Video for Homework H05.2 on Concepts from Section 5.2 Induction

Closed form Expressions of Sums

- **Introducing Closed Form Expressions for two important sums**
    - Sum of the first n positive integers
    - Sum of a Geometric Sequence
- **Using the Closed Form Expressions of Sums**
- **Using Arithmetic to Prove the Closed Form Expressions for Sums**

The Principle of Induction

Using the Principle of Induction to prove the closed form expressions for sums.

- Sum of the first n positive integers
- Sum of the first n positive perfect squares

Closed form Expressions of Sums

Sum of the first n positive integers

It is known that the following equation is true

Formula for the sum of the first n positive integers

If $n \geq 1$, then

$$1 + 2 + 3 + \cdots + n = \sum_{k=1}^{k=n} k = \frac{n(n+1)}{2}$$

[Example 1] Consider the following quantity

$$S = 1 + 2 + 3 + \cdots + 7$$

(a) Compute S directly by finding the sum.

(b) Compute S by using the formula $S = \dfrac{n(n+1)}{2}$

In both (a) and (b), count the number of operations used to compute S.

(a) $S = 1+2+3+4+5+6+7 = 28$ six operations

(b) $S = \dfrac{7(7+1)}{2} = \dfrac{7 \cdot 8}{2} = \dfrac{56}{2} = 28$ three operations

[Example 2] Consider the following quantity

$$S = 1 + 2 + 3 + \cdots + 200$$

$n=200$

(a) Compute S directly by finding the sum.

(b) Compute S by using the formula $S = \dfrac{n(n+1)}{2}$

In both (a) and (b), count the number of operations used to compute S.

(a) Too tedious! It would require 199 operations

(b) $S = \dfrac{200(200+1)}{2} = 100(201) = 20{,}100$ 3 operations

[Example 3] Consider the following quantity

$$S = 1 + 2 + 3 + \cdots + n$$

(a) How many operations are required to compute S directly by finding the sum?

(b) How many operations are required to compute S by using the formula $S = \dfrac{n(n+1)}{2}$?

(a) $n-1$ operations. (unknown, because n is unknown)

(b) 3 operations

Definition of Closed Form Expression

A ***closed form expression*** is a mathematical expression that involves a known (finite) number of standard operations.

Notice that the expression $1 + 2 + 3 + \cdots + n$ contains $n - 1$ operations. This is a finite but unknown number. So the expression is *not* a closed form expression.

But the expression $\frac{n(n+1)}{2}$ contains exactly three operations. It *is* a closed form expression.

The equation $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ gives us a closed form expression for computing the value of the sum. The equation is useful because

- When n is known, the formula enables us to compute the value of the sum with fewer operations, as in **[Examples 1,2].**
- When n is unknown, the equation enables us to replace the sum that has an unknown number of operations with an expression that has a known (finite) number of operations.

Another equation giving a closed form expression that is equal to a sum:

Formula for the Sum of a Geometric Sequence

If $r \in \boldsymbol{R}, r \neq 0{,}1$ and $n \geq 0$, then

$$1 + r + r^2 + r^3 + \cdots + r^n = \sum_{k=0}^{k=n} r^k = \frac{r^{n+1} - 1}{r - 1}$$

[Example 4] Consider the following quantity

$r=2$ $\qquad 64 = 2^7$ so $n=7$

$$S = 1 + 2 + 4 + \cdots + 64$$

(a) Compute S directly by finding the sum.

(b) Compute S by using the formula $S = \dfrac{r^{n+1} - 1}{r - 1}$

(a) $S = 1 + 2 + 4 + 8 + 16 + 32 + 64 = 127$

(b) $S = \dfrac{2^{7+1} - 1}{2 - 1} = \dfrac{2^8 - 1}{1} = 128 - 1 = 127$

[Example 5] Consider the following quantity

$$S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$$

$r = \frac{1}{2}$

$\left(\frac{1}{8}\right) = \left(\frac{1}{2}\right)^3$ so $n = 3$

(a) Compute S directly by finding the sum.

(b) Compute S by using the formula $S = \dfrac{r^{n+1} - 1}{r - 1}$

(a) $S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{8}{8} + \frac{4}{8} + \frac{2}{8} + \frac{1}{8} = \frac{15}{8}$

(b) $S = \dfrac{\left(\frac{1}{2}\right)^{3+1} - 1}{\left(\frac{1}{2}\right) - 1} = \dfrac{\left(\frac{1}{2}\right)^4 - 1}{-\frac{1}{2}} = \dfrac{\frac{1}{16} - 1}{-\frac{1}{2}} =$

$= \dfrac{-\frac{15}{16}}{-\frac{1}{2}} = -\frac{15}{\cancel{16}} \cdot \left(-\frac{\cancel{2}}{1}\right) = \frac{15}{8}$

Using the formulas for the standard sums to find values for sums that are not standard

[Example 6] (a) Find the sum $15 + 20 + 25 + \cdots + 2000$

$S = 15 + 20 + 25 + \cdots + 2000$

factor out 5

$= 5(3 + 4 + 5 + \cdots + 400)$

Trick

$= 5(1 + 2 + 3 + 4 + 5 + \cdots + 400 - 3)$

$= 5(\underbrace{1 + 2 + 3 + \cdots + 400}_{\text{sum of the first 400 positive integers}}) - 5(3)$

$= 5\left(\frac{400(400+1)}{2}\right) - 15$

$= 5(200)401 - 15 = 1000(401) - 15 = 401{,}000 - 15$

$= 400{,}985$

(b) Find the sum $18 + 54 + 162 + \cdots + 13122$

Solution

$S = 18 + 54 + 162 + \cdots + 13{,}122$

factor out a 2

$= 2(9 + 27 + 81 + \cdots + 6561)$

$= 2(3^2 + 3^3 + 3^4 + \cdots + 3^8)$

trick

$= 2(1 + 3^1 + 3^2 + 3^4 + \cdots + 3^8 - 4)$

$= 2\underbrace{(1 + 3^1 + 3^2 + \cdots + 3^8)}_{\text{standard geometric sum with } r=1 \text{ and } n=8} - 2(4)$

$= 2\left(\frac{3^{8+1} - 1}{3 - 1}\right) - 8$

$= \frac{\cancel{2}(3^9 - 1)}{\cancel{2}} - 8 = (19{,}683 - 1) - 8 = 19{,}674$

Using Arithmetic to Prove the Closed Form Expressions for Sums

Some closed form expressiong for sums can be proved easily, using arithmetic.

[Example 7] Prove the formula for the sum of the first n positive integers

If $n \geq 1$, then

$$1 + 2 + 3 + \cdots + n = \sum_{k=1}^{k=n} k = \frac{n(n+1)}{2}$$

Let $S = 1 + 2 + 3 + \cdots + n$

trick

$$2S = S + S = \begin{array}{l} 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n \\ + \; n + (n-1) + (n-2) + \cdots + 3 + 2 + 1 \\ \hline \end{array}$$

$$= \underbrace{(n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1) + (n+1)}_{n \text{ of these}}$$

$$2S = n(n+1)$$

$$S = \frac{n(n+1)}{2}$$

[Example 8] Prove the formula for the sum of a geometric sequence

If $r \in \boldsymbol{R}, r \neq 0,1$ and $n \geq 0$, then

$$1 + r + r^2 + r^3 + \cdots + r^n = \sum_{k=0}^{k=n} r^k = \frac{r^{n+1} - 1}{r - 1}$$

Let $S = 1 + r + r^2 + \cdots + r^n$

then $rS = r(1 + r + r^2 + \cdots + r^n) = r + r^2 + r^3 + \cdots + r^{n+1}$

$$\begin{aligned} rS - S = \quad & r + r^2 + r^3 + \cdots + r^n + r^{n+1} \\ & - (1 + r + r^2 + r^3 + \cdots + r^n) \end{aligned}$$

$S(r-1) = r^{n+1} - 1$

$$S = \frac{r^{n+1} - 1}{r - 1}$$

But some equations that give closed form expressiong for sums must be proven using the ***Principle of Induction*.**

New Rule of Inference: The Principle of Induction

$P(a)$is true

For all integers $k \geq a$ if $P(k)$ is true, then $P(k+1)$ is true.

$\therefore$ For all integers $n \geq a$, $P(n)$ is true.

Usage:

- The letter a represents some fixed integer.
- The letters k and n represent variables whose domain is the set of all integers greater than or equal to a.
- The symbol $P(n)$ represents a predicate.

This new rule of inference will be used to prove statements of the form

Statement S: For all integers $n \geq a$, $P(n)$ is true.

Strategy for using the Principle of Induction

Preliminary work:

- Identify the number playing the role of a. (Introduce it.)
- Identify the predicate $P(n)$. (Introduce it in a sentence.)
- Figure out what the expressions for $P(a), P(k), P(k+1)$ look like. (Write them down.)

Build a proof of Statement *S* using the following structure:

Proof of Statement *S*:

The Inductive Hypothesis

Basis Step: Prove that $P(a)$ is true.

A bunch of steps may be involved. Usually a computation.

Inductive Step: Prove that for all integers $k \geq a$ if $P(k)$ is true, then $P(k+1)$ is true.

Proof for Inductive Step (Direct Proof)

P(k)

(1) Suppose that k is an integer such that $k \geq a$ and that $\cancel{k \geq}$ a is true.

generic particular element

*

* a bunch of steps will be involved

*

(**) $P(k+1)$ is true. (some justification goes here.)

End of Proof for the Inductive Step

Conclusion: Therefore, for all integers $n \geq a$, $P(n)$ is true. (by the ***Principle of Induction***)

End of Proof of Statement *S*

[Example 9] Use the Method of Induction to prove the formula for the sum of the first n positive integers. (This is presented as Theorem 5.2.1 on p.280 of the book.)

$$\forall n \in \mathbf{Z}, n \geq 1 \left(1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}\right)$$

Preliminary work:

a is the number 1

P(n) is the predicate $1+2+3+\cdots+n = \frac{n(n+1)}{2}$

P(k) is this equation $1+2+3+\cdots+k = \frac{k(k+1)}{2}$

P(k+1) is this equation $1+2+3+\cdots+(k+1) = \frac{(k+1)((k+1)+1)}{2}$

P(a) is P(1) which is

$$1+2+3+\cdots+1 = \frac{1(1+1)}{2}$$

$$1 = \frac{1(1+1)}{2}$$

Proof (proof that $\forall n \geq 1\,(P(n)$ is true$)$)

By method of induction

Basis Step Prove that P(a) is true

P(a) is P(1) which is the equation

$$1 = \frac{1(1+1)}{2}$$

$$1 = \frac{1(2)}{2}$$

$$1 = 1$$ true ✓

Inductive Step Prove that $\forall k \geq a$ (If $P(k)$ then $P(k+1)$)

Proof (Direct Proof)

(1) Suppose that $k \geq 1$ and $P(k)$ is true (generic particular element)

Inductive hypothesis is

(2) $1+2+3+\cdots+k = \frac{k(k+1)}{2}$ by (2)

(4) Left side of $P(k+1) = 1+2+3+\cdots+(k+1)$

$= 1+2+3+\cdots+k+(k+1)$

using the inductive hypothesis $P(k)$

$= \frac{k(k+1)}{2} + (k+1)$

$= \frac{k^2+k}{2} + \frac{2k+2}{2}$

$$= \frac{k^2+k+2k+2}{2}$$

$$= \frac{k^2+3k+2}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

= right side of $P(k+1)$

(**) $\overbrace{1+2+3+\cdots+(k+1)}^{\text{left side of } P(k+1)} = \overbrace{\frac{(k+1)((k+1)+1)}{2}}^{\text{right side of } P(k+1)}$

(*) Therefore $P(k+1)$ is true

End of proof of Inductive step.

Conclusion

Therefore, for all $n \geq 1$ $\left(1+2+3+\cdots+n = \frac{n(n+1)}{2}\right)$

By principle of induction

End of Proof

In the book on page 283, you can see a proof, using the Principle of Induction, of the formula for the sum of a geometric sequence. (Theorem 5.2.2)

If $r \in \boldsymbol{R}, r \neq 0{,}1$ then

$$\forall n \in \boldsymbol{Z}, n \geq 0 \left(1 + r + r^2 + r^3 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1} \right)$$

[Example 10] Use the Method of Induction to prove the formula for the sum of the first n positive perfect squares.

$$\forall n \in \mathbf{Z}, n \geq 1 \left(1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}\right)$$

Preliminary work

$a = 1$

$P(n)$ is $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

$P(a)$ is $P(1)$, which is

$$1^2 + 2^2 + 3^2 + \cdots + 1^2 = \frac{1(1+1)(2(1)+1)}{6}$$

$$1 = \frac{1(1+1)(2(1)+1)}{6}$$

$P(k)$ is $1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$

$P(k+1)$ is $1^2 + 2^2 + 3^2 + \cdots + (k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$

$$1^2 + 2^2 + 3^2 + \cdots + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

Build Proof

Basis step Prove that P(a) is true

P(1) is the equation

$$1 = \frac{1(1+1)(2(1)+1)}{6}$$

$$1 = \frac{1(2)(3)}{6}$$

$$1 = \frac{6}{6}$$

$$1 = 1$$

True

Inductive Step must prove that $\forall k \geq a$ (If $P(k)$ then $P(k+1)$)

Proof (Direct Proof)

(1) Suppose $k \geq 1$ and that $P(k)$ is true. (generic particular element)

Inductive hypothes

(2) $1^2 + 2^2 + 3^2 + \cdots + k^2 = \dfrac{k(k+1)(2k+1)}{6}$ (by (1))

(3) Left side of $P(k+1$ $= 1^2 + 2^2 + 3^2 + \cdots + (k+1)^2$

$$= \underbrace{1^2 + 2^2 + 3^2 + \cdots + k^2}_{\text{The left side of } P(k)} + (k+1)^2$$

use fact that $P(k)$ is true

$$= \overbrace{\frac{k(k+1)(2k+1)}{6}}^{\text{The right side of } P(k)} + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$$

$$= \frac{(k+1)(2k^2+k) + 6(k+1)(k+1)}{6}$$

$$= \frac{(k+1)(2k^2+k) + (6k+6)(k+1)}{6}$$

$$= \frac{(k+1)\left((2k^2+k) + (6k+6)\right)}{6}$$

$$= \frac{(k+1)(2k^2+7k+6)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

= right side of $P(k+1)$

$$(**) \quad \overbrace{1^2+2^2+3^2+\cdots+(k+1)^2}^{\text{Left side of } P(k+1)} = \overbrace{\frac{(k+1)(k+2)(2k+3)}{6}}^{\text{Right side of } P(k+1)}$$

(*) therefore $P(k+1)$ is true

End of proof. (of the induction step)

Conclusion

Therefore for all $n \geq 1$, $1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$

by the principle of induction

End of Proof