Solutions to Group Work GW06 Limits Involving Infinity for a Rational Function

Part I: $f(x) = \dfrac{5x^2 - 40x + 35}{3x^2 - 27x + 42} = \dfrac{5(x-1)(x-7)}{3(x-2)(x-7)}$

(a) Find $f(1)$. **Solution:** $f(1) = \frac{5((1)-1)((1)-7)}{3((1)-2)((1)-7)} = \frac{5(0)(-6)}{3(-1)(-6)} = \frac{0}{18} = 0.$

(b) Find $\lim\limits_{x\to 1} f(x)$. **Solution:** $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} \frac{5(x-1)(x-7)}{3(x-2)(x-7)} \underset{\text{Thm 1.3.3.2}}{=} \frac{5((1)-1)((1)-7)}{3((1)-2)((1)-7)} = \cdots = 0.$

Since we are taking the limit, as $x \to 1$, of a rational function, and $x = 1$ is in the domain of the rational function, Theorem 1.3.3.2 tells us that we can just substitute in $x = 1$.

(c) Based on (a),(b), what does the factor $(x-1)$ cause in the graph of $f(x)$?

The factor causes an x intercept at $(x, y) = (1,0)$

(d) Find $f(2)$. **Solution:** $f(2) = \frac{5((2)-1)((2)-7)}{3((2)-2)((2)-7)} = \frac{5(1)(-5)}{3(0)(-5)} = \frac{-10}{0}$ *does not exist*

(e) Find $\lim\limits_{x\to 2^-} f(x)$. **Solution:** $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} \frac{5(x-1)(x-7)}{3(x-2)(x-7)} \underset{\substack{\text{can}\\\text{cancel}}}{=} \lim\limits_{x\to 2^-} \frac{5(x-1)}{3(x-2)}$

- *Since $x \to 2^-$, we know that $x \neq 7$, so $x - 7 \neq 0$, so we can cancel $\frac{x-7}{x-7}$.*
- *Since $x \to 2^-$, the term $(x-1)$ will be close to 1. So numerator $5(x-1)$ will be close to 5.*
- Since x is close to 2, but less than 2, *we know that the term $(x-0)$ will be close to 0 and negative. So the denominator $3(x-2)$ will be very close to 0 and negative.*
- Therefore, the ratio $\frac{numerator}{denominator} = \frac{5(x-1)}{3(x-2)} = \frac{\text{close to 5}}{\text{close to 0 and } neg}$ will be a large, negative number.
- The as x gets closer and closer to 2 but less than 2, the value of the ratio will grow more and more negative, without bound.
- Therefore $\lim\limits_{x\to 2^-} f(x) = -\infty$.

(f) Find $\lim\limits_{x\to 2^+} f(x)$.

- *Steps similar to the steps in (e), except that the denominator will be very close to 0 and pos.*
- *Therefore, $\lim\limits_{x\to 2^+} f(x) = \infty$*

(g) Find $\lim\limits_{x\to 2} f(x)$.

Limit does not exist, because the left & right limits don't match.

(h) Based on (d),(g), what does the factor $(x-2)$ cause in the graph of $f(x)$?

The factor causes a vertical asymptote with line equation $x = 2$. The graph goes down along the left side of the asymptote and up along the right side of the asymptote.

(i) Find $f(7)$. **Solution:** $f(7) = \frac{5((7)-1)((7)-7)}{3((7)-2)((7)-7)} = \frac{5(6)(0)}{3(5)(0)} = \frac{0}{0}$ *does not exist*

(j) Find $\lim_{x\to 7} f(x)$. **Solution:** $\lim_{x\to 7} f(x) = \lim_{x\to 2^-} \frac{5(x-1)(x-7)}{3(x-2)(x-7)} \underset{\text{can cancel}}{=} \lim_{x\to 2^-} \frac{5(x-1)}{3(x-2)} \underset{\text{Thm 1.3.3.2}}{=} \frac{5((7)-1)}{3((7)-2)} = \frac{5(6)}{3(5)} = 2$

- *Since* $x \to 7$, *we know that* $x \neq 7$, *so* $x - 7 \neq 0$, *so we can cancel* $\frac{x-7}{x-7}$.
- *In the next step, we are taking the limit, as* $x \to 7$, *of a rational function, and* $x = 7$ *is in the domain of the rational function, Theorem 1.3.3.2 tells us that we can just substitute in* $x = 7$.

(k) Based on (i),(j), what do the factors $\frac{(x-7)}{(x-7)}$ cause in the graph of $f(x)$?

Question (i) tells us that there is no point on the graph with x *coordinate* $x = 7$. *But question (j) tells us the graph is heading for the location* $(x, y) = (7,2)$. *In other words, there is a hole in the graph at the location* $(x, y) = (7,2)$.

(l) Find $\lim_{x\to\infty} f(x)$. **Solution:** $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{5x^2-40x+35}{3x^2-27x+42} \underset{\text{keep dominant terms}}{=} \lim_{x\to\infty} \frac{5x^2}{3x^2} \underset{\text{can cancel}}{=} \lim_{x\to\infty} \frac{5}{3} = \frac{5}{3}$

- *Since* $x \to \infty$, *we know that* $x \neq 0$, *so we can cancel* $\frac{x^2}{x^2}$.
- *The limit is telling us to consider what happens to the value of* $\frac{5}{3}$ *when* x *gets more and more positive, without bound. Well, the value of* $\frac{5}{3}$ *never changes. So the limit is just* $\frac{5}{3}$.

(m) Based on (l), what is the behavior of the right end of the graph of $f(x)$?

The graph of $f(x)$ *has a horizontal asymptote on the right, with line equation* $y = \frac{5}{3}$.

(n) Find $\lim_{x\to-\infty} f(x)$. **Solution:** $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{5x^2-40x+35}{3x^2-27x+42} \underset{\text{keep dominant terms}}{=} \lim_{x\to-\infty} \frac{5x^2}{3x^2} \underset{\text{can cancel}}{=} \lim_{x\to-\infty} \frac{5}{3} = \frac{5}{3}$

- *Since* $x \to -\infty$, *we know that* $x \neq 0$, *so we can cancel* $\frac{x^2}{x^2}$.
- *The limit is telling us to consider what happens to the value of* $\frac{5}{3}$ *when* x *gets more and more negative, without bound. Well, the value of* $\frac{5}{3}$ *never changes. So the limit is just* $\frac{5}{3}$.

(o) Based on (n), what is the behavior of the left end of the graph of $f(x)$?

The graph of $f(x)$ *has a horizontal asymptote on the left, with line equation* $y = \frac{5}{3}$.

(p) List all the asymptotes of $f(x)$. Give their line equations and say whether they are horizontal or vertical.

Vertical asymptote at $x = 2$. *Horizontal asymptote (left and right) at* $y = \frac{5}{3}$.

Part II: $g(x) = \frac{5x^3 - 75x^2 + 315x - 245}{3x^2 - 27x + 42} = \frac{5(x-1)(x-7)^2}{3(x-2)(x-7)}$

(a) Find $g(1)$. **Solution:** $g(1) = \frac{5((1)-1)((1)-7)^2}{3((1)-2)((1)-7)} = \frac{5(0)(-6)^2}{3(-1)(-6)} = \frac{0}{18} = 0$.

(b) Find $\lim_{x\to1} g(x)$. **Solution:** $\lim_{x\to1} g(x) = \lim_{x\to1} \frac{5(x-1)(x-7)^2}{3(x-2)(x-7)} \underset{\text{Thm 1.3.3.2}}{=} \frac{5((1)-1)((1)-7)^2}{3((1)-2)((1)-7)} = \cdots = 0$.

Since we are taking the limit, as $x \to 1$, of a rational function, and $x = 1$ is in the domain of the rational function, Theorem 1.3.3.2 tells us that we can just substitute in $x = 1$.

(c) Based on (a),(b), what does the factor $(x-1)$ cause in the graph of $g(x)$?

The factor causes an x intercept at $(x,y) = (1,0)$

(d) Find $g(2)$. **Solution:** $g(2) = \frac{5((2)-1)((2)-7)^2}{3((2)-2)((2)-7)} = \frac{5(1)(-5)^2}{3(0)(-5)} = \frac{125}{0}$ *does not exist*

(e) Find $\lim_{x\to2^-} g(x)$. **Solution:** $\lim_{x\to2^-} g(x) = \lim_{x\to2^-} \frac{5(x-1)(x-7)^2}{3(x-2)(x-7)} \underset{\text{can cancel}}{=} \lim_{x\to2^-} \frac{5(x-1)(x-7)}{3(x-2)}$

- *Since $x \to 2^-$, we know that $x \neq 7$, so $x - 7 \neq 0$, so we can cancel $\frac{x-7}{x-7}$.*
- Furthermore, *since $x \to 2^-$, we know that $(x-1)$ will be very close to 1 and $(x-7)$ will be close to -5. So the numerator $5(x-1)(x-7)$ will be very close to -25.*
- Since x is very close to 2, but less than 2, *we know that the term $(x-0)$ will be very close to 0 and negative. So the denominator $3(x-2)$ will be very close to 0 and negative.*
- Therefore, the ratio $\frac{numerator}{denominator} = \frac{5(x-1)}{3(x-2)} = \frac{\text{close to -25}}{\text{close to 0 and } neg}$ will be a large, positive number.
- The as x gets closer and closer to 2 but less than 2, the value of the ratio will grow more and more positive, without bound.
- Therefore $\lim_{x\to2^-} g(x) = \infty$.

(f) Find $\lim_{x\to2^+} g(x)$.

- *Steps similar to the steps in (e), except that the denominator will be very close to 0 and pos.*
- *Therefore, $\lim_{x\to2^+} g(x) = -\infty$*

(g) Find $\lim_{x\to2} g(x)$.

Limit does not exist, because the left & right limits don't match.

(h) Based on (d),(g), what does the factor $(x-2)$ cause in the graph of $g(x)$?

The factor causes a vertical asymptote with line equation $x = 2$. The graph goes up along the left side of the asymptote and down along the right side of the asymptote.

(i) Find $g(7)$. **Solution:** $g(7) = \frac{5((7)-1)((7)-7)^2}{3((7)-2)((7)-7)} = \frac{5(6)(0)^2}{3(5)(0)} = \frac{0}{0}$ *does not exist*.

(j) Find $\lim_{x\to 7} g(x)$.

Solution: $\lim_{x\to 7} g(x) = \lim_{x\to 2^-} \frac{5(x-1)(x-7)^2}{3(x-2)(x-7)} \underset{\text{can cancel}}{=} \lim_{x\to 2^-} \frac{5(x-1)(x-7)}{3(x-2)} \underset{\text{Thm 1.3.3.2}}{=} \frac{5((7)-1)((7)-7)}{3((7)-2)} = \frac{5(6)(0)}{3(5)} = 0$

- *Since* $x \to 7$, *we know that* $x \neq 7$, *so* $x - 7 \neq 0$, *so we can cancel* $\frac{x-7}{x-7}$.
- *In the next step, we are taking the limit, as* $x \to 7$, *of a rational function, and* $x = 7$ *is in the domain of the rational function, Theorem 1.3.3.2 tells us that we can just substitute in* $x = 7$.

(k) Based on (i),(j), what do the factors $\frac{(x-7)^2}{(x-7)}$ cause in the graph of $g(x)$?

Question (i) tells us that there is no point on the graph with x coordinate $x = 7$. *But question (j) tells us the graph is heading for the location* $(x, y) = (7,0)$. *In other words, there is a hole in the graph at the location* $(x, y) = (7,0)$.

(l) Find $\lim_{x\to\infty} g(x)$. **Solution:** $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{5x^3-75x^2+315x-245}{3x^2-27x+42} \underset{\text{keep dominant terms}}{=} \lim_{x\to\infty} \frac{5x^3}{3x^2} \underset{\text{can cancel}}{=} \lim_{x\to\infty} \frac{5x}{3} = \infty$

- *Since* $x \to \infty$, *we know that* $x \neq 0$, *so we can cancel* $\frac{x^2}{x^2}$.
- *When x gets more and more positive, without bound, the value of* $\frac{5x}{3}$ *will also be getting more and more positive, without bound. So the limit is* ∞.

(m) Based on (l), what is the behavior of the right end of the graph of $g(x)$?

The right end of the graph goes up, without bound. (no norizontal asymptote)

(n) Find $\lim_{x\to-\infty} g(x)$. **Sol:** $\lim_{x\to-\infty} g(x) = \lim_{x\to-\infty} \frac{5x^3-75x^2+315x-245}{3x^2-27x+42} \underset{\text{keep dominant terms}}{=} \lim_{x\to-\infty} \frac{5x^3}{3x^2} \underset{\text{can cancel}}{=} \lim_{x\to-\infty} \frac{5x}{3} = -\infty$

- *Since* $x \to -\infty$, *we know that* $x \neq 0$, *so we can cancel* $\frac{x^2}{x^2}$.
- *When x gets more and more negative, without bound, the value of* $\frac{5x}{3}$ *will also be getting more and more negative, without bound. So the limit is* $-\infty$.

(o) Based on (n), what is the behavior of the left end of the graph of $g(x)$?

The left end of the graph goes down, without bound. (no norizontal asymptote)

(p) List all the asymptotes of $g(x)$. Give their line equations and say whether they are horizontal or vertical. **Solution:** *Vertical asymptote at* $x = 2$.

Part III: $h(x) = \dfrac{5x^2 - 40x + 35}{3x^3 - 48x^2 + 231x - 294} = \dfrac{5(x-1)(x-7)}{3(x-2)(x-7)^2}$

(a) Find $h(1)$. **Solution:** $h(1) = \frac{5((1)-1)((1)-7)}{3((1)-2)((1)-7)^2} = \frac{5(0)(-6)}{3(-1)(-6)^2} = \frac{0}{-108} = 0.$

(b) Find $\lim_{x\to 1} h(x)$. **Solution:** $\lim_{x\to 1} h(x) = \lim_{x\to 1} \frac{5(x-1)(x-7)}{3(x-2)(x-7)^2} \underset{\text{Thm 1.3.3.2}}{=} \frac{5((1)-1)((1)-7)}{3((1)-2)((1)-7)^2} = \cdots = 0.$

Since we are taking the limit, as $x \to 1$, of a rational function, and $x = 1$ is in the domain of the rational function, Theorem 1.3.3.2 tells us that we can just substitute in $x = 1$.

(c) Based on (a),(b), what does the factor $(x-1)$ cause in the graph of $h(x)$?

The factor causes an x intercept at $(x, y) = (1,0)$

(d) Find $h(2)$. **Solution:** $h(2) = \frac{5((2)-1)((2)-7)}{3((2)-2)((2)-7)^2} = \frac{5(1)(-5)}{3(0)(-5)^2} = \frac{-10}{0}$ *does not exist*

(e) Find $\lim_{x\to 2^-} h(x)$. **Solution:** $\lim_{x\to 2^-} h(x) = \lim_{x\to 2^-} \frac{5(x-1)(x-7)}{3(x-2)(x-7)^2} \underset{\substack{\text{can}\\\text{cancel}}}{=} \lim_{x\to 2^-} \frac{5(x-1)}{3(x-2)(x-7)}$

- *Since $x \to 2^-$, we know that $x \neq 7$, so $x - 7 \neq 0$, so we can cancel $\frac{x-7}{x-7}$.*
- Furthermore, *since $x \to 2^-$, we know that $(x-1)$ will be very close to 1. So the numerator $5(x-1)$ will be very close to 5.*
- Since x is very close to 2, but less than 2, *we know that $(x-2)$ will be very close to 0 and negative and and $(x-7)$ will be very close to -5. So the denominator $3(x-2)(x-7)$ will be very close to 0 and positive.*
- Therefore, the ratio $\frac{numerator}{denominator} = \frac{5(x-1)}{3(x-2)} = \frac{\text{close to 5}}{\text{close to 0 and } pos}$ will be a large, positive number.
- The as x gets closer and closer to 2 but less than 2, the value of the ratio will grow more and more positive, without bound.
- Therefore $\lim_{x\to 2^-} h(x) = \infty$.

(f) Find $\lim_{x\to 2^+} h(x)$.

- *Steps similar to the steps in (e), except that the denominator will be very close to 0 and neg.*
- *Therefore, $\lim_{x\to 2^+} g(x) = -\infty$*

(g) Find $\lim_{x\to 2} h(x)$.

Limit does not exist, because the left & right limits don't match.

(h) Based on (d),(g), what does the factor $(x-2)$ cause in the graph of $h(x)$?

The factor causes a vertical asymptote with line equation $x = 2$. The graph goes up along the left side of the asymptote and down along the right side of the asymptote.

(i) Find $h(7)$. **Solution:** $h(7) = \frac{5((7)-1)((7)-7)}{3((7)-2)((7)-7)^2} = \frac{5(6)(0)}{3(5)(0)^2} = \frac{0}{0}$ *does not exist*

(j) Find $\lim_{x\to 7^-} h(x)$. **Solution:** $\lim_{x\to 7^-} 7(x) = \lim_{x\to 7^-} \frac{5(x-1)(x-7)}{3(x-2)(x-7)^2} \underset{\text{can cancel}}{=} \lim_{x\to 2^-} \frac{5(x-1)}{3(x-2)(x-7)}$

- *Since* $x \to 7^-$, *the term* $(x-1)$ *will be close to 6. So numerator* $5(x-1)$ *will be close to 30.*
- Since x is close to 7, but less than 7, *the term* $(x-7)$ *will be close to* 0 *and negative and the term* $(x-5)$ *will be close to* 2. *So denominator* $3(x-2)(x-7)$ *will be close to* 0 *and neg.*
- Therefore, the ratio $\frac{numerator}{denominator} = \frac{5(x-1)}{3(x-2)} = \frac{\text{close to 30}}{\text{close to 0 and } neg}$ will be a large, negative number.
- The as x gets closer and closer to 7 but less than 7, the value of the ratio will grow more and more negative, without bound. Therefore $\lim_{x\to 7^-} h(x) = -\infty$.

(k) Find $\lim_{x\to 7^+} h(x)$. **Solution:** *Steps similar to the steps in (e), except that the denominator will be very close to 0 and pos. Therefore,* $\lim_{x\to 7^+} h(x) = \infty$

(l) Find $\lim_{x\to 7} h(x)$. **Solution:** *Limit does not exist, because the left & right limits don't match.*

(m) Based on (i)-(l), what do the factors $\frac{(x-7)}{(x-7)^2}$ cause in the graph of $h(x)$?

The factor causes a vertical asymptote with line equation $x = 7$. *The graph goes down along the left side of the asymptote and up along the right side of the asymptote*.

(n) Find $\lim_{x\to\infty} h(x)$. **Solution:** $\lim_{x\to\infty} h(x) = \lim_{x\to\infty} \frac{5x^2-40x+35}{3x^3-48x^2+231x-294} \underset{\text{keep dominant terms}}{=} \lim_{x\to\infty} \frac{5x^2}{3x^3} \underset{\text{can cancel}}{=} \lim_{x\to\infty} \frac{5}{3x} = 0$

When x gets more and more positive, without bound, the value of $\frac{5}{3x}$ *will get closer and closer to zero. So the limit is* 0.

(o) Based on (n), what is the behavior of the right end of the graph of $h(x)$?

The graph of $h(x)$ *has a horizontal asymptote on the right, with line equation* $y = 0$.

(p) Find $\lim_{x\to-\infty} h(x)$. **Sol:** $\lim_{x\to-\infty} h(x) = \lim_{x\to-\infty} \frac{5x^2-40x+35}{3x^3-48x^2+231x-294} \underset{\text{keep dominant terms}}{=} -\lim_{x\to\infty} \frac{5x^2}{3x^3} \underset{\text{can cancel}}{=} \lim_{x\to-\infty} \frac{5}{3x} = 0$

When x gets more and more negative, without bound, the value of $\frac{5}{3x}$ *will get closer and closer to zero. So the limit is just* 0.

(q) Based on (p), what is the behavior of the left end of the graph of $h(x)$?

The graph of $h(x)$ *has a horizontal asymptote on the left, with line equation* $y = 0$.

(r) List all the asymptotes of $h(x)$. Give their line equations and say whether they are horizontal or vertical.

Vertical asymptote at $x = 2$. *Horizontal asymptote (left and right) at* $y = 0$.