Video for Homework H08.2

Reading: Section 8.2 Reflexivity, Symmetry, Transitivity

Homework: H08.2: 8.2#2,4,17,26,35

Topics:

- **Definition of *Reflexive*, *Symmetric*, and *Transitive* Relations**
- **Lots of examples of verifying those properties for given relations**
- **Proving or disproving abstract statements about those properties**

Ordered Pairs, definition from Chapter 1

Notation

Given elements a and b, the symbol (a, b) denotes the **ordered pair** consisting of a and b together with the specification that a is the first element of the pair and b is the second element. Two ordered pairs (a, b) and (c, d) are equal if, and only if, $a = c$ and $b = d$. Symbolically:

$$(a, b) = (c, d) \quad \text{means that} \quad a = c \text{ and } b = d.$$

The *Cartesian Product* of Sets, definition from Chapter 1

Definition

Given sets $A_1, A_2, \ldots, A_n$, the **Cartesian product** of $A_1, A_2, \ldots, A_n$, denoted $\mathbf{A_1 \times A_2 \times \cdots \times A_n}$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1$, $a_2 \in A_2, \ldots, a_n \in A_n$.

Symbolically:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

In particular,

$$A_1 \times A_2 = \{(a_1, a_2) \mid a_1 \in A_1 \text{ and } a_2 \in A_2\}$$

is the Cartesian product of A_1 and A_2.

Definition of *Relation*, from Section 1.3

Definition

Let A and B be sets. A **relation R from A to B** is a subset of $A \times B$. Given an ordered pair (x, y) in $A \times B$, **x is related to y by R**, written $x \, R \, y$, if, and only if, (x, y) is in R. The set A is called the **domain** of R and the set B is called its **co-domain**.

The notation for a relation R may be written symbolically as follows:

$$\boldsymbol{x \, R \, y} \quad \text{means that} \quad \boldsymbol{(x, y) \in R.}$$

The notation $x \not{R} \, y$ means that x is not related to y by R:

$$\boldsymbol{x \not{R} \, y} \quad \text{means that} \quad \boldsymbol{(x, y) \notin R.}$$

Definition of *Relation on a Set*, Section 8.1

Definition

A **relation on a set** A is a relation from A to A.

a relation R from A to A
is a subset, $R \subseteq A \times A$

Inverse Relations

Definition

Let R be a relation from A to B. Define the inverse relation R^{-1} from B to A as follows:

$$R^{-1} = \{(y, x) \in B \times A \mid (x, y) \in R\}.$$

This definition can be written operationally as follows:

$$\text{For all } x \in A \text{ and } y \in B, \quad (y, x) \in R^{-1} \quad \Leftrightarrow \quad (x, y) \in R.$$

In Section 8.2, we will study three properties that relations may or may not have

Definition of a *Reflexive Relation*

Words: *Relation R on set A is **reflexive***

Meaning: Every element of A is related to itself.

Meaning Written Formally:

using *xRy* notation: $\forall a \in A(aRa)$

using ordered pair notation: $\forall a \in A\big((a,a) \in R\big)$

Words: *R is **not reflexive***

Meaning: There exists an element of A that is *not* related to itself.

Meaning Written Formally:

using *xRy* notation: $\exists a \in A(\not{R}a)$ $(a \not{R} a)$

using ordered pair notation: $\exists a \in A\big((a,a) \notin R\big)$

Definition of a *Symmetric Relation*

Words: *Relation R on set A is* ***symmetric***

Meaning: If one element of A is related to any second element of A, then the second element is also related to the first.

Meaning Written Formally:

using *xRy* notation: $\forall a, b \in A(If\ \ aRb\ \ then\ bRa)$

using ordered pair notation: $\forall a \in A(If\ \ (a,b) \in R\ \ then\ \ (b,a) \in R)$

$\forall a, b \in A$

Words: *R is* ***not symmetric***

Meaning: There exist two elements of A such that the first is related to the second but the second is *not* related to the first.

Meaning Written Formally:

using *xRy* notation: $\exists a, b \in A(aRb\ \ \ and\ \ b\not{R}a)$ $(b,a) \notin R$

using ordered pair notation: $\exists a, b \in A\big((a,b) \in R\ \ and\ \ (a,a) \notin R\big)$

Definition of a *Transitive Relation*

Words: *Relation R on set A is **transitive***

Meaning: If one element of A is related to any second element and that second element is related to any third element, then the first element is also related to the third.

Meaning Written Formally:

using *xRy* notation: $\forall a,b,c \in A(If\ (aRb\ \ and\ bRc)\ \ then\ \ aRc)$

using ordered pair notation:

$$\forall a,b,c \in A\Big(If\ \ \big((a,b) \in R\ and\ (b,c) \in R\big)\ \ then\ \ (a,c) \in R\Big)$$

Words: *R is **not transitive***

Meaning: There exist three elements of A such that the first is related to the second and the second is related to the third, but the first is *not* related to the third.

Meaning Written Formally:

$a,b,c \in A$

using *xRy* notation: $\exists a,b \in A((aRb\ \ and\ bRc)\ \ and\ \ a \not R c)$

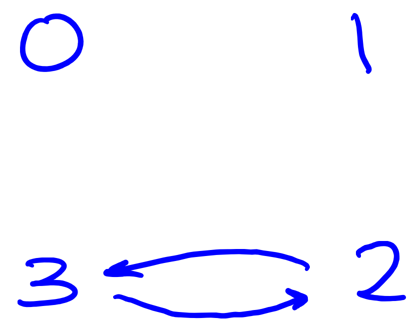
using ordered pair notation:

$$\exists a,b,c \in A\Big(\big((a,b) \in R\ \ and\ (b,c) \in R\big)\ \ and\ \ (a,c) \notin R\Big)$$

[Example 1] $A = \{0,1,2,3\}$ Define relation R on A by $R = \{(2,3),(3,2)\}$

(a) Illustrate R using a table and a directed graph.

	0	1	2	3
0				
1				
2				X
3			X	



(b) Is R *reflexive*? If not, give a counterexample.

No 1 is not related to itself

(c) Is R *symmetric* ? If not, give a counterexample.

yes Symmetric

(d) Is R *transitive*? If not, give a counterexample.

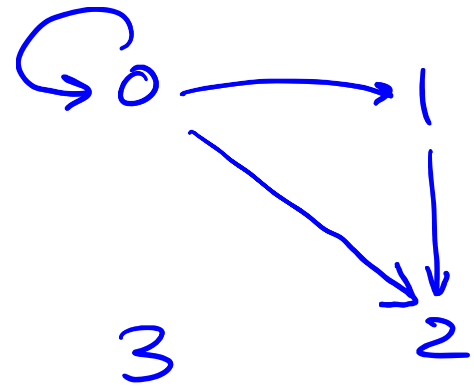
Observe $2R3$ and $3R2$ but $2 \not R 2$

not transitive

[Example 2] $A = \{0,1,2,3\}$ Define relation R on A by $R = \{(0,0), (0,1), (0,2), (1,2)\}$

(a) Illustrate R using a table and a directed graph.

	0	1	2	3
0	X	X	X	
1			X	
2				
3				



(b) Is R *reflexive*? If not, give a counterexample.

No   1 is not related to itself

(c) Is R *symmetric* ? If not, give a counterexample.

No $0 R 1$ but $1 \not R 0$

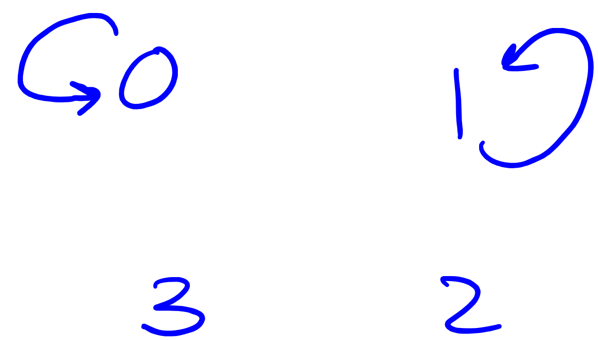
(d) Is R *transitive*? If not, give a counterexample.

yes transitive

[Example 3] $A = \{0,1,2,3\}$ Define relation R on A by $R = \{(0,0), (1,1)\}$

(a) Illustrate R using a table and a directed graph.

	0	1	2	3
0	X			
1		X		
2				
3				



(b) Is R *reflexive*? If not, give a counterexample.

no 2 is not related to itself

(c) Is R *symmetric* ? If not, give a counterexample.

yes table is symmetric across the main diagonal

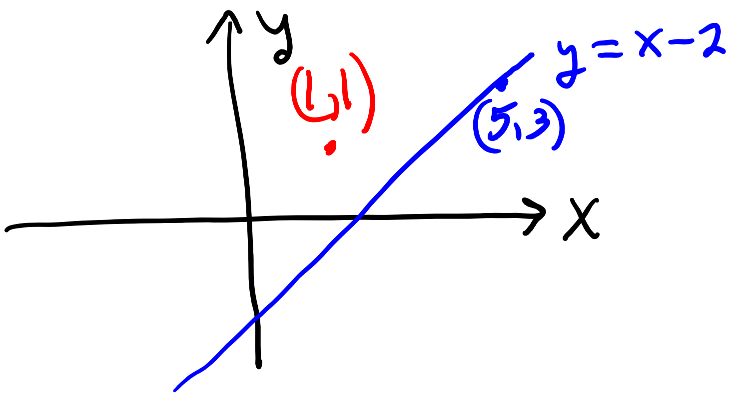
(d) Is R *transitive*? If not, give a counterexample.

yes R is transitive. There are no examples that show it failing to be transitive

[Example 4] Define relation R on $\boldsymbol{R}$ by saying xRy means $x - y = 2$

$y = x - 2$

(a) Illustrate R using a graph.



$3 = 5 - 2$

(b) Is R *reflexive*? If not, give a counterexample.

No 1 is not related to 1

$1 = 1 - 2$ is false

(c) Is R *symmetric* ? If not, give a counterexample.

(5,3) is on the graph but (3,5) is not

So R is <u>not</u> symmetric

(d) Is R *transitive*? If not, give a counterexample.

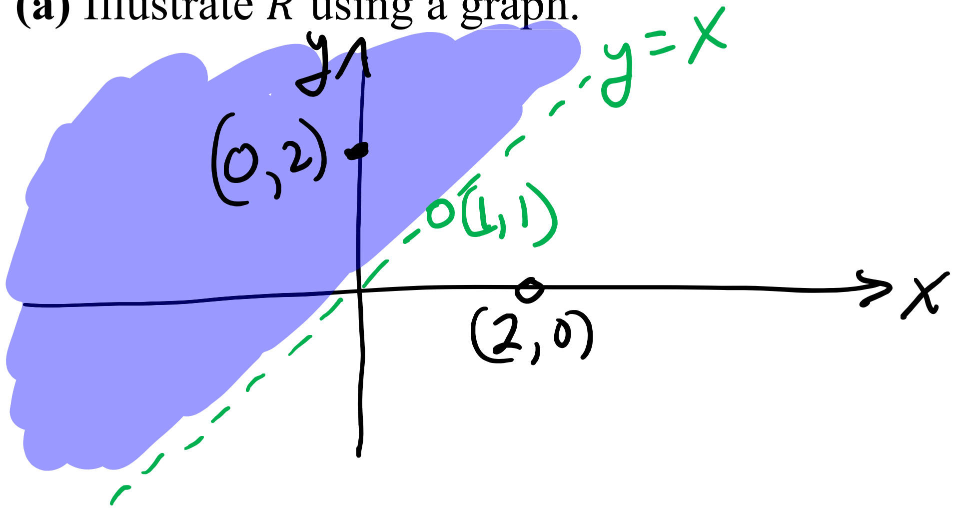
No $(7,5) \in R$ and $(5,3) \in R$ but $(7,3) \notin R$

$5 = 7 - 2$ $\quad$ $3 = 5 - 2$ $\quad$ $3 \neq 7 - 2$

$7R5$ and $5R3$ but $7 \not R 3$

[Example 5] Define relation R on $\boldsymbol{R}$ by saying xRy means $x < y$

(a) Illustrate R using a graph.



(b) Is R *reflexive*? If not, give a counterexample.

no 1 is not related to itself $1<1$ is false

(c) Is R *symmetric* ? If not, give a counterexample.

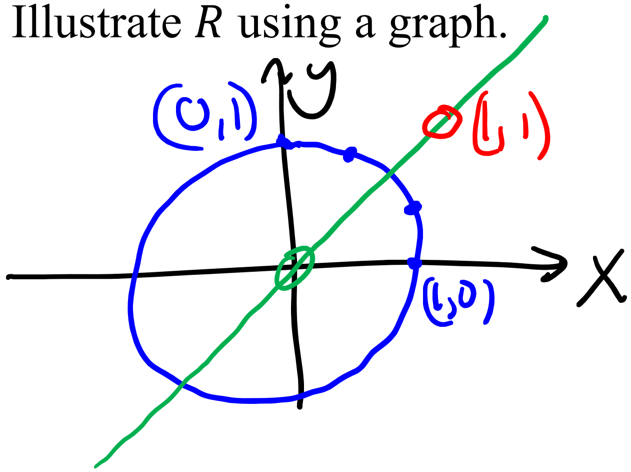
no $(0,2) \in R$ but $(2,0) \notin R$ $0<2$ is true $2<0$ is false

(d) Is R *transitive*? If not, give a counterexample.

yes if $a<b$ and $b<c$ then $b<c$

[Example 6] Define relation R on $\boldsymbol{R}$ by saying xRy means $x^2 + y^2 = 1$

(a) Illustrate R using a graph.



(b) Is R *reflexive*? If not, give a counterexample.

no 0 is not related to itself (0,0) is not on the graph

(c) Is R *symmetric* ? If not, give a counterexample.

yes the graph is symmetric across the main diagonal

If $x^2+y^2=1$ then $y^2+x^2=1$ If xRy then yRx

(d) Is R *transitive*? If not, give a counterexample.

no $(1,0) \in R$ and $(0,1) \in R$ but $(1,1) \notin R$

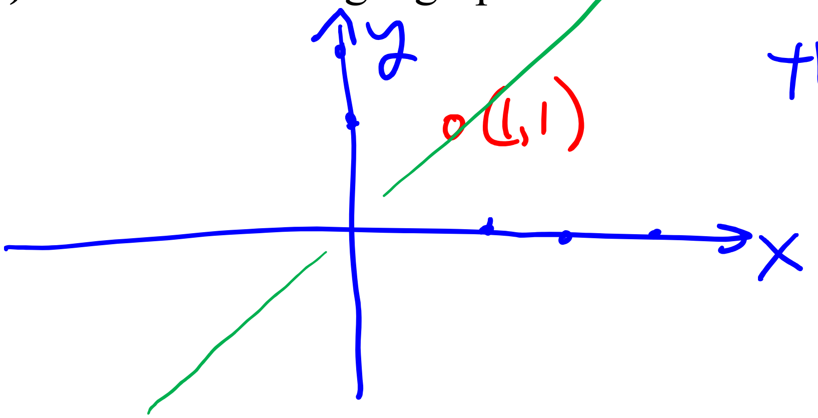
$1R0$ and $0R1$ but $1 \not R 1$

[Example 7] Define relation R on $\boldsymbol{R}$ by saying xRy means $xy = 0$

this only happens when $x=0$ or $y=0$

(a) Illustrate R using a graph.

the coordinate axes are the graph



(b) Is R *reflexive*? If not, give a counterexample.

no $(1,1)$ is not on the graph $1R1$ $1\cdot 1 \neq 0$

(c) Is R *symmetric* ? If not, give a counterexample.

yes Symmetric If $xy=0$ then $yx=0$

If xRy then yRx

(d) Is R *transitive*? If not, give a counterexample.

no $(1,0)\in R$ and $(0,1)\in R$ but $(1,1)\notin R$

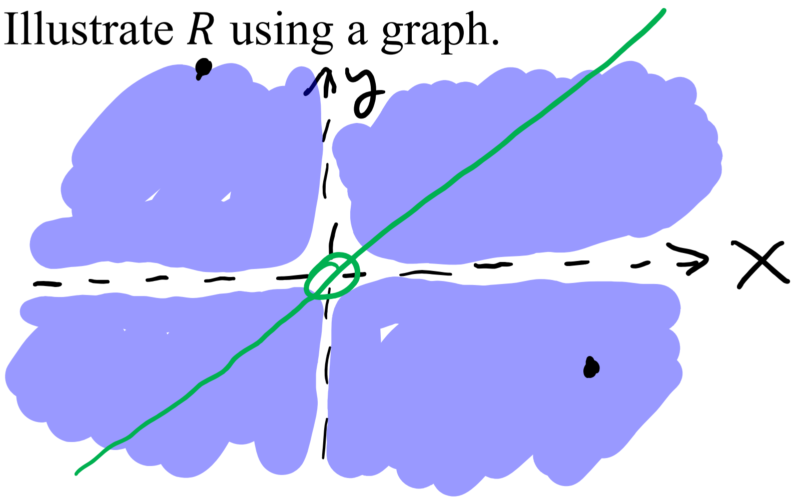
$1\cdot 0=0$ and $0\cdot 1=0$ but $1\cdot 1\neq 0$

$1R0$ and $0R1$ but $1 \cancel{R} 1$

[Example 8] Define relation R on $\boldsymbol{R}$ by saying xRy means $xy \neq 0$

neither x nor y is zero

(a) Illustrate R using a graph.



(b) Is R *reflexive*? If not, give a counterexample.

No! $(0,0) \notin R$ 0 is not related to itself. $0 \cdot 0 \neq 0$ is false!!

(c) Is R *symmetric* ? If not, give a counterexample.

yes. if $xy \neq 0$ then $yx \neq 0$ so if xRy then yRx

(d) Is R *transitive*? If not, give a counterexample.

yes If $xy \neq 0$ and $yz \neq 0$ then $xz \neq 0$

because both are non zero

If xRy (neither are zero) and yRz (neither are zero) then xRz.

[Example 9] Prove or disprove: If a relation R on a set A is reflexive, then R^{-1} is reflexive.

True

(1) Suppose R is a relation on set A and R is Reflexive

(generic particular element)

(2) Suppose $a \in A$

(3) Then aRa is true (by (1) and definition of reflexive)

(4) Therefore $aR^{-1}a$ is true (by (3) and def of R^{-1})

(5) for all $a \in A$, $aR^{-1}a$ is true (by (2) and (4))

(6) Therefore R^{-1} is Reflexive (by (5) and def of reflexive)

End of Proof.

[Example 10] Prove or disprove: If a relation R on a set A is transiive, then R^{-1} is transitive.

Proof

(1) Suppose that relation R on set A is transitive (generic particular element)

(2) If cRb and bRa then cRa (by (1) and def of transitive)

(3) If bRa and cRb then cRa commutativity of AND

(4) If $aR^{-1}b$ and $bR^{-1}c$ then $aR^{-1}c$ (by (3) and definition of R^{-1})

(5) Therefore R^{-1} is transitive (by (4) and definition of transitive

End of Proof