Negating Quantified Statements

Reading: Section 3.2 Intro to Predicates and Quantified Statements II

Homework: 3.2 # 4, 10, 15, 17, 25, 27, 38, 44 HO3.2

Concepts and tools from previous sections that we will use:

Recall that in the video for Section 2.1, it it was mentioned about negations:

> In the coming week, we will encounter this idea again, that there may be a simple way to construct the sentence for a negation, but that simple way of constructing the sentence might not be the clearest. We will look for clearer ways of writing the sentence.

Definition *logically equivalent statement forms* (From Section 2.1)

symbols: $P \equiv Q$

spoken: P

is logically equivalent to Q

usage: P, Q are statement forms

meaning: For all possible substitutions of statements for their statement variables, the resulting truth values of P, Q match.

De Morgan's Laws (from Section 2.1)

$$\sim(p \wedge q) \equiv \sim p \vee \sim q$$

$$\sim(p \vee q) \equiv \sim p \wedge \sim q$$

Definition of the *Conditional Statement Form* (from Section 2.2)

symbols: $p \to q$ also denoted $IF\ p\ THEN\ q$

spoken: *if p then q*

meaning: $p \to q$ is a statement form whose truth is given by the following table

truth of p	truth of q	truth of $p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

additional terminology: p is called the *hypothesis*, q is called the *conclusion*.

The *Negation of the Conditional Statement Form* (from Section 2.2)

For the *conditional statement form **S***: $p \to q$

The ***negation of S*** is the statement form: $p \wedge {\sim}q$

That is, ${\sim}(p \to q) \equiv p \wedge {\sim}q$

Definition of the *Converse, the Inverse, and the Contrapositive* (from Section 2.2)

For a statement form ***S***: $p \to q$

Symbol: *contrapositive(S)*

- **Spoken:** the *contrapositive of S*
- **Meaning:** the statement form: ${\sim}q \to {\sim}p$

Symbol: *converse(S)*

- **Spoken:** the *converse of S*
- **Meaning:** the statement form: $q \to p$

Symbol: *inverse(S)*

- **Spoken:** the *inverse of S*
- **Meaning:** the statement form: ${\sim}p \to {\sim}q$

Definition of *Universally Quantified Statement* From section 3.1

Symbol: $\forall x \in D\big(P(x)\big)$

Symbol used in book: $\forall x \in D, P(x)$

Spoken: *For all x in D, $P(x)$.*

Usage: x is a variable with domain D, and $P(x)$ is a *predicate* with variable x.

Meaning: The domain D has the property that every element x in D, when substituted into predicate $P(x)$, will turn $P(x)$ into a *true* statement.

Remark: $\forall x \in D\big(P(x)\big)$ is a *statement* about the domain D and the predicate $P(x)$.

Additional Terminology: If there is an element x in the domain D that, when substituted into predicate $P(x)$, turns $P(x)$ into a *false* statement, then the statement $\forall x \in D\big(P(x)\big)$ is a *false* statement, as well. An example of an $x \in D$ that does this is called a ***counterexample*** for the statement $\forall x \in D\big(P(x)\big)$.

Additional Terminology: The phrase *for all x in D*, denoted by the symbol $\forall x \in D$, is called the ***universal quantifier***.

Definition of *Existentially Quantified Statement* from Section 3.1

Symbol: $\exists x \in D\big(P(x)\big)$

Symbol used in book: $\exists x \in D \text{ such that } P(x)$

Spoken: *There exists an x in D such that $P(x)$.*

Usage: x is a variable with domain D, and $P(x)$ is a *predicate* with variable x.

Meaning: The domain D has the property that there is an element x in D (at least one) that, when substituted into predicate $P(x)$, will turn $P(x)$ into a *true* statement.

Remark: $\exists x \in D\big(P(x)\big)$ is a *statement* about the domain D and the predicate $P(x)$.

Additional Terminology: The phrase *there exists an x in D such that*, denoted by the symbol $\exists x \in D$, is called the ***existential quantifier***.

Negating a *Universally Quantified Statement*

[Example 1] Let A be the *universally quantified statement*

Every car in the Morton Hall parking lot is silver.

What is $\sim A$?

There exists a car in the morton Hall lot that is <u>not</u> silver.

More generally, let A be the *universally quantified statement*

$$\forall x \in D\big(Q(x)\big)$$

What is $\sim A$?

$$\exists x \in D\big(\sim Q(x)\big)$$

[Example 2] Let B be the statement

Every elephant at Ohio University is purple

Is this statement true or false?

Let D be the set of elephants at Ohio University

Let x be a variable with domain D

Let $P(x)$ be the predicate "x is purple"

Then statement B would be written "formally" (that is, abbreviated in symbols) as

statement B: $\forall x \in D (P(x))$

$\sim B$: $\exists x \in D (\sim P(x))$

There exists an Elephant at Ohio University that is not purple.

We see that $\sim B$ is false. So B is true. "Vacuously true"

Negating an Existentially Quantified Statement

[Example 3] Let C be the existentially quantified statement

There exists a car in the Morton Hall parking lot that is neon green.

What is $\sim C$?

For every car in the Morton Hall lot, the car is <u>not</u> neon green.

More generally, let C be the *existentially quantified statement*

$$\exists x \in D\big(Q(x)\big)$$

What is $\sim C$?

$$\forall x \in D\big(\sim Q(x)\big)$$

Negating a *Universal Conditional Statement*

Let D be the *universal conditional statement*,

$$\underline{\forall x \in D, IF\ P\ (x)\ THEN\ Q(x)}$$

What is $\sim D$?

D: $\forall x \in D$ (IF p(x) THEN q(x))

$\sim D \equiv \sim(\ \forall x \in D$ (IF p(x) THEN q(x)))

$\equiv\ \exists x \in D(\sim$ (IF p(x) THEN q(x)))

$\equiv\ \exists x \in D$ (p(x) AND $\sim$q(x))

[Example 4] Let E be the *universal conditional statement* introduced in the video for Homework ~~H02.1:~~ H03.1

$$E \equiv \forall x \in \boldsymbol{R}(x \leq 5 \rightarrow x^2 \leq 25)$$

Find the negation $\sim E$.

$$\sim E \equiv \sim\left(\forall x \in \mathbb{R}\left((x \leq 5) \rightarrow (x^2 \leq 25)\right)\right)$$

$$\equiv \exists x \in \mathbb{R}\left(\underbrace{\sim\left((x \leq 5) \rightarrow (x^2 \leq 25)\right)}_{\text{negate conditional statement form}}\right)$$

$$\equiv \exists x \in \mathbb{R}\left((x \leq 5) \text{ AND } \sim(x^2 \leq 25)\right)$$

$$\equiv \exists x \in \mathbb{R}\left((x \leq 5) \text{ AND } x^2 > 25\right)$$

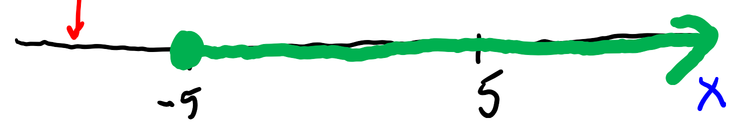
For the specific example given, which statement is true? E or $\sim E$? Explain

Recall that in the video for Homework ~~H02.1~~ H03.1, $P(x)$ was the predicate

$$x \le 5 \to x^2 \le 25$$

The truth set for this predicate was the set

bad x value $x = -7$



$[-5, \infty)$

Observe the truth set is not the set of all real numbers,

Because of that, we know that the universal conditional statement E

$$\forall x \in \boldsymbol{R}(x \le 5 \to x^2 \le 25)$$ is false

A counterexample is

$$x = -7$$

[Example 5] Let G be the *universally quantified statement*

Every prime number is odd.

Find the negation, $\sim G$.

There exists a prime number that is <u>not</u> odd.

Which is true, G or $\sim G$? Explain.

$\sim G$ is true, because $x=2$ is an example of a prime number that is <u>not</u> odd.

Start over.

Rewrite the original statement G as a *universal conditional statement* (also called G).

$G \equiv$ For all integers x, if x is prime then x is odd

Find the *negation* of the *universal conditional statement* (The negation is denoted $\sim G$).

$\sim G \equiv \sim($For all integers x, if x is prime then x is odd$)$

$\equiv$ There exists an integer x such that $\sim($if x is prime then x is odd$)$

$\equiv$ There exits an integer x such that x is prime and x is odd

Contrapositive, Converse, and Inverse of Universal Conditional Statement

[Example 6] Return to the *universal conditional statement E* discussed earlier:

$$\forall x \in \boldsymbol{R}(\underbrace{x \leq 5}_{p} \rightarrow \underbrace{x^2 \leq 25}_{q})$$

Write the *contrapositive*, *converse*, and *inverse* of Statement *E*

E: $\forall x \in R(p \rightarrow q)$

Contrapositive(E): $\forall x \in \mathbb{R}(\sim q \rightarrow \sim p)$

$\forall x \in \mathbb{R}(x^2 > 25 \rightarrow x > 5)$

Converse(E): $\forall x \in \mathbb{R}(q \rightarrow p)$

$\forall x \in \mathbb{R}(x^2 \leq 25 \rightarrow x \leq 5)$

Inverse(E) $\forall x \in \mathbb{R}(\sim p \rightarrow \sim q)$

$\forall x \in \mathbb{R}(x > 5 \rightarrow x^2 > 25)$

Which of the statements E, contrapositive(E), converse(E), inverse(E) are true and which are false?

We know from [Example 4] that E is false.

We know that Contrapositive(E) $\equiv$ E,

So contrapositive(E) is also false.

It is easy to see that inverse(E) is true.

The inverse(E) $\equiv$ converse(E), So that tells us that converse(E) is also true.

End of Video