Video for Homework H08.1

Reading: Section 8.1 Relations on Sets

Homework: H08.1: 8.1# 4,6,7,9,11,17,20

Topics:

- **Definition of Relation on a Set**
- **Illustrating Relations on Finite Sets**
  - **Using Tables**
  - **Using Arrow Diagrams**
  - **Using Diricted Graphs** Directed Graphs
- **Inverse Relations**
- **Unions and Intersections of Relations**

Ordered Pairs, definition from Chapter 1

Notation

Given elements a and b, the symbol (a, b) denotes the **ordered pair** consisting of a and b together with the specification that a is the first element of the pair and b is the second element. Two ordered pairs (a, b) and (c, d) are equal if, and only if, $a = c$ and $b = d$. Symbolically:

$$(a, b) = (c, d) \quad \text{means that} \quad a = c \text{ and } b = d.$$

The *Cartesian Product* of Sets, definition from Chapter 1

Definition

Given sets $A_1, A_2, \ldots, A_n$, the **Cartesian product** of $A_1, A_2, \ldots, A_n$, denoted $\mathbf{A_1 \times A_2 \times \cdots \times A_n}$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1$, $a_2 \in A_2, \ldots, a_n \in A_n$.

Symbolically:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

In particular,

$$A_1 \times A_2 = \{(a_1, a_2) \mid a_1 \in A_1 \text{ and } a_2 \in A_2\}$$

is the Cartesian product of A_1 and A_2.

Definition of *Relation*, from Section 1.3

Definition

Let A and B be sets. A **relation R from A to B** is a subset of $A \times B$. Given an ordered pair (x, y) in $A \times B$, **x is related to y by R**, written $x\,R\,y$, if, and only if, (x, y) is in R. The set A is called the **domain** of R and the set B is called its **co-domain**.

The notation for a relation R may be written symbolically as follows:

$$\boldsymbol{x\,R\,y} \quad \text{means that} \quad \boldsymbol{(x, y) \in R}.$$

The notation $x \not R\, y$ means that x is not related to y by R:

$$\boldsymbol{x \not R\, y} \quad \text{means that} \quad \boldsymbol{(x, y) \notin R}.$$

Definition of *Relation on a Set*, Section 8.1

Definition

A **relation on a set** A is a relation from A to A.

[Example 1] (8.1#15) Let $A = \{2,3,4,5,6,7,8\}$

Define a relation R on A by saying that xRy means x and y have a common prime factor

(a) Is $2R6$? yes. 2 & 6 have a common prime factor of 2

Is $6R2$? yes. 6&2 have a common prime factor of 2.

Is $5R5$? yes! 5&5 both have a prime factor of 5.

Is $2R5$? No. 2&5 do not have any common prime factors.

Describe R explicitly by listing its elements in *set roster notation*.

$\{(2,2),(2,4),(2,6),(2,8),(3,3),(4,2),(4,4),(4,6),(4,8),(5,5),$
$(6,2),(6,4),(6,6),(6,8),(7,7),(3,6),(6,3),$
$(8,2),(8,4),(8,6),(8,8)\}$

Illustrating Relations on Finite Sets Using Tables

(b) Illustrate R from **[Example 1]** using a *table*.

(a_1, a_2)	2	3	4	5	6	7	8
2	X		X		X		X
3		X			X		
4	X		X		X		X
5				X			
6	X	X	X		X		X
7						X	
8	X		X		X		X

Illustrating Relations on Finite Sets Using Arrow Diagrams

Arrow Diagram of a Relation

Suppose R is a relation from a set A to a set B. The **arrow diagram for R** is obtained as follows:

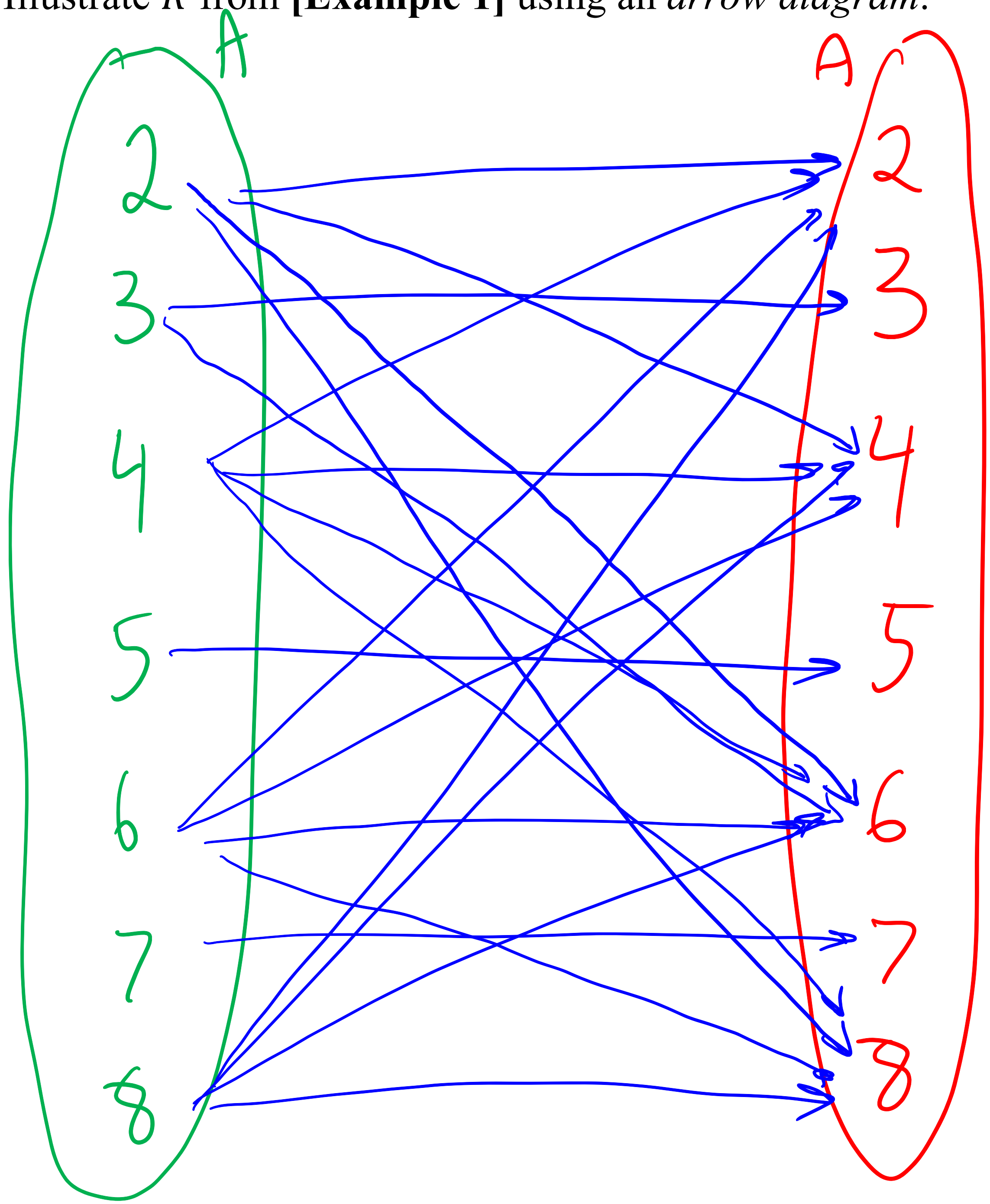
1. Represent the elements of A as points in one region and the elements of B as points in another region.
2. For each x in A and y in B, draw an arrow from x to y if, and only if, x is related to y by R. Symbolically:

Draw an arrow from x to y

if, and only if, $\quad x \, R \, y$

if, and only if, $\quad (x, y) \in R$.

(c) Illustrate *R* from **[Example 1]** using an *arrow diagram*.



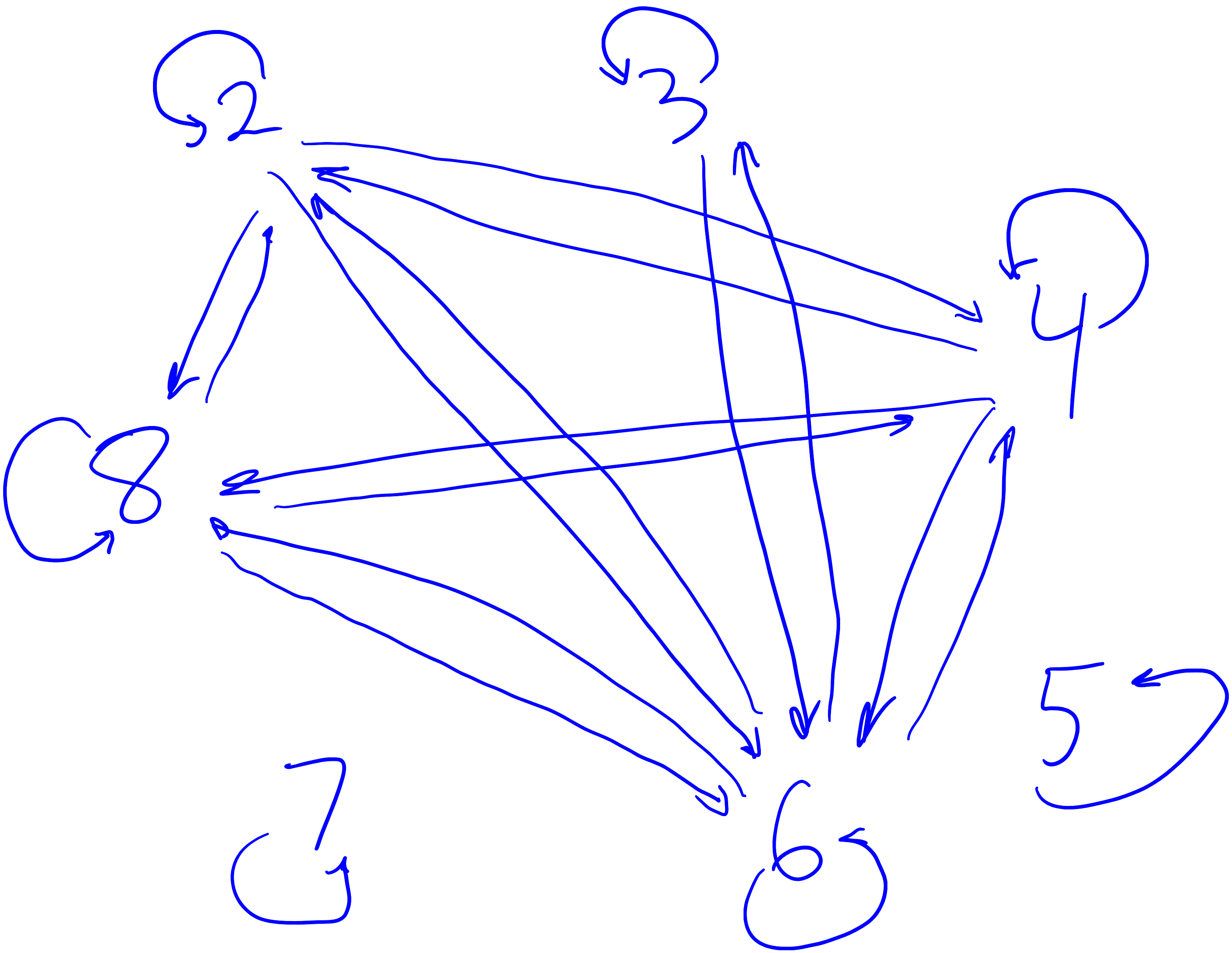
Illustrating Relations on Finite Sets Using Directed Graphs

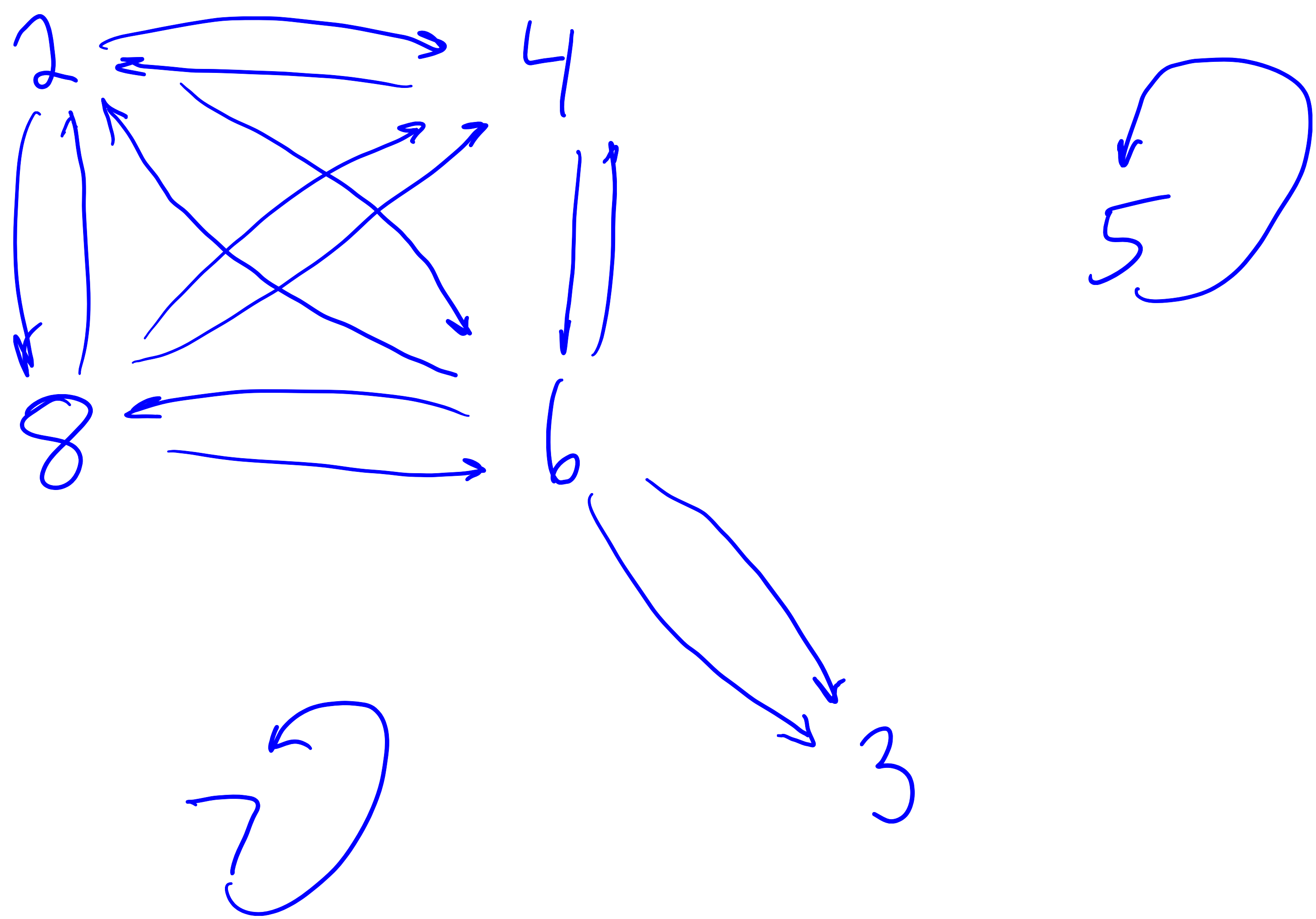
When a relation R is defined *on* a set A, the arrow diagram of the relation can be modified so that it becomes a **directed graph**. Instead of representing A as two separate sets of points, represent A only once, and draw an arrow from each point of A to each related point. As with an ordinary arrow diagram,

> For all points x and y in A,
>
> $$\text{there is an arrow from } x \text{ to } y \quad \Leftrightarrow \quad x\,R\,y \quad \Leftrightarrow \quad (x, y) \in R.$$

If a point is related to itself, a loop is drawn that extends out from the point and goes back to it.

(d) Illustrate *R* from **[Example 1]** using a *directed graph*.




2
4
5
8
6
3
7

[Example 2] (8.1#18) Let $A = 0,1,3,4,5,6$

Define a relation V on A by saying that xVy means $5|(x^2 - y^2)$

(a) Describe V explicitly by listing its elements in *set roster notation*.

$0^2 - 0^2 = 0$, and $0 = 5 \cdot 0$ so $5|(0^2 - 0^2)$ $(0,0) \in V$

Similarly $(1,1), (3,3), (4,4), (5,5), (6,6) \in V$

$5|(5^2 - 0^2)$ and $5|(0^2 - 5^2)$ So $(5,0)$ and $(0,5) \in V$

$5|(4^2 - 1^2)$ and $5|(1^2 - 4^2)$ $(4,1)$ and $(1,4) \in V$

3 is not related to anything

$5|(6^2 - 4^2)$ and $5|(4^2 - 6^2)$ So $(6,4)$ and $(4,6) \in V$

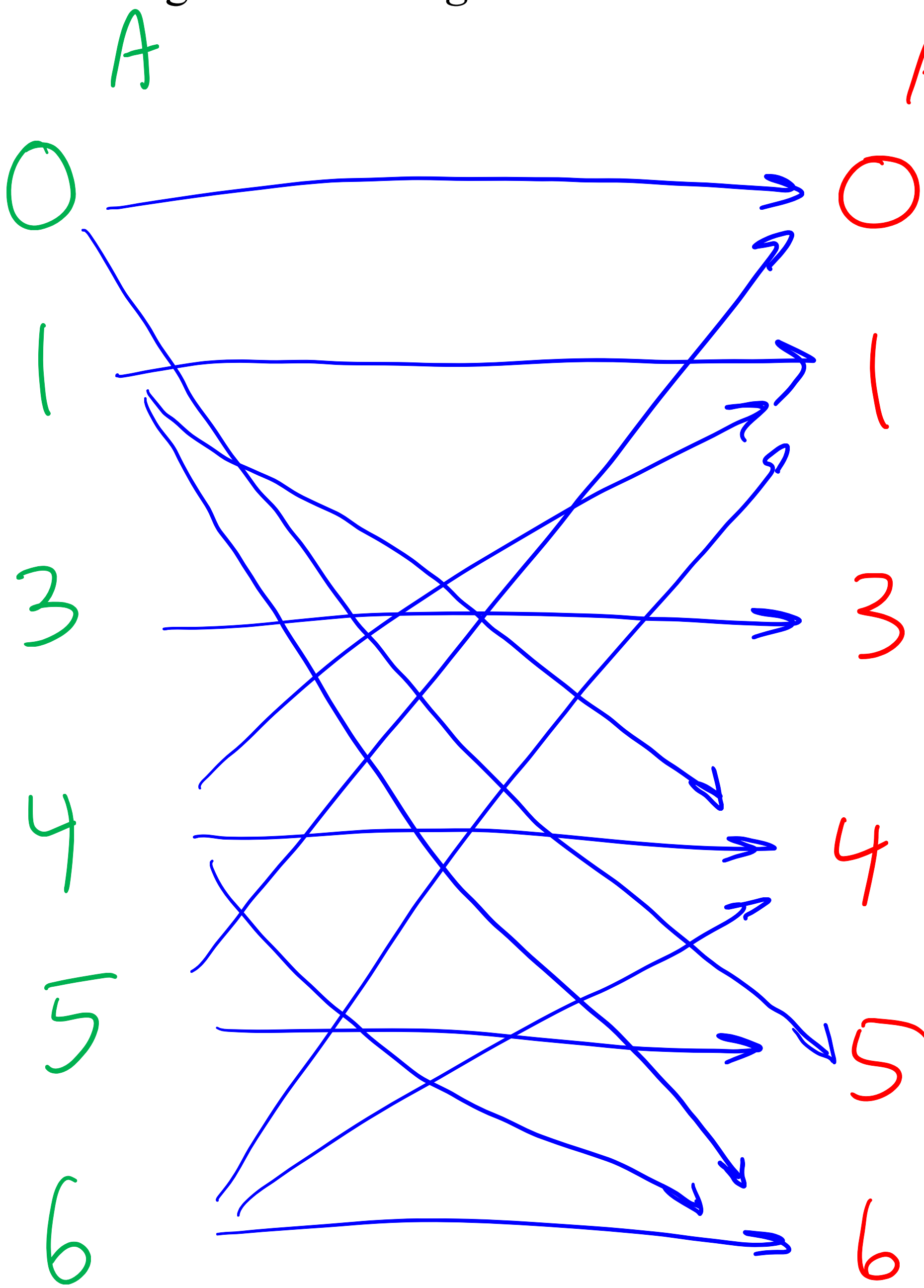
$5|(6^2 - 1^2)$ and $5|(1^2 - 6^2)$ So $(6,1)$ and $(1,6) \in V$

$V = \{(0,0), (0,5), (1,1), (1,4), (1,6), (3,3), (4,1), (4,4), (4,6), (5,0), (5,5), (6,1), (6,4), (6,6)\}$

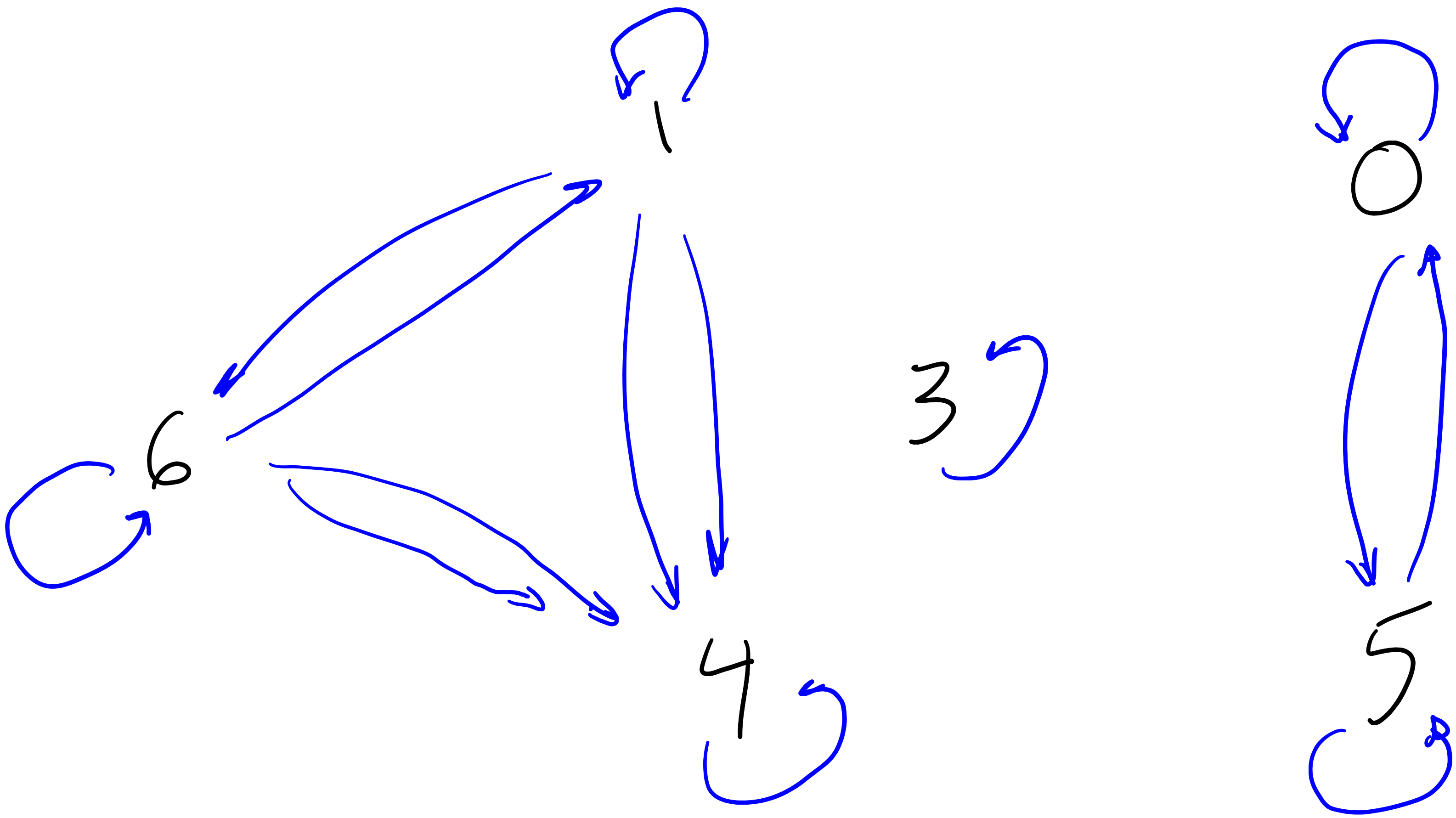
(b) Illustrate V using a *table*.

				A			
	(a_1, a_2)	0	1	3	4	5	6
	0	X				X	
	1		X		X		X
A	3			X			
	4		X		X		X
	5	X				X	
	6		X		X		X

(c) Illustrate V using an *arrow diagram*.



(d) Illustrate *V* using a *directed graph*.



Relations on sets that are described abstractly, rather than listed explicitly

[Example 3] Let $X = \{a, b, c\}$.

Recall that $\mathcal{P}(X)$ denotes the *power set of* X, which is the set of all subsets of X.

Define relation $\boldsymbol{S}$ on $\mathcal{P}(X)$ by saying that

$A\boldsymbol{S}B$ means that *set A has the same number of elements as set B*

(a) Is $\{c\}\boldsymbol{S}\{b\}$? yes because each set has 1 element.

(b) Is $\{c\}\boldsymbol{S}\{b, c\}$? no! The set $\{c\}$ has 1 element. the set $\{b,c\}$ has 2.

(c) Is $X\boldsymbol{S}X$? yes. X has same number of elements as itself.

(d) Is $\phi\boldsymbol{S}\phi$? true because $0=0$.

ϕ is the empty set. Number of elements in ϕ is 0.

Inverse Relations

Definition

Let R be a relation from A to B. Define the inverse relation R^{-1} from B to A as follows:

$$R^{-1} = \{(y, x) \in B \times A \mid (x, y) \in R\}.$$

$R^{-1} \subseteq B \times A$

This definition can be written operationally as follows:

For all $x \in A$ and $y \in B$, $\quad (y, x) \in R^{-1} \quad \Leftrightarrow \quad (x, y) \in R.$

[Example 4] (8.1#10) Let $A = \{3,4,5\}$ and $B = \{4,5,6\}$

Let R be the "less than relation" from A to B. That is, xRy means $x < y$

(a) State explicitly which ordered pairs are in R and R^{-1} using *set roster notation*.

$A = (3,4,5)$ $B = (4,5,6)$

$R = \{(3,4),(3,5),(3,6),(4,5),(4,6),(5,6)\}$

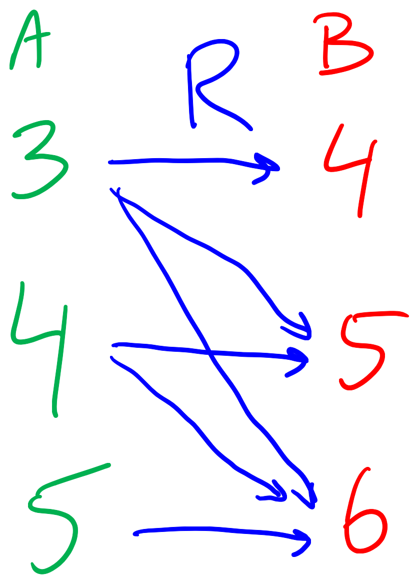
$R^{-1} = \{(4,3),(5,3),(6,3),(5,4),(6,4),(6,5)\}$

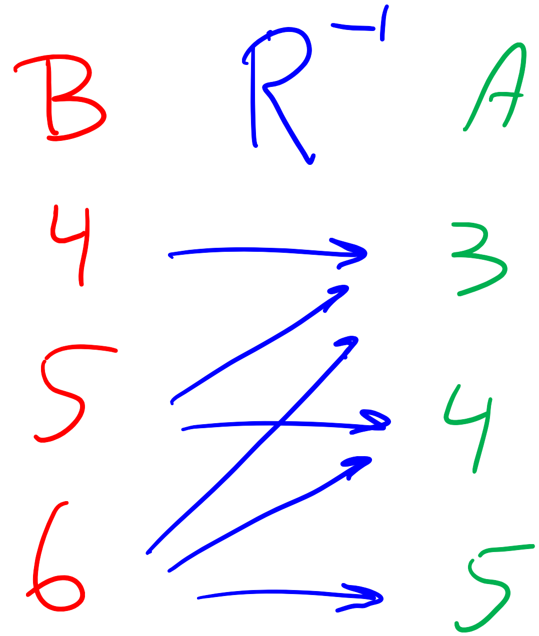
(b) Illustrate R and R^{-1} using *tables*.

xRy	4	5	6
3	X	X	X
4		X	X
5			X

$yR^{-1}x$	3	4	5
4	X		
5	X	X	
6	X	X	X

(c) Illustrate R and R^{-1} using *arrow diagrams*.





Unions and Intersections of Relations

[Example 5] (8.1#19) Let $A = \{2,4\}$ and $B = \{6,8,10\}$

Define relations R and S from A to B as follows.

Define relation R by saying that xRy means $x|y$

Define relation S by saying that xRy means $y - 4 = x$

State Explicitly what ordered pairs are in $A \times B,\ R,\ S,\ R \cup S,\ and\ \ R \cap S$

$A \times B = \{(2,6),(2,8),(2,10),(4,6),(4,8),(4,10)\}$

$R = \{(2,6),(2,8),(2,10),(4,8)\}$

$S = \{(2,6),(4,8)\}$

$R \cup S = R = \{(2,6),(2,8),(2,10),(4,8)\}$

$R \cap S = S = \{(2,6),(4,8)\}$