<u>Reference R04: Limits Involving Infinity</u>

Infinite Limits

Expanded Definition of Limit: ***Infinite Limits***

Symbol: $\lim_{x \to c} f(x) = \infty$

Spoken: The limit, as x approaches c, of $f(x)$ is infinity.

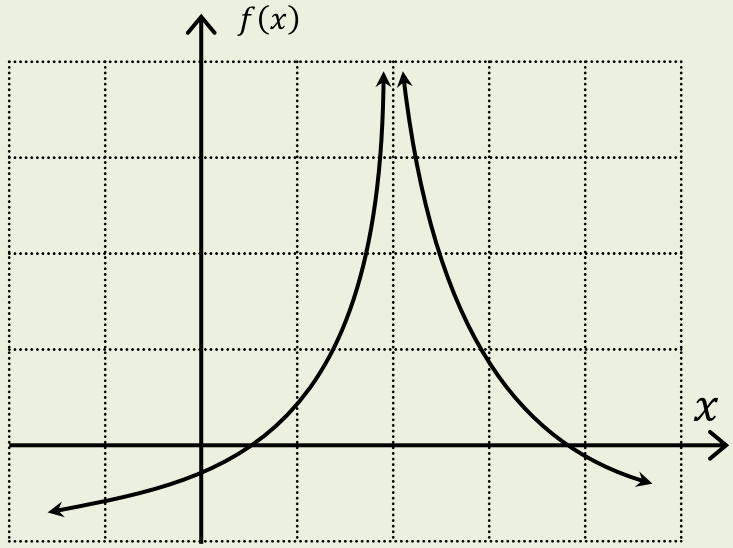
Usage: x is a variable, c is a real number constant, and f is a function.

Informal Meaning: As x gets closer and closer to c but not equal to c, the values of $f(x)$ get more and more positive without bound.

Precise Meaning: There is an open interval I containing the number c, with function f defined on all of I, except possibly at c, and such that given any number $N > 0$, there exists a number $\delta > 0$ such that for all x in I, where $x \neq c$, if $|x - c| < \delta$ then $f(x) > N$.

Graphical Significance: The graph of f has a vertical asymptote at $x = c$, and the graph goes up along both sides of the asymptote. The line equation for the asymptote is $x = c$.

Graphical Example:



$\lim_{x \to 2} f(x) = \infty$

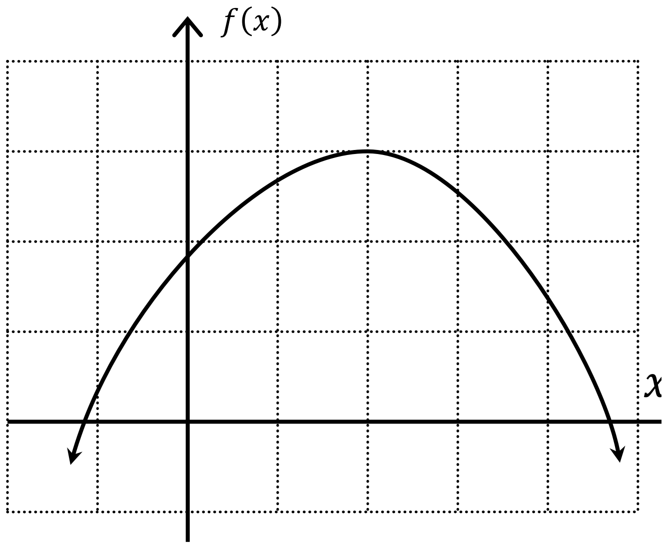
Remark: (a very important point) We have *expanded* the definition of limit!

Note the phrase "…*without bound*…" in the definition of *Infinite Limit*:

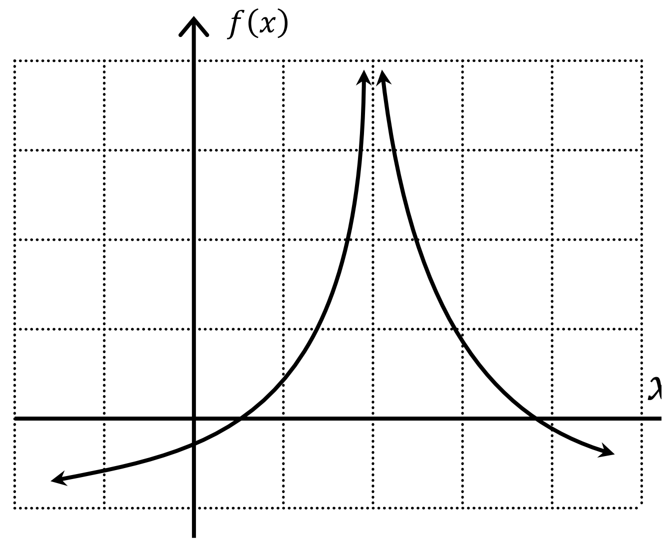
As x gets closer and closer to c but not equal to c, the values of $f(x)$ get more and more positive without bound.

What is the significance of that phrase *without bound*?

To understand the significance of the phrase, it is helpful to consider two examples:



$\lim_{x \to 2} f(x) = 3$



$\lim_{x \to 2} f(x) = \infty$

In both graphs, as x approaches 2, the y values get *more and more positive*. But in the graph on the left, the y values are getting more and more positive and *approaching* 3, while in the graph on the right, the y values are getting more and more positive *without bound*.

Obvious variations:

Expanded Definition of Limit: One-Sided Infinite Limit from the Right

Symbol: $\lim_{x \to c^+} f(x) = \infty$

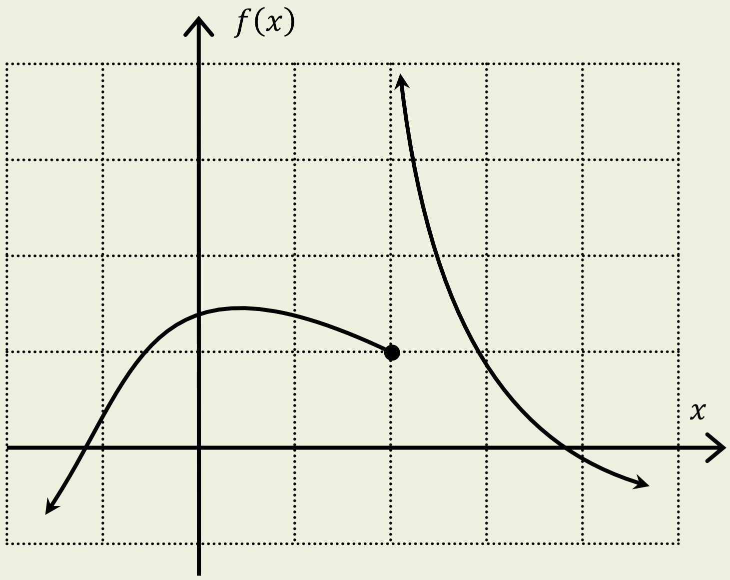
Spoken: The limit, as x approaches c from the right, of $f(x)$ is infinity.

Meaning: As x approaches c from the right, the values of $f(x)$ get more and more positive without bound.

Precise Meaning: There is an open interval I containing the number c, with function f defined on all of I, except possibly at c, and such that given any number $N > 0$, there exists a number $\delta > 0$ such that for all x in I, where $x < c$, if $|x - c| < \delta$ then $f(x) > N$.

Graphical Significance: The graph of f has a vertical asymptote at $x = c$, and the graph goes up along the left side of the asymptote. The line equation for the asymptote is $x = c$.

Graphical Example:



$\lim_{x \to 2^-} f(x) = 1$

$\lim_{x \to 2^+} f(x) = \infty$

$\lim_{x \to 2} f(x) \quad DNE$

Obvious variations that I won't discuss:

Expanded Definition of Limit: One Sided Infinite Limit from the Left

Symbol: $\lim_{x \to c^+} f(x) = \infty$

Expanded Definition of Limit: Negative Infinite Limits

Symbol: $\lim_{x \to c} f(x) = -\infty$

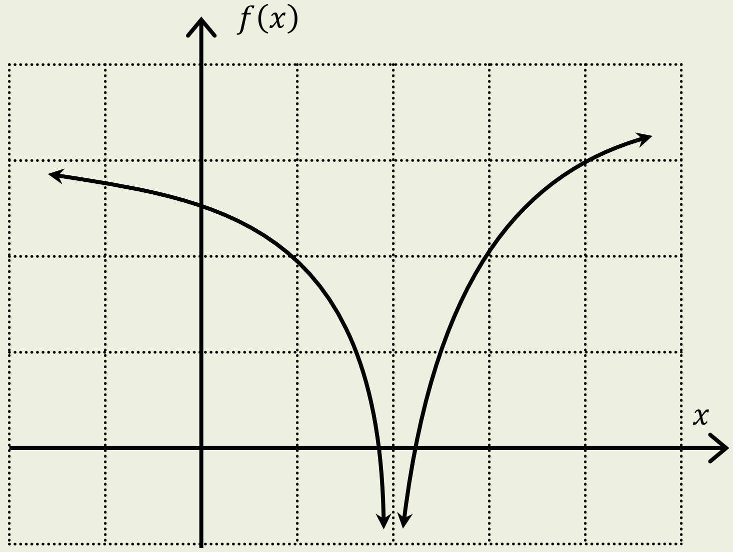
Spoken: The limit, as x approaches c, of $f(x)$ is negative infinity.

Informal Meaning: As x gets closer and closer to c but not equal to c, the values of $f(x)$ get more and more negative without bound.

Precise Meaning: There is an open interval I containing the number c, with function f defined on all of I, except possibly at c, and such that given any number $N < 0$, there exists a number $\delta > 0$ such that for all x in I, where $x \neq c$, if $|x - c| < \delta$ then $f(x) < N$.

Graphical Significance: The graph of f has a vertical asymptote at $x = c$, and the graph goes down along both sides of the asymptote. The line equation for the asymptote is $x = c$.

Graphical Example:



$\lim_{x \to 2} f(x) = -\infty$

Obvious variations that I won't discuss:

Expanded Definition of Limit: One Sided Negative Infinite Limits

Symbol: $\lim_{x \to c^-} f(x) = -\infty$ and $\lim_{x \to c^+} f(x) = -\infty$

Observations:

With the definition of infinite limits, we now have a shorthand notation to describe the situation where the y values of a function are getting more and more positive (or negative) without bound. This is a new kind of trend.

Limits at Infinity

Expanded Definition of Limit: Definition of Limits at Infinity

Symbol: $\lim_{x\to\infty} f(x) = L$

Spoken: The limit, as x goes to infinity, of $f(x)$ is L.

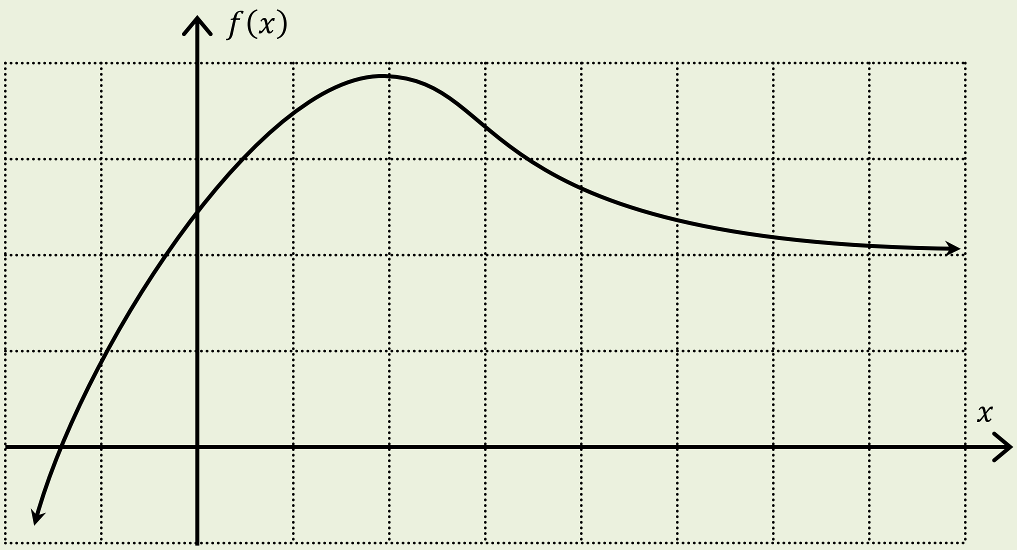
Usage: x is a variable, f is a function, and L is a real number.

Meaning: As x gets more and more positive without bound, the values of $f(x)$ get closer and closer to L, and may actually equal L.

Precise Meaning: There is an open interval (a, ∞) for some number a, with function f defined on all of the interval, and such that given any number $\epsilon > 0$, there exists a number $M > a$ such that if $x > M$ then $|f(x) - L| < \epsilon$.

Graphical Significance: Graph of f has a horizontal asymptote on the right at $y = L$. The line equation for the asymptote is $y = L$.

Graphical Example:



$\lim_{x\to\infty} f(x) = 2$

Obvious variations that I won't discuss:

Expanded Definition of Limit: Definition of Limit at Negative Infinity

Symbol: $\lim_{x\to-\infty} f(x) = L$

We can combine the idea of infinite limits and limits at infinity in the obvious way

Expanded Definition of Limit: Definition of Infinite Limits at Infinity

Symbol: $\lim_{x\to\infty} f(x) = \infty$

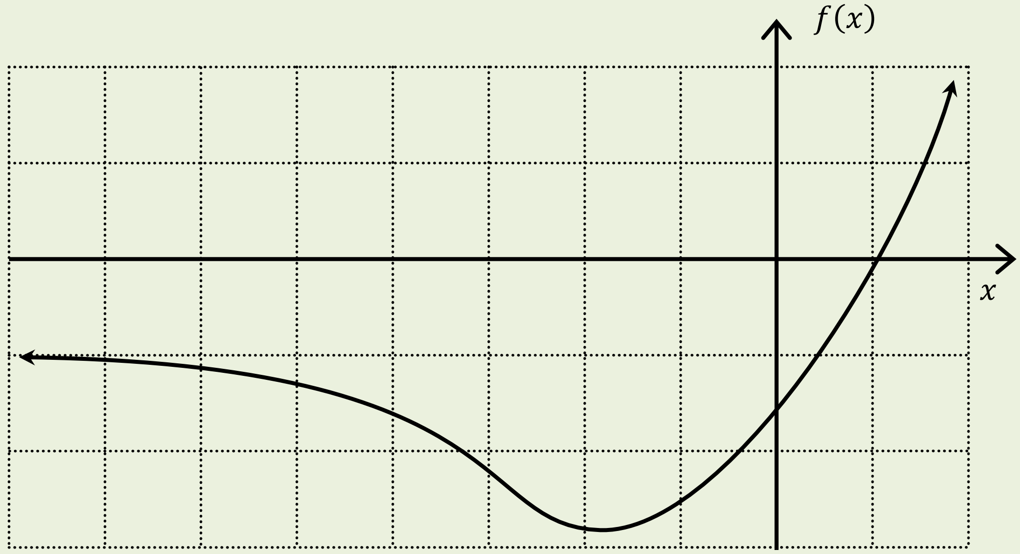
Spoken: The limit, as x goes to infinity, of $f(x)$ is infinity

Meaning: As x gets more and more positive without bound, the values of $f(x)$ get more and more positive without bound.

Precise Meaning: There is an open interval (a, ∞) for some number a, with function f defined on all of the interval, and such that given any number $N > 0$, there exists a number $M > a$ such that if $x > M$ then $f(x) > N$.

Graphical Significance: The graph of f goes up on the right without bound.

Graphical Example:



$\lim_{x\to-\infty} f(x) = -1$

$\lim_{x\to\infty} f(x) = \infty$

Remark about End Behavior: The term *limit at infinity* (or *limit at negative infinity*) refers to what the right (or left) end of the graph is doing. This is called the ***end behavior*** of the graph

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