Statements with Multiple Quantifiers

Reading: Section 3.3 Statements with Multiple Quantifiers

Homework: 3.3 # 2, 3, 6, 17, 19, 20, 23, 26, 30, 38 HO3.3

Important Result from Section 3.2: Negating Quantified Statements

Negating a Universally Quantified Statement

$$\sim\left(\forall x \in D\left(P(x,y)\right)\right) \equiv \exists x \in D\left(\sim P(x,y)\right)$$

Negating an Existentially Quantified Statement

$$\sim\left(\exists x \in D\left(P(x,y)\right)\right) \equiv \forall x \in D\left(\sim P(x,y)\right)$$

Negating Statements with Multiple Quantifiers

[Example 1] Consider Statement S:

There is a program that gives the correct answer to every question posed to it.

(a) Rewrite the statement formally (in symbols) using variables and quantifiers.

(b) Find the negation of the formal statement.

(c) Rewrite the negation informally (in words).

(a) Let A be the set of computer programs

Let p be a variable with domain A.

Let B be the set of all questions.

Let q be a variable with domain B

Let $C(p,q)$ be the predicate "program p correctly answers question q"

S can be abbreviated $\exists p \in A \left(\forall q \in B \left(C(p,q) \right) \right)$ Statement A rewritten "formally" (in symbols)

(b) $\sim S \equiv \sim(\exists p \in A(\forall q \in B(C(p,q))))$

$\equiv \forall p \in A(\sim(\forall q \in B(C(p,q))))$

$\equiv \forall p \in A(\exists q \in B(\sim C(p,q)))$

This is the formal negation.

(c) Rewrite this informally

at least one question q

For every program p, there is a question q such that p gives the wrong answer to q.

Changing the Order of Multiple Quantifiers

[Example 2] In the following examples, Consider Statement A, and Statement B obtained by changing the order of the quantifiers in Statement A.

For each pair of statements, do the following:

i. Which of the statements are true? (might be none, one, or both) Explain

ii. Some of the statements are famous properties of real numbers. Which statements, and what is the name of the property? Explain

iii. Find the negation of any of the statements that are false.

[Example 2a]

Statement A: $\underbrace{\forall x \in \boldsymbol{R}}\big(\underbrace{\exists y \in \boldsymbol{Z}}(x < y)\big)$

Statement B: $\overbrace{\exists y \in \boldsymbol{Z}}\big(\overbrace{\forall x \in \boldsymbol{R}}(x < y)\big)$

Statement A is true.

Given any real number x

Let $y = \lceil x \rceil + 1$, then y will be an integer and $x < y$.

ceiling function

least integer

Statement B says there exists an integer y that is greater than every real number. This is <u>false.</u> Given any integer y, we could let $x = y + 1$. Then x is a real number and x is not less than y.

Find the negation of B

$$\sim B \equiv \sim\left[\exists y \in \mathbb{Z} \left(\forall x \in \mathbb{R} \left(x < y\right)\right)\right]$$

$$\equiv \forall y \in \mathbb{Z} \left(\sim\left(\forall x \in \mathbb{R} \left(x < y\right)\right)\right)$$

$$\equiv \forall y \in \mathbb{Z} \left(\exists x \in \mathbb{R} \left(\sim\left(x < y\right)\right)\right)$$

$$\equiv \forall y \in \mathbb{Z} \left(\exists x \in \mathbb{R} \left(y \leq x\right)\right)$$

[Example 2b]

Statement A: $\forall x \in \boldsymbol{R}(\exists y \in \boldsymbol{R}(x + y = x))$

Statement B: $\exists y \in \boldsymbol{R}(\forall x \in \boldsymbol{R}(x + y = x))$ The Existence of an Additive Identity Element

Statement A is true: Let $y=0$

Statement B is true: Let $y=0$

[Example 2c]

Statement A: $\forall x \in \boldsymbol{R}\left(\exists y \in \boldsymbol{R}(x + y = 0)\right)$

Statement B: $\exists y \in \boldsymbol{R}\left(\forall x \in \boldsymbol{R}(x + y = 0)\right)$

Statement A is true!

Given a real number x,

let $y = -x$

Then $x + y = x + (-x) = 0$

Statement A is the statement of an additive inverse for each real number.

B is false! Find its negation

$$\sim B \equiv \sim\left(\exists y \in \mathbb{R}\left(\forall x \in \mathbb{R}\left(x+y=0\right)\right)\right)$$

$$\equiv \forall y \in \mathbb{R}\left(\sim\left(\forall x \in \mathbb{R}\left(x+y=0\right)\right)\right)$$

$$\equiv \forall y \in \mathbb{R}\left(\exists x \in \mathbb{R}\left(\sim\left(x+y=0\right)\right)\right)$$

$$\equiv \forall y \in \mathbb{R}\left(\exists x \in \mathbb{R}\left(x+y \neq 0\right)\right)$$

[Example 2d]

Statement A: $\forall x \in \boldsymbol{R}^{*}\left(\exists y \in R^{*}(xy = 1)\right)$

Statement B: $\exists y \in \boldsymbol{R}^{*}\left(\forall x \in R^{*}(xy = 1)\right)$

Statement A is true! Given any $x \in \mathbb{R}^*$, let $y = \frac{1}{x}$.

Then $y \in \mathbb{R}^*$ and $x \cdot y = x \cdot \frac{1}{x} = 1$

This is the property of the existence of a multiplicative inverse for every non-zero real number.

Statement B is false!

Statement B says that there is one single special real number y that is the additive inverse for every real number x. Not true!

$$\sim B \equiv \forall y \in \mathbb{R}^*\left(\exists x \in \mathbb{R}^*\left(xy \neq 1\right)\right)$$

Changing the Domain in Quantifiers

[Example 3] Consider statement S:

$$\exists x \in D\big(\forall y \in D(xy < y)\big)$$

Write the negation for S.

$$\sim S \equiv \sim\big(\exists x \in D\big(\forall y \in D(xy < y)\big)\big)$$

$$\equiv \forall x \in D\big(\sim \forall y \in D(xy < y)\big)$$

$$\equiv \forall x \in D\big(\exists y \in D\big(\sim(xy < y)\big)\big)$$

$$\equiv \forall x \in D\big(\exists y \in D(xy \geq y)\big)$$

Is Statement S true when the domain is $D = \boldsymbol{R}^+$? Explain

S: $\exists x \in \mathbb{R}^+ (\forall y \in \mathbb{R}^+ (xy < y))$

This is true!

Let $x = \frac{1}{2}$, for example

Then $x < 1$ is a true inequality

Let y be any positive real number.

Multiply both sides of the inequality by y

$x \cdot y < 1 \cdot y$

$xy < y$ a new true inequality

Is Statement S true when the domain is $D = \boldsymbol{R}^*$? Explain

Statement S is not true!

The trick of letting $x = \frac{1}{2}$ won't work.

Then if $y = -3$, we find that

$$x \cdot y = \frac{1}{2} \cdot (-3) = -\frac{3}{2}$$

So the inequality $xy < y$ would

become $-\frac{3}{2} < -3$, which is false!

I think you can see why other possible values of x won't work either.

Is Statement S true when the domain is $D = \mathbf{Z}^+$? Explain

$S: \exists x \in \mathbb{Z}^+ (\forall y \in \mathbb{Z}^+ (xy < y))$

Notice: we won't be able to choose x between 0 & 1,
So we won't be able to find an x that works

So S is false.

Consider $\sim S$

$\sim S \equiv \forall x \in \mathbb{Z} (\exists y \in \mathbb{Z}^+ (xy \geq y))$

This is true. Let x be any positive integer
then $x \geq 1$
Then let $y = 2$ for example.
Multiply both sides of our true inequality by $y=2$

$x \cdot 2 \geq 1 \cdot 2$

$x \cdot 2 \geq 2$

So $x \cdot y \geq y$ is true. So $\sim S$ is true
so S is false.

Interchanging $\forall, \exists$ in multiple quantifiers

[Example 4]

Consider Statement A, and Statement B obtained by interchanging $\forall, \exists$ in Statement A.

Statement A: $\forall x \in R^+\big(\exists y \in R^+(y < x)\big)$

Statement B: $\exists x \in R^+\big(\forall y \in R^+(y < x)\big)$

Is either of these statements true? Explain.

Statement A is true! Given some $x \in \mathbb{R}^+$

Let $y = \frac{1}{2}x$

Then $y < x$ will be true.

Statement B says that there exists a positive real number x that is greater than all positive real numbers y.

This is false! To see why, write the negation of B and show that $\sim B$ is true.

$$\sim B \equiv \sim\left[\exists x \in \mathbb{R}^+ \left(\forall y \in \mathbb{R}^+ \left(y < x\right)\right)\right]$$

$$\equiv \forall x \in \mathbb{R}^+ \left(\exists y \in \mathbb{R}^+ \left(y \geq x\right)\right)$$

To see why $\sim B$ is true,

Suppose $x \in \mathbb{R}^+$ is given

Let $y = x$

Then $y \geq x$ is true.

This shows that $\sim B$ is <u>true</u>, so B is false.

Interchanging x and y in multiple quantifiers

[Example 5]

Consider Statement A, and Statement B obtained by interchanging x, y in Statement A.

Statement A: $\forall x \in D\big(\exists y \in D(y = 2x + 1)\big)$

Statement B: $\forall y \in D\big(\exists x \in D(y = 2x + 1)\big)$

Let the domain D be the set $\boldsymbol{R}$. Is either of the statements A, B true? Explain.

Statement A is true. Given any x, just let $y = 2x+1$

Then the equation $y = 2x+1$ is true!

Statement B is also true

Given a y, what should you use for x?

Solve the equation $y=2x+1$ for x in terms of y

$$y = 2x+1$$

$$y-1 = 2x$$

$$\frac{y-1}{2} = x$$

then the equation $y = 2x+1$ becomes

$$y = 2\left(\frac{y-1}{2}\right)+1$$

$$= (y-1)+1$$

$$= y \quad \text{this is true!}$$

Let the domain D be the set $\boldsymbol{Z}$. Is either of the statements A, B true? Explain.

Statement A is still true

given any $x \in \mathbb{Z}$, let $y = 2x+1$

Then y is an integer, and the equation $y = 2x+1$ is true.

Statement B is not true

Given $y \in \mathbb{Z}$, the only x that can possibly work is $x = \frac{y-1}{2}$

But that might not be an integer!

For instance when $y=4$, x would have to be $x = \frac{4-1}{2} = \frac{3}{2}$

and $\frac{3}{2}$ is not an integer.

So for $y=4$, there is no x that will work.

End of Video