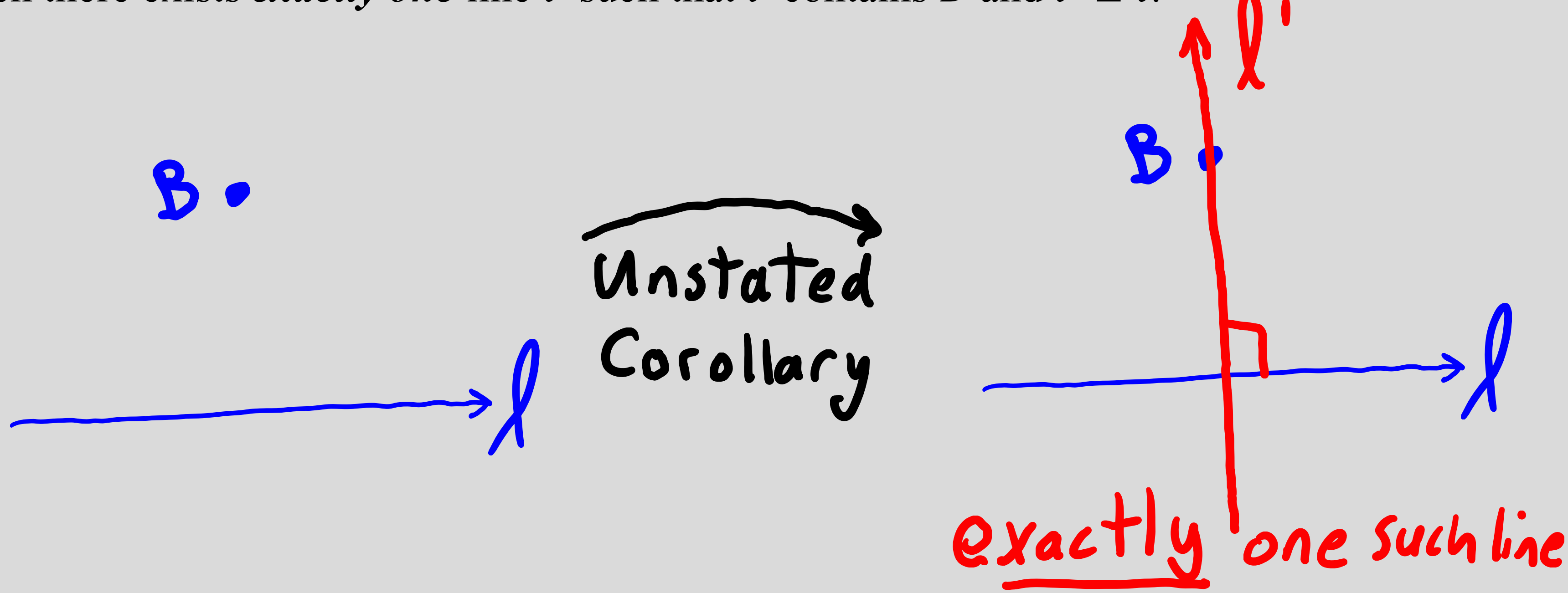
The unstated Corollary just presented allows us to make a more precise statement than Theorem 6.2.5 about a line l, a point B not on l, and exactly how many lines exist that contain B and are perpendicular to l. This more precise statement amounts to a new Corollary of both Theorem 6.2.5 and the Neutral Exterior Angle Theorem. But the new, more precise statement is not presented in the book.

Unstated Corollary of Theorem 6.2.5 and of the Neutral Exterior Angle Theorem

Existence of Exactly One Perpendicular to a Line through a Point *Not On the Line*

In a neutral geometry, if B is a point not on a line l,

then there exists ***exactly one*** line l' such that l' contains B and $l' \perp l$.

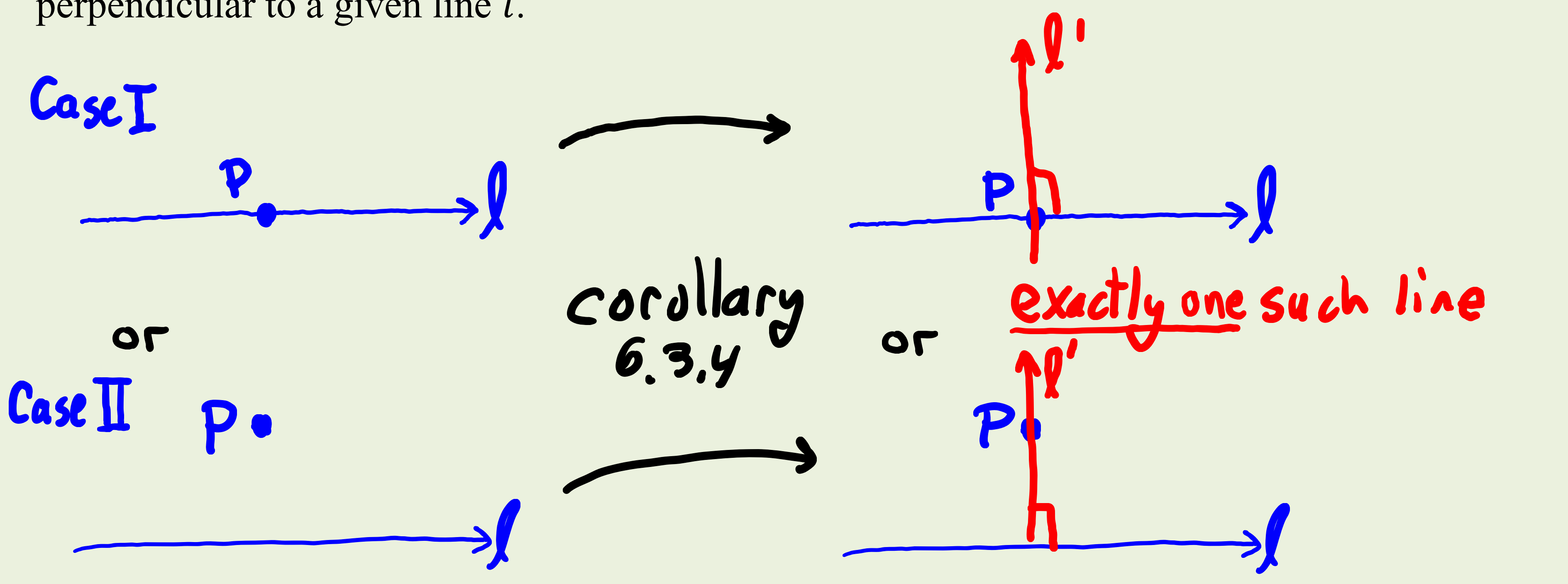


The book does present a Corollary 6.3.4 about the existence and uniqueness of lines that pass through a given point P and are perpendicular to a given line l. It is important to realize that the book's corollary really includes two cases.

Case I: Point P is on line l. (This case is the subject of Theorem 5.3.5)

Case II: Point P is not on line l. (This case is the subject of the unstated Corollary.)

Corollary 6.3.4 In neutral geometry, there is *exactly one* line through a given point P perpendicular to a given line l.



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