Video for Homework H02.1

Reading: Epp's Section 2.1 Logical Form and Logical Equivalence

Definition of Statement

A statement is a sentence that is true or false, but not both.

Examples

Halloween is in September A false statement.

Chocolate is the best flavor of ice cream A sentence, but not a statement. Can't be proven true or false.

$3^2 + 4^2$ not a sentence (no verb), so not a statement.

$3^2 + 4^2 = 5^2$ A true statement.

Definition of Statement Variable

A variable that represents a statement.

Building Statements From Other Statements

The Simplest Way: The Negation

Definition of the negation

symbol: $\sim p$

spoken: the negation of p or not p

usage: p is some statement.

meaning: $\sim p$ is a new statement whose truth is given by the following table.

truth of p	truth of $\sim p$
T	F
F	T

Example involving an actual statement

false

The negation of the statement Halloween is in September.

Would be denoted $\sim$(Halloween is in September)

Would be converted to a sentence most simply by just putting the words "It is not the case that" in front.

It is not the case that Halloween is in September.

But this could be expressed more clearly as

Halloween is not in September. true.

In the coming week, we will encounter this idea again, that there may be a simple way to construct the sentence for a negation, but that simple way of constructing the sentence might not be the clearest. We will look for clearer ways of writing the sentence.

Building Statements from Other Statements in More Complicated Ways: Compound Statements

One Compound Statement: The Conjunction (the Logical *AND*)

Definition of the logical *AND*

symbols: p AND q $\quad\quad$ $p \wedge q$

spoken: p and q $\quad\quad$ p wedge q

usage: p, q are statements.

meaning: $p \wedge q$ is a new statement, whose truth is given by this table.

truth of p	truth of q	truth of $p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Another Compound Statement: The Disjunction (the Logical *OR*)

Definition of the *logical OR*

symbols: p OR q or $p \vee q$

spoken: p or q p vee q

usage: p, q are statements

meaning: $p \vee q$ is a new statement whose truth is given by this table.

truth of p	truth of q	truth of $p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Remark: the *logical OR* is also sometimes called the *inclusive OR*.

Another Compound Statement: The *exclusive OR*

Definition of the *exclusive OR*

symbols: p XOR q or $p \oplus q$

spoken: p exclusive or q

usage: p, q are statements

meaning: $p \oplus q$ is a new statement whose truth is given by this table.

truth of p	truth of q	truth of $p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

We won't be using the *exclusive OR* much in this course.

Remark: An important thing to be aware of in this course is that the terminology and notation is defined by definitions, not by common usage or by what makes sense. Common usage of the same words is often not the same as the defnitions of the terms that we use in math. And common usage of the same words often includes more than one usage for a particular term.

For example, when the waiter says *"you can have a baked potato or fries"*, they do not mean that you can choose one or the other or both. They mean that you can choose one or the other but not both. That is, what the waiter means would be described in our terminology as *"you can have a baked potato exclusive or fries"*. Of course, nobody speaks that way. But you get the idea. In this course, the terminology has the meaning that is specified in the definitions, regardless of what common usage you might be aware of, or might prefer.

Statement Forms

Remember that a ***statement variable*** is a *variable* that represents a *statement*.

A ***statement form*** is an expression made up of *statement variables* and *logical symbols* (so far, our only logical symbols are $\sim, \wedge, \vee, \oplus$) that becomes a *statement* when actual statements are substituted for the *statement variables.*

We have already seen examples of *statement forms*.

[Example] consider the symbol $p \wedge q$ is a statement form

If we

- let p be the statement Halloween is in september
- let q be the statement $3^2 + 4^2 = 5^2$

Then $p \wedge q$ is the following statement:

$$\underbrace{\underbrace{\text{Halloween is in september}}_{F} \quad \text{and} \quad \underbrace{3^2+4^2=5^2}_{T}}_{F}$$

But if we

- let p be the statement The capital of Ohio is Columbus $\leftarrow T$
- let q be the statement $3^2+4^2=5^2$ $\leftarrow T$

Then $p \wedge q$ is the following statement:

$$\underbrace{\underbrace{\text{The capital of Ohio is Columbus}}_{T} \quad \text{and} \quad \underbrace{3^2+4^2=5^2}_{T}}_{T}$$

End of [Example]

Just as it is useful to use letters as statement variables, it is also useful to use letters to represent statement forms.

That can be confusing, because a statement form is, itself, made up of a bunch of statement variables.

A handy convention will be

- use *lower case* letters for *statement variables*
- use *upper case* letters to denote *statement forms*

Truth Tables

The ***truth table*** for a *statement form* displays the truth values that correspond to all possible combinations of truth values ~~fror~~ its *statement variables*.

for

We have already seen examples of *truth tables*, in the definitions of ~,∧,∨, ⊕

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

Logical Equivalence

Definition of *logically equivalent statement forms*

symbols: $P \equiv Q$ (capital letters)

spoken: P is logically equivalent to Q

usage: P, Q represent <u>statement forms</u>.

meaning:

For all possible substitutions of statements for their statement variables, the resulting truth values of P and Q match

To test whether two statement forms P, Q are logically equivalent,

Make a truth table with one column for the truth values of P and another column for the truth values of Q

- If all entries in the column for P match the corresponding entries in the column for Q, then $P \equiv Q$.
- If any entries don't match then $P \not\equiv Q$.

[Example] Are $\overbrace{(p \wedge q) \vee r}^{P}$ and $\overbrace{p \wedge (q \vee r)}^{Q}$ logically equivalent? Explain using a truth table

Solution: We need $2 \cdot 2 \cdot 2 = 8$ rows

p	q	r	$p \wedge q$	$\overbrace{(p \wedge q) \vee r}^{P}$	$q \vee r$	$\overbrace{p \wedge (q \vee r)}^{Q}$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	F	T	T	T
T	F	F	F	F	F	F
F	T	T	F	T	T	F
F	T	F	F	F	T	F
F	F	T	F	T	T	F
F	F	F	F	F	F	F

Since these columns dont match, we conclude that $(p \wedge q) \vee r$ is <u>not</u> logically equivalent to $p \wedge (q \vee r)$

DeMorgan's Laws

Two famous logical equivalences having to do with negating AND, OR statements

De Morgan's Laws

$$\sim(p \wedge q) \equiv \sim p \vee \sim q$$

$$\sim(p \vee q) \equiv \sim p \wedge \sim q$$

Note: The negation of an AND statement is an OR Statement.

The negation of an OR statement is an AND statement

[Example] Use De Morgan's Laws to negate the statement

$$x < 3 \; OR \; 7 \leq x$$

Note: I always try to use $<$ or $\leq$, not $>$ or $\geq$.

Solution:

$$\sim\left((x<3) \text{ OR } (7 \leq x)\right) \equiv \sim(x<3) \text{ AND } \sim(7 \leq x)$$

$$\equiv (3 \leq x) \text{ AND } (x < 7)$$

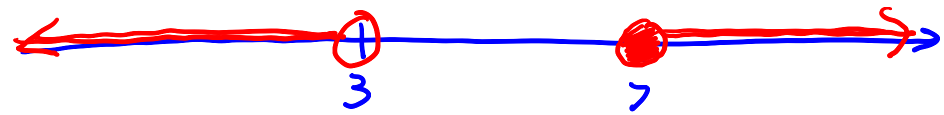
rather than $x \geq 3$

$$\equiv 3 \leq x < 7$$

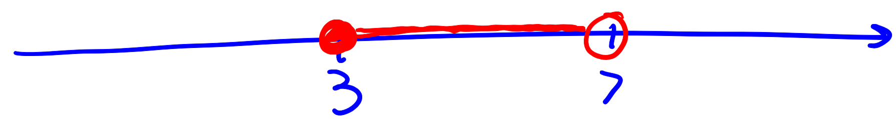
End of [Example]

Remark 1: The sets described by these expressions can be visualized on the number line:

The set described by $x < 3$ OR $7 \leq x$ would be



The solution set to the inequality $3 \leq x < 7$ would be

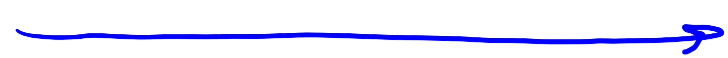


Remark 2: Can the symbol $x < 3$ OR $7 \leq x$ be further shortened to $7 \leq x < 3$?

No! The reason is that the symbol $7 \leq x < 3$ means $7 \leq x$ AND $x < 3$

that is $x < 3$ AND $7 \leq x$ impossible

That is impossible. The solution set would be empty.



Tautologies

A ***tautology*** is a *statement form* that is always *true*, regardless of the truth values of the statement variables.

A ***tautological statement*** is a statement that has the form of a *tautology*.

For example, the statement form $p \vee \sim p$

is a ***tautology***.

the statement

Halloween is in September or Halloween is not in September.

is a ***tautological statement.***

In a truth table, a tautology can be represented by a column with heading t and with every row filled with entry T.

Contradictions

A ***contradiction*** is a *statement form* that is always *false*, regardless of the truth values of the statement variables.

A ***contradictory statement*** is a statement that has the form of a *contradiction*.

For example, the statement form $p \wedge \sim p$

is a ***contradiction***.

The statement

Halloween is in September and Halloween is not in september.

is a ***contradictory statement.***

In a truth table, a contradiction can be represented by a column with heading c and with every row filled with entry F.

[Example] Use a truth table to establish whether the following statement form is a *tautology* or *contradiction* or *neither*.

$$(p \wedge \sim q) \wedge (\sim p \vee q)$$

Solution:

p	q	$\sim p$	$\sim q$	$p \wedge \sim q$	$\sim p \vee q$	$(p \wedge \sim q) \wedge (\sim p \vee q)$
T	T	F	F	F	T	F
T	F	F	T	T	F	F
F	T	T	F	F	T	F
F	F	T	T	F	T	F

Conclusion: The entries in this column are all false, so the statement form is a contradiction

End of [Example]

End of Video for H02.1