Video 6.3b Four Cool Consequences of The Neutral Exterior Angle Theorem

produced by Mark Barsamian, 2022.04.05

for Ohio University MATH 3110/5110 College Geometry

Topics for this Video

- **The *Side-Angle-Angle Theorem***
- **The *BS* ➔ *BA* Theorem**
- **The *BA* ➔ *BS* Theorem**
- **The *Triangle Inequality of Neutral Geometry***

138 – 140

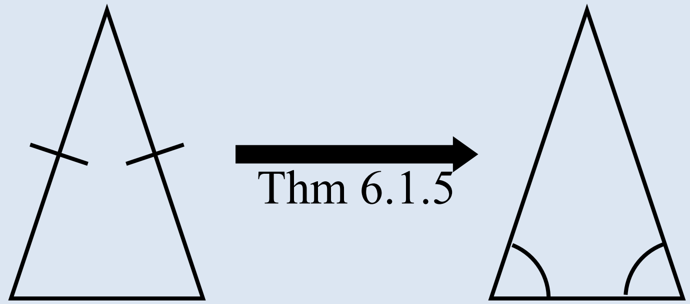
Reading: Pages ~~135~~ – 137 of Section 6.3 The Exterior Angle Theorem and Its Consequences in *Geometry: A Metric Approach with Models, Second Edition* by Millman & Parker

Suggested Exercises: Section 6.3 # 1, 2, 3, 4, 5, 8, 10

Recall Theorems about Congruent Parts of Triangles from Sections 6.1 and 6.2:

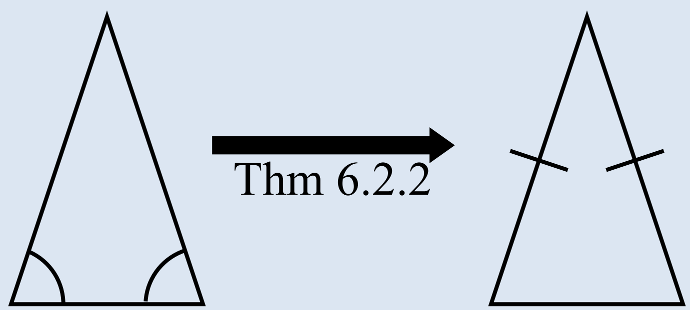
Theorem 6.1.5 (*Pons Asinorum*) (*Isosceles Triangle Theorem*) (*CS* ➔ *CA* Theorem)

In Neutral geometry, if two sides of a triangle are congruent, then the angles opposite those sides are also congruent. That is, in a triangle, if *CS* then *CA*.



Theorem 6.2.2 (*Converse of the Statement of Pons Asinorum)* (*CA* ➔ *CS* Theorem)

In Neutral geometry, if two angles of a triangle are congruent, then the sides opposite those angles are also congruent. That is, in a triangle, if *CA* then *CS*.



Recall the Neutral Exterior Angle Theorem, discussed in Video 6.3a

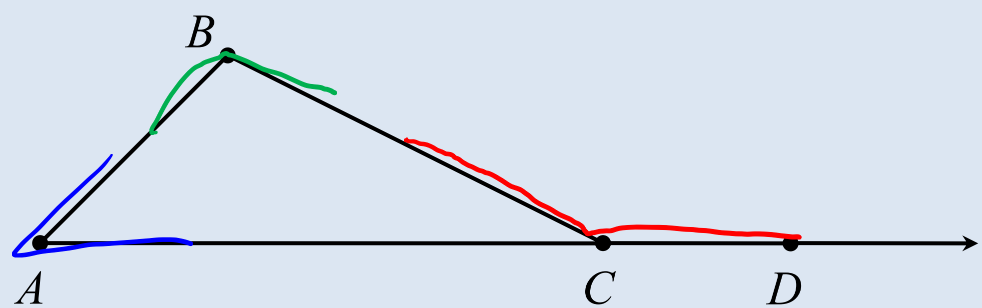
Definition of Exterior Angle and Remote Interior Angles

Words: *exterior angle of a triangle*

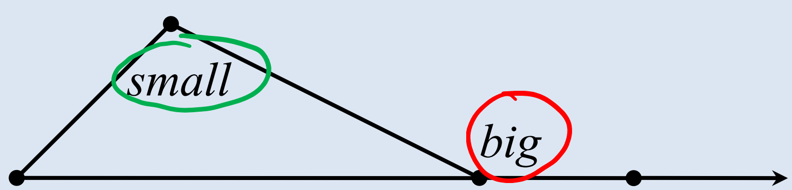
Usage: There is a protractor geometry and a triangle in the discussion.

Meaning: An angle that can be labeled $\angle BCD$ where ΔABC is the given triangle and $A - C - D$.

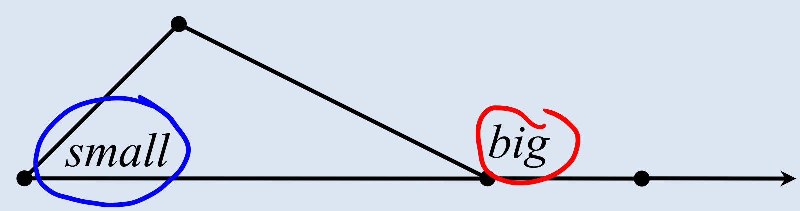
Related Terminology: for the exterior angle $\angle BCD$ of ΔABC, the two angles $\angle ABC$ and $\angle BAC$ are called *remote interior angles for the exterior angle* $\angle BCD$.



Theorem 6.3.3 (*Neutral Exterior Angle Theorem*) In a neutral geometry, the measure of an exterior angle of a triangle is greater than the measure of either of its remote interior angles.



and



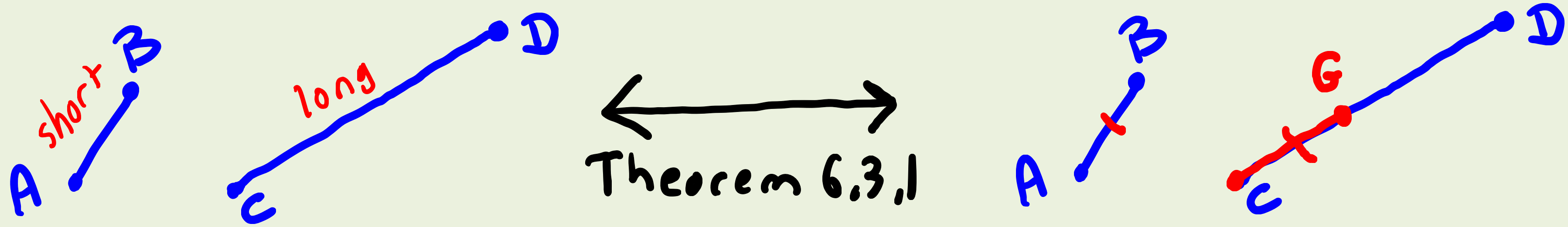
Two Useful Theorems from the Beginning of Section 6.3

We will use these theorems in the current video, but I won't discuss their proofs because the proofs only use concepts from Chapter 3 and Chapter 5.

Theorem 6.3.1: In a metric geometry, the following are equivalent (TFAE)

(i) $AB < CD$

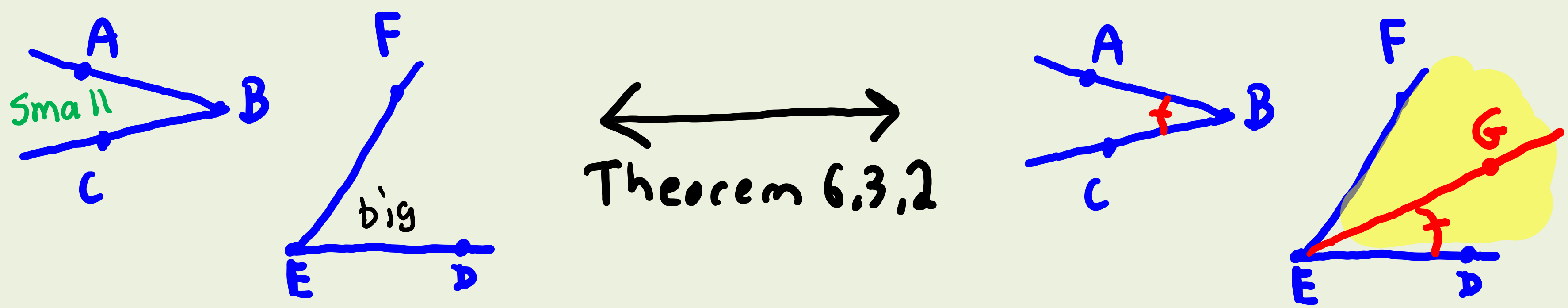
(ii) There exists a point $G \in \text{int}(\overline{CD})$ such that $\overline{AB} \simeq \overline{CG}$



Theorem 6.3.2: In a metric geometry, the following are equivalent (TFAE)

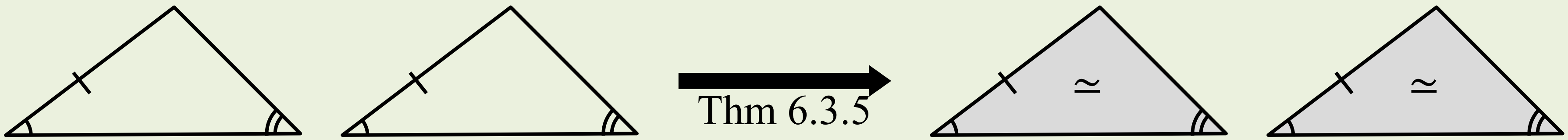
(i) $m(\angle ABC) < m(\angle DEF)$

(ii) There exists a point $G \in \text{int}(\angle DEF)$ such that $\angle ABC \simeq \angle DEG$



First Cool Consequence of the Neutral Exterior Angle Theorem: The *SAA Theorem*

Theorem 6.3.5 (*Side-Angle-Angle, SAA*): In Neutral Geometry, if there is a correspondence between parts of two triangles such that two angles and a non-included side of one triangle are congruent to the corresponding parts of the other triangle, then the triangles are congruent.



Proof

(1) Suppose that in Neutral Geometry, triangles ΔABC and ΔDEF have $\overline{AB} \simeq \overline{DE}$ and $\angle A \simeq \angle D$ and $\angle C \simeq \angle F$.

(2) There exists a point G on ray $\overrightarrow{DF}$ such that $\overline{DG} \simeq \overline{AC}$. **(Justify.)**

(3) $\Delta ABC \simeq \Delta DEG$. **(Justify.)**

(4) $\angle C = \angle ACB \simeq \angle DGE$. **(Justify.)**

(5) $\angle DGE \simeq \angle F$. **(Justify.)**

(6) There are three possibilities for where point G can be on ray $\overrightarrow{DF}$:

(i) $D - G - F$, or (ii) $D - F - G$, or (iii) $G = F$. **(Justify.)**

Case (i) $D - G - F$.

(7) Assume that $D - G - F$. **(Illustrate.)**

(8) Then $\angle DGE$ is an exterior angle for ΔEFG, and $\angle EFG = \angle F$ is one of its remote interior angles.

(9) $m(\angle DGE) > m(\angle EFG)$ **(Illustrate.) (Justify.)**

(10) We have reached a contradiction. **(Explain the contradiction.)** Therefore, the assumption in step (7) was wrong.So Case (i) is impossible.

Case (ii) $D - F - G$.

(11) – (14) Assume that $D - F - G$. **(Make a new Illustration. Fill in the proof steps to show that we reach a contradiction, so that Case (ii) is also impossible.)**

Conclusion of Cases

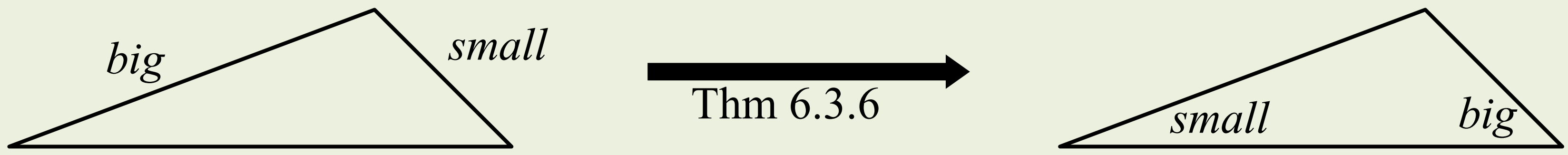
(15) Since Cases (i) and (ii) are impossible, we conclude that only Case (iii) is possible. That is, it must be that $G = F$.

(16) Therefore, $\Delta ABC \simeq \Delta DEF$. **(Illustrate.) (Justify.)**

End of proof

Second Consequence of the Neutral Exterior Angle Theorem: The *BS* ➔ *BA* Theorem

Theorem 6.3.6 (The *BS* ➔ *BA* Theorem): In Neutral Geometry, if one side of a triangle is longer than another side, then the angle opposite the longer side is bigger than the angle opposite the shorter side. That is, in a triangle, *if BS then BA*. In symbols, *BS* ➔ *BA*.



Proof

(1) Suppose that neutral geometry triangle ΔABC has $AB > AC$. **(Illustrate.)**

(2) There exists a point $G \in \text{int}(\overline{AB})$ such that $\overline{AG} \simeq \overline{AC}$. **(Illustrate.) (Justify.)** (**Hint:** Use a theorem from Section 6.3.)

(3) $G \in \text{int}(\angle ACB)$. **(Illustrate.) (Justify.)** (**Hint:** Use a theorem from Section 4.4)

(4) $m(\angle ACB) > m(\angle ACG)$. **(Illustrate.) (Justify.)** (**Hint:** Use the *Angle Measurement Axioms* (the *Definition of Angle Measure*) from Chapter 5.)

(5) $m(\angle ACG) = m(\angle AGC)$. **(Illustrate.) (Justify.)** (**Hint:** Use a theorem from Section 6.1.)

(6) $m(\angle AGC) > m(\angle ABC)$ **(Illustrate.) (Justify.)** (**Hint:** Use a theorem from Section 6.3.)

(8) $m(\angle ACB) > m(\angle ABC)$ **(Illustrate.) (Justify.)**

End of Proof

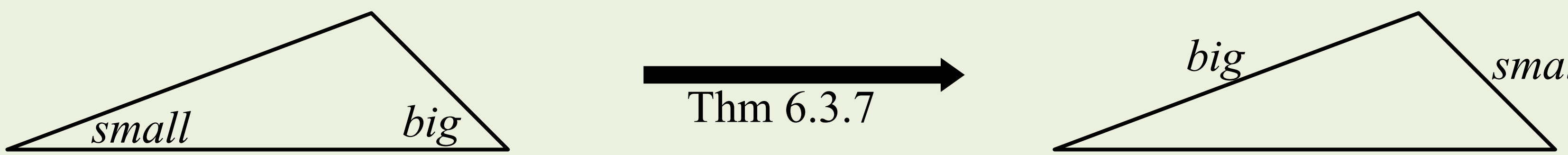
Third Consequence of the Neutral Exterior Angle Theorem: The *BA* → *BS* Theorem

We now have three very similar theorems about congruent parts of triangles and about bigger and smaller parts of triangles.

- **Thm 6.1.5 (*CS* → *CA*)** Clever proof using *SAS*.
- **Thm 6.2.2 (*CA* → *CS*)** Clever proof using *ASA*.
- **Thm 6.3.3 (*BS* → *BA*)** Proof involved many steps, used the Neutral Exterior Angle Theorem.

There is, not surprisingly, one more theorem to add to this collection of theorems:

Theorem 6.3.7 (The *BA* → *BS* Theorem): In Neutral Geometry, if one angle of a triangle is bigger than another angle, then the side opposite the bigger angle is longer than the side opposite the smaller angle. That is, in a triangle, *if BA then BS*. In symbols, *BA* → *BS*.



What is surprising about Theorem 6.3.7 is that it can be proved *very simply* by making clever use of two of Theorems 6.1.5 and 6.3.3 listed above. (The proof is so simple that Theorem 6.3.7 could really be considered a *Corollary* of those theorems.) The key to the proof's simplicity is that we will prove the *contrapositive* version of Theorem 6.3.7. The contrapositive version of the theorem is easiest to state, and most clear, if we use the version of Theorem 6.3.7 that has vertices named.

Theorem 6.3.7 (The $BA \Rightarrow BS$ Theorem) (with vertices named)

(Original) In Neutral Geometry triangle ΔABC, if $m(\angle ACB) > m(\angle ABC)$ then $AB > AC$.

(Contrapositive) In Neutral Geometry triangle ΔABC, if $AB \ngtr AC$ then $m(\angle ACB) \ngtr m(\angle ABC)$.

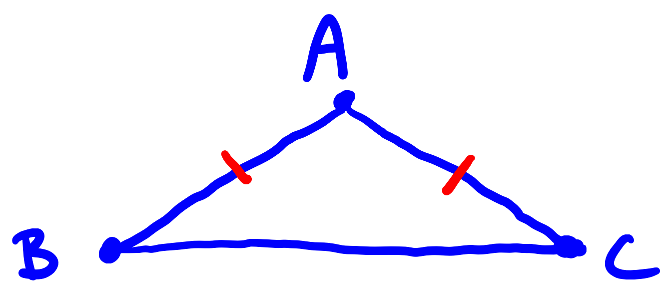
Proof (Prove the contrapositive.)

(1) Suppose that neutral geometry triangle ΔABC has $AB \ngtr AC$.

(2) Then $AB = AC$ or $AC > AB$.

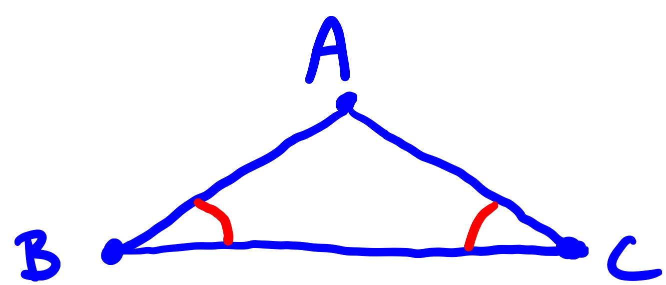
Case 1: $\boldsymbol{AB = AC}$

(3) Suppose $AB = AC$.



(4) Then $m(\angle ACB) = m(\angle ABC)$

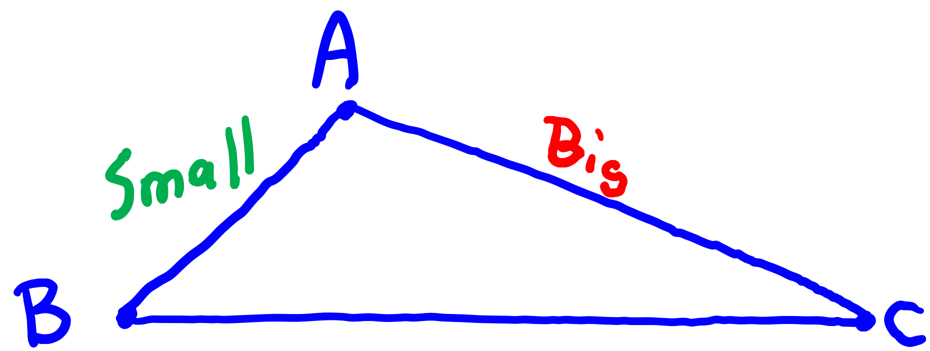
(by Theorem 6.1.5 ($CS \Rightarrow CA$))



(5) Observe that $m(\angle ACB) \ngtr m(\angle ABC)$ in this case.

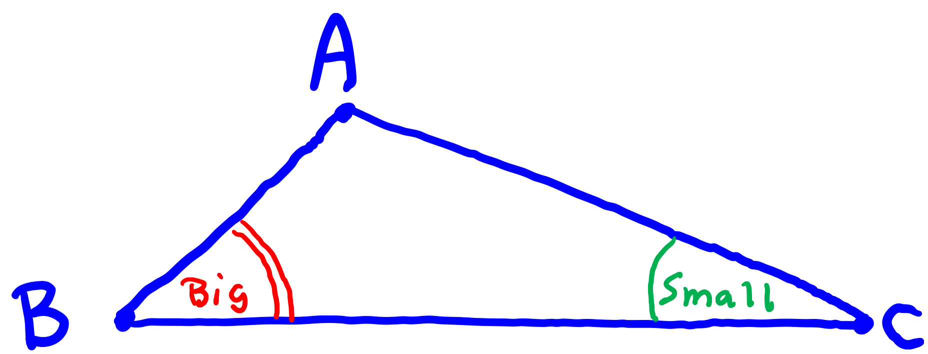
Case 2: $AC > AB$

(6) Suppose $AC > AB$.



(7) Then $m(\angle ABC) > m(\angle ACB)$

(by Theorem 6.3.3 (*BS* ➔ *BA*))



(8) Observe that $m(\angle ACB) \ngtr m(\angle ABC)$ in this case, as well

Conclusion of Cases

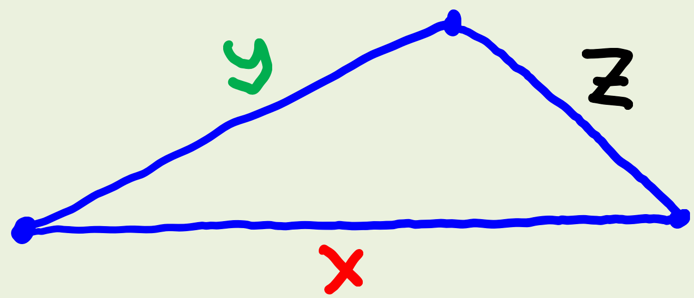
(9) Conclude that $m(\angle ACB) \ngtr m(\angle ABC)$ (because it is true in every case)

End of Proof

Third Consequence of the Neutral Exterior Angle Theorem: The *Triangle Inequality*

Theorem 6.3.8 (The *Triangle Inequality of Neutral Geometry*)

Version without vertices named: In Neutral Geometry, the length of one side of a triangle is strictly less than the sum of the lengths of the other two sides.



$x < y + z$

Here is a Proof, but without Illustrations or Justifications.

Proof of Theorem 6.3.8 (The *Triangle Inequality of Neutral Geometry*)

(1) Suppose that a neutral geometry triangle is given, and one of the sides of the triangle has been chosen. (Our goal is to prove that the length of that one side is strictly less than the sum of the lengths of the other two sides.) Label the vertices ΔABC, so that ~~$\overline{AB}$~~ $\overline{AC}$ is the chosen side. (Now our goal is to prove that the inequality $AC < AB + BC$ is true.)

(2) There exists a point D such that $C - B - D$ and $\overline{BD} \simeq \overline{AB}$. **(Justify)(Illustrate)**

(3) $CB + BD = CD$ **(Justify)(Illustrate)**

(4) $CB + AB = CD$ **(Justify)**

(5) $\angle BDA \simeq \angle BAD$ **(Justify)(Illustrate)**

(6) $B \in \text{int}(\angle CAD)$. **(Justify)(Illustrate)**

(7) $m(\angle BAD) < m(\angle CAD)$ **(Justify)(Illustrate)**

(8) $m(\angle CDA) < m(\angle CAD)$ **(Justify)(Illustrate)**

(9) $CA < CD$ **(Justify)(Illustrate)**

(10) $CA < CB + AB$ **(Justify)(Illustrate)**

End of Proof

End of Video.