Video for Homework H05.1b

Topics from the second part of Section 5.1: Sequences

- **Factorial**
- **n Choose r**
- **Converting between base 10 & base 2**
- **Tracing an Algorithm**

Factorial Notation

Here is the book's definition of the factorial symbol, along with the author's remark.

Definition

For each positive integer n, the quantity ***n* factorial** denoted $\boldsymbol{n!}$, is defined to be the product of all the integers from 1 to n:

$$n! = n\cdot(n-1)\cdots 3\cdot 2\cdot 1.$$

Zero factorial, denoted 0!, is defined to be 1:

$$0! = 1.$$

The definition of zero factorial as 1 may seem odd, but, as you will see when you read Chapter 9, it is convenient for many mathematical formulas.

This amounts to a piecewise definition:

$$n! = \begin{cases} n(n-1)\cdots 3\cdot 2\cdot 1 & when\ n \geq 1 \\ 1 & when\ n = 0 \end{cases}$$

This sort of definition for factorial is common, but it inevitably leads to some confusion. The fact that $0! = 1$ seems strange and arbitrary. Many people would expect that $0! = 0$.

I prefer a different definition:

Mark's Definition of Factorial

Symbol: $n!$

Spoken: n *factorial*

Usage: $n \in \boldsymbol{Z}^{\boldsymbol{nonneg}}$ That is, $n \in \boldsymbol{Z}$ and $n \geq 0$

Meaning: $n! = 1 \cdot \underbrace{1 \cdot 2 \cdots (n-1) \cdot n}_{n \ consecutive \ integers}$

For example:

$4! = 1 \cdot \underbrace{1 \cdot 2 \cdot 3 \cdot 4}_{4 \text{ integers}} = 24$

$3! = 1 \cdot \underbrace{1 \cdot 2 \cdot 3}_{3 \text{ integers}} = 6$

$2! = 1 \cdot 1 \cdot 2 = 2$

$1! = 1 \cdot \underbrace{1}_{1 \text{ integer}} = 1$

$0! = 1$

↑
0 integers next to the 1.

One can see that the definition that I suggest does take insure that $0! = 1$ without having any sort of special case for $n = 0$. But that can still be unsatisfying. The obvious question is, why have $0! = 1$ when a more obvious definition would be that $0! = 0$?

The reason that the factorial is defined the way it is has to do with the sequence of numbers that it produces. The sequence

$$1, 1, 2, 6, 24, 120, \ldots$$

produced by the factorial operation is a sequence of numbers that occurs often in math.

By contrast, this sequence

$$0, 1, 2, 6, 24, 120, \ldots$$

does not occur often in math.

The factorial notation was introduced to correspond to the sequence of numbers that occurs often. There would be no need to invent a symbol to correspond to a sequence that does not often occur.

In fact, it is possible to give an alternate definition of factorial that also serves as an example of one situation where the sequence

$$1, 1, 2, 6, 24, 120, \ldots$$

occurs in math.

Alternate Definition of Factorial

Symbol: $n!$

Spoken: n *factorial*

Usage: $n \in \boldsymbol{Z}^{\boldsymbol{nonneg}}$ That is, $n \in \boldsymbol{Z}$ and $n \geq 0$

Meaning: the n^{th} derivative of x^n

Meaning in symbols: $\left(\frac{d}{dx}\right)^n x^n$

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Using this definition,

$$4! = \left(\frac{d}{dx}\right)^4 x^4 = \left(\frac{d}{dx}\right)^3 4\cdot x^3 = \left(\frac{d}{dx}\right)^2 3\cdot 4\cdot x^2 = \left(\frac{d}{dx}\right) 2\cdot 3\cdot 4\cdot x = 2\cdot 3\cdot 4 = 24$$

$$3! = \left(\frac{d}{dx}\right)^3 x^3 = \left(\frac{d}{dx}\right)^2 3x^2 = \left(\frac{d}{dx}\right) 2\cdot 3x = 2\cdot 3 = 6$$

$$2! = \left(\frac{d}{dx}\right)^2 x^2 = \frac{d}{dx} 2x = 2$$

$$1! = \left(\frac{d}{dx}\right)^1 x^1 = 1$$

$$0! = \left(\frac{d}{dx}\right)^0 x^0 = x^0 = 1$$

[Example 1] (similar to 5.1# 62,64,67,70) Calculations involving factorial.

Compute the following. (In the expressions that involve variables, assume that the values of the variables are restricted so that the expressions are defined.)

(a) $\frac{7!}{5!} = \frac{7\cdot 6\cdot \cancel{5}\cdot \cancel{4}\cdot \cdots \cancel{2}\cdot 1}{\cancel{5}\cdot \cancel{4}\cdot \cdots \cancel{2}\cdot 1} = 7\cdot 6 = 42$

(b) $\frac{7!}{0!} = \frac{7!}{1} = 7! = 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = \cdots = 5040$

(c) $\frac{m!}{(m-4)!} = \frac{m(m-1)(m-2)(m-3)\cancel{(m-4)}\cancel{(m-5)}\cdots \cancel{2}\cdot 1}{\cancel{(m-4)}\cancel{(m-5)}\cdots \cancel{2}\cdot 1} = m(m-1)(m-2)(m-3)$

(d) $\frac{m!}{(m-p+2)!} = \frac{m(m-1)\cdots\cdots(m-p+3)\cancel{(m-p+2)}\cancel{(m-p+1)}\cdots \cancel{2}\cdot 1}{\cancel{(m-p+2)}\cancel{(m-p+1)}\cdots \cancel{2}\cdot 1}$

$= m(m-1)\cdots(m-p+3)$

$\boldsymbol{n}$ *Choose* $\boldsymbol{r}$

Definition of $\boldsymbol{n}$ *Choose* $\boldsymbol{r}$

Symbol: $\binom{n}{r}$

Alternate symbols: $nCr,\ C(n,r), C_{n,r}$

Spoken: $n\ \ choose\ \ r$

Also spoken: $n\ \ take\ \ r$

Usage: $n, r \in \boldsymbol{Z}$ and $0 \leq r \leq n$

Meaning: the number $\frac{n!}{r!(n-r)!}$

One Interpretation: the number of subsets with r elements that can be chosen from a set with n elements.

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[Example 2] (similar to 5.1#72,73,74,76) Calculations involving $\binom{n}{r}$

Compute the following. (In the expressions that involve variables, assume that the values of the variables are restricted so that the expressions are defined.)

(a) $\binom{7}{4} = \frac{7!}{4!(7-4)!} = \frac{7!}{4!3!} = \frac{7\cdot\cancel{6}\cdot 5\cdot\cancel{4\cdot 3\cdot 2\cdot 1}}{(\cancel{4\cdot 3\cdot 2\cdot 1})(\cancel{3\cdot 2\cdot 1})} = 7\cdot 5 = 35$

(b) $\binom{7}{7} = \frac{7!}{7!(7-7)!} = \frac{7!}{7!\,0!} = \frac{\cancel{7!}}{\cancel{7!}(1)} = 1$

(c) $\binom{7}{0} = \frac{7!}{0!(7-0)!} = \frac{7!}{(1)\,7!} = 1$

(d) $\binom{m+2}{m-3} = \frac{(m+2)!}{(m-3)!\,((\cancel{m}+2)-(\cancel{m}-3))!} = \frac{(m+2)!}{(m-3)!\,(2-(-3))!} = \frac{(m+2)!}{(m-3)!\,5!}$

$= \frac{(m+2)(m+1)m(m-1)(m-2)\cancel{(m-3)}\cancel{(m-4)}\cdots\cancel{(2)}(1)}{(\cancel{(m-3)}\cancel{(m-4)}\cdots\cancel{(2)}(1))\cdot(5\cdot 4\cdot 3\cdot 2\cdot 1)}$

$= \frac{(m+2)(m+1)m(m-1)(m-2)}{120}$

Base notation

Definition of *base notation*

Symbol: $(r_k r_{k-1} \cdots r_2 r_1 r_0)_b$

Usage:

- $b \in \boldsymbol{Z}$ and $b \geq 2$
- $k \in \boldsymbol{Z}$ and $k \geq 0$
- $r_0, r_1, r_2, \ldots, r_{k-1}, r_k \in \boldsymbol{Z}$ with each r_m satisfying the inequality $0 \leq r_m < b$

Meaning: the number $r_k \cdot b^k + r_{k-1} \cdot b^{k-1} + \cdots + r_2 \cdot b^2 + r_1 \cdot b + r_0$

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[Example 3] Calculations involving base notation

(a) If we consider the symbol 105 as representing a base 10 number, then the symbol means

$$1 \cdot 10^2 + 0 \cdot 10^1 + 5 \cdot 10^0 = 1 \cdot 100 + 0 \cdot 10 + 5 \cdot 1 = 100 + 0 + 5$$

Using base notation, we would write

$$(105)_{10} = 1 \cdot 10^2 + 0 \cdot 10^1 + 5 \cdot 10^0 = 105$$

(b) But if we consider the symbol 105 as representing a base 7 number, then the symbol stands for a different number. We can convert this number to a base 10 representation.

$(105)_7$ means $1\cdot 7^2 + 0\cdot 7^1 + 5\cdot 7^0 = 1\cdot 49 + 0\cdot 7 + 5\cdot 1$

$= 54$

(c) Convert $(101101)_2$ to a base 10 representation.

$(101101)_2 = 1\cdot 2^5 + 0\cdot 2^4 + 1\cdot 2^3 + 1\cdot 2^2 + 0\cdot 2^1 + 1\cdot 2^0$

$= 1\cdot 32 + 0\cdot 16 + 1\cdot 8 + 1\cdot 4 + 0\cdot 2 + 1\cdot 1$

$= 32 + 8 + 4 + 1$

$= (45)_{10}$

Converting Base 10 to Base 2

As you can see, given a number in some other base representation, it is easy to convert it into base 10 representation.

The reverse process, however is more tedious. That is,

Given a number n expressed in base 10 notation, how can one find the coefficients that would express the same number in base 2 notation?

The process is described in the book on pages 272 – 273. Essentially, one sets up a sequence of equations of the form

$$m = 2q + r$$

where $0 \leq r < 2$. (That is, $r = 0$ or $r = 1$.) I call this a ***QRT*** equation.

The starting equation is set up by letting $m = n$, and finding the q, r that work. Use the symbols q_0, r_0 to be the values of q, r that work. The *QRT* equation is

$$n = 2q_0 + r_0$$

I'll call this the 0^{th} equation.

Then another equation is set up by letting $m = q_0$, and finding the q, r that work. (Find them by dividing q_0 by 2 and to find the quotient q and the remainder r.) Use the symbols q_1, r_1 to be the values of q, r that work. The *QRT* equation is

$$q_0 = 2q_1 + r_1$$

I'll call this the 1^{st} equation.

The procedure continues until one reaches an equation where the quotient q turns out to be zero. If we call this the k^{th} equation, then the QRT equation is

$$q_{k-1} = 2 \cdot 0 + r_k$$

The whole list of QRT equations is

$$\begin{aligned} n &= 2q_0 + r_0 \\ q_0 &= 2q_1 + r_1 \\ q_1 &= 2q_2 + r_2 \\ &\vdots \\ q_{k-1} &= 2 \cdot 0 + r_k \end{aligned}$$

By substituting each equation into the one previous, we can obtain the following equation that expresses n as a sum of powers of 2.

$$n = r_k \cdot 2^k + r_k \cdot 2^k + \cdots + r_2 \cdot 2^2 + r_1 \cdot 2 + r_0$$

Expressing this using base notation, this would be written

$$(n)_{10} = (r_k r_{k-1} \cdots r_2 r_1 r_0)_2$$

The book presents a method of doing repeated divisions by 2, and shows a concise way of doing the repeated divisions without taking up much space, by leaving out a lot of symbols. This sort of shorthand presentation is nifty, much in the same way that synthetic division is nifty. The problem is, the meaning of the concise symbols is easy to forget, and so it is possible to get incorrect results by misreading them. Also, if you want to present a conversion calculation to someone else, it is best to not use a presentation that leaves out meaningful symbols.

That is why I prefer to do base conversions as I described on the previous pages. I don't even find it helpful to use division by 2. I simply find the q, r that work. My method is

To convert a number n in base 10 notation to a base 2 notation

- Build a list of QRT equations.
- Write the equation that expresses n as a sum of powers of 2.
- Express that equation using base notation

[Example 4] (similar to 5.1#83)

$n = 109$

(a) Write an equation that expresses 109 as a sum of powers of 2.

(b) Convert 109 from base 10 to base 2.

Build the QRT equations, each with $d=2$

starting with $n=109$

0th equation $109 = 2 \cdot 54 + 1$ $\quad r_0 = 1$

$54 = 2 \cdot 27 + 0$ $\quad r_1 = 0$

$27 = 2 \cdot 13 + 1$ $\quad r_2 = 1$

$13 = 2 \cdot 6 + 1$ $\quad r_3 = 1$

$6 = 2 \cdot 3 + 0$ $\quad r_4 = 0$

$3 = 2 \cdot 1 + 1$ $\quad r_5 = 1$

6th equation $1 = 2 \cdot \boxed{0} + 1$ $\quad r_6 = 1$

$q = 0$

(a) $\underline{109} = 1\cdot 2^6 + 1\cdot 2^5 + 0\cdot 2^4 + 1\cdot 2^3 + 1\cdot 2^2 + 0\cdot 2^1 + 1\cdot 2^0$

(b) $(109)_{10} = (1101101)_2$

An algorithm for converting base 10 to base 2

On pages 272 – 273, the author presents a *Decimal to Binary Conversion Algorithm*.

Algorithm 5.1.1 Decimal to Binary Conversion Using Repeated Division by 2

[In Algorithm 5.1.1 the input is a nonnegative integer a. The aim of the algorithm is to produce a sequence of binary digits $r[0], r[1], r[2], \ldots, r[k]$ *so that the binary representation of n is*

$$(r[k]r[k-1]\cdots r[2]r[1]r[0])_2.$$

That is,

$$a = 2^k \cdot r[k] + 2^{k-1} \cdot r[k-1] + \cdots + 2^2 \cdot r[2] + 2^1 \cdot r[1] + 2^0 \cdot r[0].]$$

Input: a *[a nonnegative integer]*

Algorithm Body:

$q := a, i := 0$

[Repeatedly perform the integer division of q by 2 until q becomes 0. Store successive remainders in a one-dimensional array $r[0], r[1], r[2], \ldots, r[k]$*. Even if the initial-value of q equals 0, the loop should execute one time (so that r[0] is computed). Thus the guard condition for the* **while** *loop is* $i = 0$ *or* $q \neq 0$*.]*

while ($i = 0$ or $q \neq 0$)

$r[i] := q \bmod 2$

$q := q \operatorname{div} 2$

[r[i] and q can be obtained by calling the division algorithm.]

$i := i + 1$

end while

[After execution of this step, the values of $r[0], r[1], \ldots, r[i-1]$ *are all 0's and 1's, and* $a = (r[i-1]r[i-2]\cdots r[2]r[1]r[0])_2$*.]*

Output: $r[0], r[1], r[2], \ldots, r[i-1]$ *[a sequence of integers]*

r[2]: q mod 2
q: q div 2
i := i+1

Although I prefer my clunky style of doing the decimal to binary conversion, it is worthwhile to do the book's exercise about tracing the operation of this algorithm, as preparation for future discussions about tracing algorithms.

[Example 5] (similar to 5.1#86) Make table to trace the action of the *Decimal to Binary Conversion Algorithm* when it is used to convert the number 109 from decimal to binary.

iteration	0	1	2	3	4	5	6	7
a	109	109	109	10				
i	0	1	2	3	4	5	6	7
q	109	54	27	13	6	3	1	0
r[0]		1						
r[1]			0					
r[2]				1				
r[3]					1			
r[4]						0		
r[5]							1	
r[6]								1