Video for Homework H05.1a

Topics from the first part of Section 5.1: Sequences

Sequences

Summation Notation

Product Notation

Definition of Sequence

A ***sequence*** is a list of numbers

If the list ends, the sequence is called a ***finite sequence***.

If the list goes on forever, the sequence is called an ***infinite sequence***.

The numbers on the list are called the ***terms*** of the sequence.

The first term of the sequence is called the ***initial term***.

If the last term of a finite sequence is called the ***final term***.

Two sequences are said to be the same if they are the same list of numbers.

For example, here is a finite sequence: $2, 4, 6, \ldots, 20$

and here is an infinite sequence: $2, 4, 6, \ldots$

Remark: The three dots, called an ***ellipsis***, indicate that the established pattern continues.

Abstract Notation for Sequences

The two sequences on the previous page were presented as lists of actual numbers.

It is helpful to have an abstract notation for sequences, either for the purpose of abbreviating the list or for the purpose of describing the pattern.

It is common to use a letter that serves as the name of the sequence, along with a ***subscript*** that serves as to create a uniqe label for each term. The subscript is also called the ***index***.

For instance, the finite sequence

$$2,4,6,\ldots,20$$

presented on the previous page could be denoted abstractly as

$$a_1, a_2, a_3, \ldots, a_{10}$$

Realize that more than one abstract description of a particular sequence is possible.
The sequence above could also be denoted

$$b_0, b_1, b_2, \ldots, b_9$$

Same sequence, just different symbols.

If a sequence has a pattern, then it is often helpful to describe the pattern using a mathematical formula.

An ***explicit formula*** or a ***general formula*** for a sequence consists of two things.

- A *mathematical formula* that shows how a_k depends on the index k.
- The *domain* of the index k.

For a finite sequence, the domain of the index k will always be of the form

$$starting\ value \leq k \leq ending\ value$$

For an infinite sequence, the domain of the index k will always be of the form

$$starting\ value \leq k$$

$k \geq$ starting value

It is important to note that the *domain* of the index is an essential part of the explicit formula.

Also note that explicit formulas that look very similar might not describe the same sequence. And note that very dissimilar-looking explicit formulas might describe the same sequence.

For example, the explicit formula

$$a_k = 2k \text{ for integers } 1 \le k \le 10$$

describes this sequence $a_1, a_2, a_3, \ldots, a_{10}$

$2(1), 2(2), 2(3), \ldots, 2(10)$

$2, 4, 6, \ldots, 20$

while the explicit formula

$$a_k = 2k \text{ for integers } 0 \le k \le 10$$

describes this sequence $a_0, a_1, a_2, a_3, \ldots, a_{10}$

$2(0), 2(1), 2(2), 2(3), \ldots, 2(10)$

$0, 2, 4, 6, \ldots, 20$

and the explicit formula

$$b_k = 2(k-3) \text{ for integers } 4 \le k \le 13$$

describes this sequence $2(4-3), 2(5-3), 2(6-3), \ldots, 2(13-3)$

$2, 4, 6, \ldots, 20$

The first two explicit formulas look very similar, but they do not describe the same sequence.

The third explicit formula describes the same sequence as the first formula.

[Example 1] Write the first five terms of the sequence defined by the explicit formula.

$$a_k = \frac{(-1)^k}{3^k} \text{ for integers } 0 \leq k$$

$$a_0 = \frac{(-1)^0}{3^0} = \frac{1}{1} = 1$$

$$a_1 = \frac{(-1)^1}{3^1} = \frac{-1}{3}$$

$$a_2 = \frac{(-1)^2}{3^2} = \frac{1}{9}$$

$$a_3 = \frac{(-1)^3}{3^3} = -\frac{1}{27}$$

$$a_4 = \frac{(-1)^4}{3^4} = \frac{1}{81}$$

An ***alternating sequence*** is one in which the *signs* of the terms *alternate*.

[Example 2] Consider the sequence

$$3,6,12,24,48,96$$

(A) Find an explicit formula for the sequence, using a starting index of 0.

(b) Find an explicit formula for the sequence, using a starting index of 1.

(a) $3(1), 3(2), 3(4), 3(8), 3(16), 3(32)$

$a_k = 3 \cdot 2^k \quad 0 \le k \le 5$

(b) $b_n = 3 \cdot 2^{(n-1)} \quad 1 \le n \le 6$

Summations

Definition

If m and n are integers and $m \leq n$, the symbol $\sum_{k=m}^{n} a_k$, read the **summation from k equals m to n of a-sub-k**, is the sum of all the terms $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$. We say that $a_m + a_{m+1} + a_{m+2} + \cdots + a_n$ is the **expanded form** of the sum, and we write

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n.$$

We call k the **index** of the summation, m the **lower limit** of the summation, and n the **upper limit** of the summation.

[Example 3]

(a) Write the first five terms of the sequence defined by the explicit formula.

$$a_k = 1 + k^3 \text{ for integers } 0 \leq k$$

$a_0 = 1 + 0^3 = 1$

$a_1 = 1 + 1^3 = 1 + 1 = 2$

$a_2 = 1 + 2^3 = 1 + 8 = 9$

$a_3 = 1 + 3^3 = 1 + 27 = 28$

$a_4 = 1 + 4^3 = 1 + 64 = 65$

first 5 terms

$1, 2, 9, 28, 65, \ldots$

sequence

(b) Write the summation in expanded form and find its value

$$\sum_{k=1}^{4} 1 + k^3 = (1+(1)^3) + (1+(2)^3) + (1+(3)^3) + (1+(4)^3)$$

$$= 2 + 9 + 28 + 65$$

$$= 104$$

(c) Write the summation in expanded form and find its value

$$\sum_{k=3}^{3} 1 + k^3 \qquad = \qquad 1 + (3)^3 \qquad = 9$$

Telescoping Sums

[Example 4] Find the value of the summation

$$\sum_{n=1}^{5}\left(\frac{1}{n}-\frac{1}{n+1}\right) =$$

$$= \left(\frac{1}{(1)}-\frac{1}{(1)+1}\right)+\left(\frac{1}{(2)}-\frac{1}{(2)+1}\right)+\left(\frac{1}{(3)}-\frac{1}{(3)+1}\right)+\left(\frac{1}{(4)}-\frac{1}{(4)+1}\right)+\left(\frac{1}{(5)}-\frac{1}{(5)+1}\right)$$

$$= \left(1-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{5}}\right)+\left(\cancel{\frac{1}{5}}-\frac{1}{6}\right)$$

$$= 1-\frac{1}{6}$$

$$= \frac{5}{6}$$

Summation Notation where the Upper Limit Involves a Variable

[Example 5]

(a) Write the summation in expanded form.

$$\sum_{k=1}^{n}(-2)^k = (-2)^1 + (-2)^2 + (-2)^3 + \cdots + (-2)^n$$

(b) Evaluate the summation (that is, find its value) when ~~$n = 3$~~. $n=2$

$$(-2)^1 + (-2)^2 + (-2)^3 + \cdots + (-2)^2$$

This means $(-2)^1 + (-2)^2 = (-2) + (4) = 2$

using summation notation

$$\sum_{k=1}^{2}(-2)^k = (-2)^1 + (-2)^2$$

Expanded form probably looks less intimidating, but it can be tricky.

[Example 6] Consider this sum that is presented in expanded form.

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$$

(a) Find the value of the sum when $n = 2$

$$\frac{1}{1^2} + \frac{1}{2^2} = 1 + \frac{1}{4} = \frac{5}{4}$$

(b) Express the original sum (with unknown n) in summation notation.

$$\sum_{k=1}^{k=n} \frac{1}{k^2}$$

(c) Express the result from (a) in summation notation

$$\sum_{k=1}^{k=2} \frac{1}{k^2}$$

Separating off the final term

[Example 7] Consider the summation

$$\sum_{m=1}^{n+1} m(m+1)$$

(a) Write the summation in expanded form.

$$1(1+1) + 2(2+1) + 3(3+1) + \cdots + (n+1)((n+1)+1)$$

(b) In the expanded form, "unhide" the second-to-last term.

$$\underbrace{1(1+1) + 2(2+1) + 3(3+1) + \cdots + n(n+1)} + \underbrace{(n+1)((n+1)+1)}_{\text{final term}}$$

(c) Rewrite the original summation (in summation notation) by separating off the final term.

$$\sum_{m=1}^{n} m(m+1) + \underbrace{(n+1)((n+1)+1)}_{\text{final term}}$$

Product Notation

Definition

If m and n are integers and $m \le n$, the symbol $\prod_{k=m}^{n} a_k$, read the **product from k equals m to n of a-sub-k**, is the product of all the terms $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$. We write

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot a_{m+2} \cdots a_n.$$

[Example 8] Compute the products.

Capital pi symbol $\prod$

(a) $\prod_{k=2}^{4} k^2 = 2^2 \cdot 3^2 \cdot 4^2 = 576$

(b) $\prod_{k=2}^{2} \left(1 - \frac{1}{k}\right) = \left(1 - \frac{1}{2}\right) = \frac{1}{2}$

only one term!

Telescoping Products

[Example 9] Compute the product.

$$\prod_{m=3}^{6} \frac{(m-1)(m+1)}{(m-2)m} =$$

$$= \frac{(3-1)(3+1)}{(3-2)3} \cdot \frac{(4-1)(4+1)}{(4-2)4} \cdot \frac{(5-1)(5+1)}{(5-2)5} \cdot \frac{(6-1)(6+1)}{(6-2)6}$$

$$= \frac{\cancel{2(4)}}{1(3)} \cdot \frac{\cancel{(3)(5)}}{\cancel{2(4)}} \cdot \frac{\cancel{(4)(6)}}{\cancel{(3)5}} \cdot \frac{(5)(7)}{\cancel{(4)(6)}}$$

$$= \frac{35}{3}$$

Expanded form probably looks less intimidating, but it can be tricky.

[Example 10] Consider this product that is presented in expanded form.

$$\left(\frac{2\cdot 4}{1\cdot 3}\right)\left(\frac{3\cdot 5}{2\cdot 4}\right)\left(\frac{4\cdot 6}{3\cdot 5}\right)\cdots\left(\frac{(m-1)(m+1)}{(m-2)m}\right)$$

(a) Find the value of the product when $m = 3$

$$\frac{2\cdot 4}{1\cdot 3} = \frac{8}{3}$$

(b) Express the original product (with unknown m) in product notation.

$$\prod_{k=3}^{k=m} \frac{(k-1)(k+1)}{(k-2)k}$$

(c) Express the result from (a) in product notation.

$$\prod_{k=3}^{k=3} \frac{(k-1)(k+1)}{(k-2)k}$$