Section 3.1: Section 3.1 Introduction to Predicates and Quantified Statements I

Helpful Notation for sets of numbers

typeset symbol	handwritten symbol	meaning
$\boldsymbol{R}$	$\mathbb{R}$	the set of all real numbers
$\boldsymbol{R}^*$		non-zero real numbers
$\boldsymbol{R}^+$		positive real numbers $x>0$
$\boldsymbol{R}^{nonneg}$		non-negative real numbers $x\geq 0$
$\boldsymbol{Q}$	$\mathbb{Q}$	the set of rational numbers $\frac{m}{n}$, m,n integers, $n\neq 0$
$\boldsymbol{Z}$	$\mathbb{Z}$	the integers $\ldots -2,-1,0,1,2,\ldots$
$\boldsymbol{W}$	$\mathbb{W}$	the whole numbers $0,1,2,\ldots$ this is just $\mathbb{Z}^{nonneg}$
	$\mathbb{N}$	the natural numbers, $1,2,\ldots$ this is just $\mathbb{Z}^+$
	$\mathbb{C}$	the complex numbers

Recall this definition from Section 2.1

> A ***statement form*** is an expression made up of *statement variables* and *logical symbols* (so far, our only logical symbols are $\sim, \wedge, \vee, \oplus$) that becomes a *statement* when actual statements are substituted for the *statement variables.*

A similar concept that is new for Section 3.1 is that of a *predicate*:

> A ***predicate*** is a sentence that contains a finite number of variables and that becomes a statement when specific values are substituted in for the variables. The ***domain*** of a predicate variable is the set of all values that may be substituted in place of the variable.

We will be interested in identifying the values of the variables that will make a predicate *true*.

> If $P(x)$ is a predicate and x has a domain D, the ***truth set*** of $P(x)$ is the set of all elements of D that make $P(x)$ true when they are substituted for x. The truth set of $P(x)$ is denoted $\{x \in D \mid P(x)\}$. This is spoken *The set of all x in D such that $P(x)$*.

[Example 1] *Predicate* $A(x)$ is the sentence $x^2 \geq x$.

Let variable x have domain the set $D = \{-3, -2, 2, 3\}$. What is the truth set of *predicate* $A(x)$?

x	$A(x)$	
-3	$(-3)^2 \geq -3$	true
-2	$(-2)^2 \geq -2$	true
2	$(2)^2 \geq 2$	true
3	$(3)^2 \geq 3$	true

the truth set is all of D. $\{-3, -2, 2, 3\}$

Now let variable x have domain the set $D = \mathbf{Z}$. What is the truth set of predicate $A(x)$?

$x^2 \geq x$

x	$A(x)$
$x < 0$ x negative	x^2 will be positive, so $x^2 \geq x$ will be true (positive) (negative)
$x = 0$	$x^2 = (0)^2 = 0$, so $x^2 \geq x$ will be true.
$0 < x$	$x^2 \geq x$ will be true, by the calculation below

Since x is an integer, and we know $0 < x$, we know $1 \leq x$

multiply both sides by the positive number x.

$$1 \leq x$$

$$x \cdot 1 \leq x \cdot x$$

$$x \leq x^2$$

So the truth set is the set of all integers, $\mathbb{Z}$.

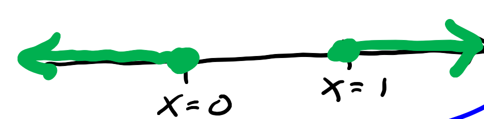
Let variable x have domain the set $D = \boldsymbol{R}$. What is the truth set of predicate $A(x)$?

$x^2 \geq x$

Make truth table

x	$A(x)$
$x < 0$ x negative	x^2 will be > 0, so $x^2 \geq x$ will be true
$x = 0$	$x^2 = (0)^2 = 0$, so $x^2 \geq x$ will be true
$0 < x < 1$	since $x < 1$ is true, we can multiply both sides by the pos number x $x \cdot x < 1 \cdot x$ $x^2 < x$ So $x^2 \geq x$ will be false
$1 \leq x$	Since $1 \leq x$ is true we can multiply both sides by the positive number x $1 \cdot x \leq x \cdot x$ $x \leq x^2$ will be true

So the solution set is $(-\infty, 0] \cup [1, \infty)$



Quantified Statements

A predicate $P(x)$ is a *sentence* about the variable x. The sentence is neither true nor false until one substitutes an actual value in for x.

It is often useful to make a summary *statement* about *how many* x in D make $P(x)$ true. In some situations, for a certain predicate $P(x)$ and domain D, *every* element x in D will turn $P(x)$ into a true statement. And in some situations, there are *no* elements x in D that will turn $P(x)$ into a true statement. And of course, in many situations, it is useful to know if there is *at least one* element x in D that will turn $P(x)$ into a true statement.

Statements about *how many* x in D make $P(x)$ true are called *quantified statements*. There are two types of quantified statements that are used in math: *universally quantified* statements and *existentially quantified* statements. Here are the definitions

$P(x)$ is not enclosed in parentheses in the book's symbol. And book's symbol has a comma.

Definition of *Universally Quantified Statement*

Symbol: $\forall x \in D\big(P(x)\big)$

Symbol used in book: $\forall x \in D, P(x)$

Spoken: *For all x in D, $P(x)$.*

Usage: x is a variable with domain D, and $P(x)$ is a *predicate* with variable x.

Meaning: The domain D has the property that every element x in D, when substituted into predicate $P(x)$, will turn $P(x)$ into a *true* statement.

Remark: $\forall x \in D\big(P(x)\big)$ is a *statement* about the domain D and the predicate $P(x)$.

Additional Terminology: If there is an element x in the domain D that, when substituted into predicate $P(x)$, turns $P(x)$ into a *false* statement, then the statement $\forall x \in D\big(P(x)\big)$ is a *false* statement, as well. An example of an $x \in D$ that does this is called a ***counterexample*** for the statement $\forall x \in D\big(P(x)\big)$.

Additional Terminology: The phrase *for all x in D*, denoted by the symbol $\forall x \in D$, is called the ***universal quantifier***.

Definition of *Existentially Quantified Statement*

In books symbol, $P(x)$ is not in parentheses, and the words <u>such that</u> are inserted.

Symbol: $\exists x \in D\big(P(x)\big)$

Symbol used in book: $\exists x \in D$ *such that* $P(x)$

Spoken: *There exists an x in D such that $P(x)$.*

Usage: x is a variable with domain D, and $P(x)$ is a *predicate* with variable x.

Meaning: The domain D has the property that there is an element x in D (at least one) that, when substituted into predicate $P(x)$, will turn $P(x)$ into a *true* statement.

Remark: $\exists x \in D\big(P(x)\big)$ is a *statement* about the domain D and the predicate $P(x)$.

Additional Terminology: The phrase *there exists an x in D such that*, denoted by the symbol $\exists x \in D$, is called the ***existential quantifier***.

Remark:

- It is easy to see that a predicate $P(x)$ is a *sentence* and *not a statement*, because it contains a variable. The sentence is *about* x, and the sentence does not become true or false (does not become a *statement*) until an actual value is substituted in for the variable x.
- Less obvious is that a symbol like $\forall x \in D\big(P(x)\big)$ is a *statement* and not a *predicate*. After all, the sentence does include the letter x. But the difference is that in the quantified statement $\forall x \in D\big(P(x)\big)$, something is being said *about the domain* D *and the predicate* $P(x)$. The letter x is needed to construct the sentence, but the x is playing a different role in the sentence than it plays in the sentence a predicate. We say that the x is a ***bound variable*** in the sentence $\forall x \in D\big(P(x)\big)$. A *bound variable* is a different thing from a *variable*.

[Example 2] Which of the statements A, B, C, D, E is true, and which is false? Explain.

Let A be the statement $\forall x \in \boldsymbol{Z}(x^2 \geq x)$.

Domain is $\mathbb{Z}$, predicate $P(x)$ is $x^2 \geq x$.

In earlier example, we found that for domain $\mathbb{Z}$ and predicate $x^2 \geq x$, the truth set was all integers. (All of the domain)

So $\forall x \in \mathbb{Z}\,(x^2 \geq x)$ is true

Let B be the statement $\forall x \in \boldsymbol{R}(\underbrace{x^2 \geq x})$.

Domain is $\mathbb{R}$ Predicate $P(x)$ is $x^2 \geq x$

Observe that $x = \frac{1}{2}$ is in $\mathbb{R}$, and when

we substitute x into $P(x)$ we get

$$\left(\frac{1}{2}\right)^2 \geq \frac{1}{2}$$

$$\frac{1}{4} \geq \frac{1}{2}$$ which is false

So the universal statement $\forall x \in \mathbb{R}(x^2 \geq x)$ is false,

an the counterexample $x = \frac{1}{2}$ shows it.

Let C be the statement $\exists x \in \boldsymbol{Z}(x^2 \geq x)$.

There exists an x in the integers such that $x^2 \geq x$

This is true.

For example let $x = 2$

then $x \in \mathbb{Z}$

and when we substitute x into the predicate, we get

$$(2)^2 \geq 2$$

$$4 \geq 2$$ which is true.

Let D be the statement $\exists x \in \boldsymbol{R}(x^2 \geq x)$.

This existential statement is true.

For example, let $x = 2$,

Let E be the statement $\exists x \in \boldsymbol{Z}(x^2 < x)$.

This is false!

There is no integer that makes $x^2 < x$ true

because we already know that every

integer makes $x^2 \geq x$ true

Quantified Conditional Statements

Consider the sentence

$$IF\ \ x \leq 5\ \ THEN\ \ x^2 \leq 25$$

One might be tempted to say that this is a *false statement*. For instance, the number -7 is less than or equal to 5, but $(-7)^2 = 49$, which is *not* less than or equal to 25.

But in the context of this course, the sentence above is not a *statement*. It is a *predicate* containing the variable x. The sentence does not become true or false until we substitute an actual value in for x. Let's name this predicate.

Let $P(x)$ be the predicate $IF\ \ x \leq 5\ \ THEN\ \ x^2 \leq 25$.

And $P(x)$ can be abbreviated in symbols $x \leq 5 \rightarrow x^2 \leq 25$

But you must be careful not to write $P(x) = x \leq 5 \rightarrow x^2 \leq 25$

.

Don't write this!!

Let's make a truth table for $P(x)$, where we substitute in different values for x and see what the resulting truth value of $P(x)$. Since x is a *variable*, it represents a number, and so it can take on many different values. That is different from a *statement variable*, that just represents a statement, and so can only be true or false. So the column for x in our truth table will be more complicated than the column for a statement variable in a truth table.

x	$x \le 5$	$x^2 \le 25$	$P(x)$ $(x \le 5) \to (x^2 \le 25)$
$x < -5$	T	F	F
$-5 \le x \le 5$	T	T	T
$5 < x$	F	F	T

.

We see that there are some values of x for which $P(x)$ is *false*—we already knew that—but there are also some values of x for which $P(x)$ is *true*.

So, to reiterate, $P(x)$ is a *predicate*. One cannot say whether $P(x)$ is true or false, because it contains a variable.

But there are some *quantified conditional statements* that we can build involving the predicate $P(x)$, and these do have a definite truth value.

This existential statement

$$\exists x \in \mathbb{R} \left(\text{IF } x \leq 5 \text{ THEN } x^2 \leq 25 \right)$$

is true!

But this universal statement

$$\forall x \in \mathbb{R} \left(\text{IF } x \leq 5 \text{ THEN } x^2 \leq 25 \right)$$

is false!

Implicitly Quantified Statements

It is worth returning to the sentence that started this discussion, to summarize what we have discussed about what this predicate means in this course and outside of this course.

Sentence	What it means to most mathematicians	What it means in our course
$x \leq 5 \rightarrow x^2 \leq 25$	*false statement*.	*predicate* that can be true or false
$\forall x \in \boldsymbol{R}(x \leq 5 \rightarrow x^2 \leq 25)$	huh? What is that?	A *false statement*.

It is very important to keep in mind, ***in this course***, that a sentence like $x \leq 5 \rightarrow x^2 \leq 25$ that contains a variable is to be interpreted as a *predicate*.

But ***outside of this course***, it will be important to keep in mind that to most mathematicians, the sentence $x \leq 5 \rightarrow x^2 \leq 25$ is really meant to mean what we would express with the sentence $\forall x \in \boldsymbol{R}(x \leq 5 \rightarrow x^2 \leq 25)$. We would say that when most mathematicians write $x \leq 5 \rightarrow x^2 \leq 25$, the quantifier is *implicit*. That is, the sentence $x \leq 5 \rightarrow x^2 \leq 25$ is ***implicitly quantified***, and really means $\forall x \in \boldsymbol{R}(x \leq 5 \rightarrow x^2 \leq 25)$.

Equivalent forms of Universal and Universal Conditional Statements

Let x be a variable with domain the set of integers, $\boldsymbol{Z}$.

Let $P(x)$ be the predicate ~~$x^2 \geq x$~~. $x^2 > 0$

Let's make a truth table for $P(x)$

x	$P(x)$ $x^2 > 0$
x negative	$x^2 > 0$ is true
$x = 0$	$x^2 > 0$ is false
x positive	$x^2 > 0$ is true

.

Observe that any *non-zero* integer x will make $P(x)$ true.

We can express that observation with the following ***universal statement***:

$$\forall x \in \mathbb{Z}^* \left(x^2 > 0 \right)$$

But we can also express the observation with this ***universal conditional statement***:

$$\forall x \in \mathbb{Z} \left(\text{IF } x \neq 0 \text{ THEN } x^2 > 0 \right)$$

The two statements are both about x that have two qualifications.

- $x \in \mathbb{Z}$
- $x \neq 0$

In the universal statement, both qualifications are ensured by using the domain $\boldsymbol{Z}^*$.

In the universal conditional statement, the one qualification is ensured by using the domain $\boldsymbol{Z}$ and the other qualification is ensured by having the hypothesis $x \neq 0$.

It might seem that the *universal statement* is the better way of expressing the observation, because it is more concise. But it will turn out that there are situations where the *universal conditional statement* is more helpful. (When we get to the part of the course where we are building proofs, it will turn out that having the statement expressed in universal conditional form will often make it easier to frame a proof structure.)

In your homework exercises, you have a problem that involves rewriting a *univeral statement* as a *universal conditional statement*.

End of Video