Video for Homework H08.3 Equivalence Relations

Reading: Section 8.3 Equivalence Relations

Homework: H08.3: 8.3#4,6,9,10,14,15,30

Topics:

- **Definition of *Equivalence Relations* and *Equivalence Classes***
- **The *Relation Induced by a Partition***
- **Examples of Equivalence Relations**
- **The Partition Induced by an Equivalence Relation**
- **Congruence Modulo *d***

Recall Disjoint Sets, Mutually Disjoint Sets, Partitions of Sets from Section 6.1

Definition

Two sets are called **disjoint** if, and only if, they have no elements in common.
Symbolically:

$$A \text{ and } B \text{ are disjoint} \quad \Leftrightarrow \quad A \cap B = \emptyset.$$

Definition

Sets $A_1, A_2, A_3, \ldots$ are **mutually disjoint** (or **pairwise disjoint** or **nonoverlapping**) if, and only if, no two sets A_i and A_j with distinct subscripts have any elements in common. More precisely, for all integers i and $j = 1, 2, 3, \ldots$

$$A_i \cap A_j = \emptyset \quad \text{whenever } i \neq j.$$

Definition

A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3, \ldots\}$ is a **partition** of a set A if, and only if,

1. A is the union of all the A_i;
2. the sets $A_1, A_2, A_3, \ldots$ are mutually disjoint.

In Section 8.2, we learned about three properties that relations may or may not have

Definition of a *Reflexive Relation*

Words: *Relation R on set A is **reflexive***

Meaning: Every element of A is related to itself.

Meaning Written Formally:

using *xRy* notation: $\forall a \in A(aRa)$

using ordered pair notation: $\forall a \in A\big((a, a) \in R\big)$

Words: *R is **not reflexive***

Meaning: There exists an element of A that is *not* related to itself.

Meaning Written Formally:

using *xRy* notation: $\exists a \in A(a \not R a)$

using ordered pair notation: $\exists a \in A\big((a, a) \notin R\big)$

Definition of a *Symmetric Relation*

Words: *Relation R on set A is* ***symmetric***

Meaning: If one element of A is related to any second element of A, then the second element is also related to the first.

Meaning Written Formally:

using $\boldsymbol{xRy}$ notation: $\forall a, b \in A(If\ \ aRb\ \ then\ bRa)$

using ordered pair notation: $\forall a, b \in A(If\ \ (a, b) \in R\ \ then\ \ (b, a) \in R)$

Words: *R is* ***not symmetric***

Meaning: There exist two elements of A such that the first is related to the second but the second is *not* related to the first.

Meaning Written Formally:

using $\boldsymbol{xRy}$ notation: $\exists a, b \in A(aRb\ \ \ and\ \ b\not{R}a)$

using ordered pair notation: $\exists a, b \in A\big((a, b) \in R\ \ and\ \ (b, a) \notin R\big)$

Definition of a *Transitive Relation*

Words: *Relation R on set A is* ***transitive***

Meaning: If one element of A is related to any second element and that second element is related to any third element, then the first element is also related to the third.

Meaning Written Formally:

using xRy notation: $\forall a,b,c \in A(If\ \ (aRb\ \ and\ bRc)\ \ then\ \ aRc)$

using ordered pair notation:

$$\forall a,b,c \in A\Big(If\ \ \big((a,b) \in R\ and\ (b,c) \in R\big)\ \ then\ \ (a,c) \in R\Big)$$

Words: *R is* ***not transitive***

Meaning: There exist three elements of A such that the first is related to the second and the second is related to the third, but the first is *not* related to the third.

Meaning Written Formally:

using xRy notation: $\exists a,b,c \in A((aRb\ \ and\ bRc)\ \ and\ \ a\not{R}c)$

using ordered pair notation:

$$\exists a,b,c \in A\Big(\big((a,b) \in R\ \ and\ (b,c) \in R\big)\ \ and\ \ (a,c) \notin R\Big)$$

In Section 8.3, we are interested in relations that have all three of those properties

Definition of *Equivalence Relation*

Words: R *is an* ***requivalence relation*** equivalence relation

Usage: R is a relation on a set A

Meaning: R is *reflexive* and *symmetric* and *transitive*.

Additional terminology and notation

Words: *the* ***equivalence class*** *of* a

Symbol: $[a]$

Usage: $a \in A$

Meaning: the set of all elements x in A such that x is related to a.

Meaning in symbols: $[a] = \{x \in A | xRa\}$

[Example 1] The ***Relation Induced by a Partition***

Definition

Given a partition of a set A, the **relation induced by the partition**, R, is defined on A as follows: For every $x, y \in A$,

$x\,R\,y \iff$ there is a subset A_i of the partition such that both x and y are in A_i.

Theorem 8.3.1

Let A be a set with a partition and let R be the relation induced by the partition. Then R is reflexive, symmetric, and transitive.

R is an equivalence relation

For example, let $A = \{0,1,2,3,4\}$ and let $A_1 = \{0,4\}$ and $A_2 = \{1,3\}$ and $A_3 = \{2\}$.

Then $\{A_1, A_2, A_3\}$ is a *partition* of set A.

The *relation induced by the partition* is the relation R on A defined by saying that

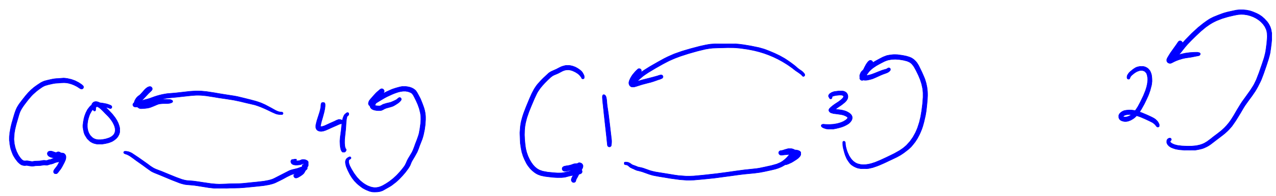
$xRy \Leftrightarrow$ there is a subset A_i of the partition such that both x and y are in A_i

Relation R is an *equivalence relation*.

(a) Write R as a set of ordered pairs.

$R = \{(0,0),(0,4),(4,0),(4,4),(1,1),(1,3),(3,1),(3,3),(2,2)\}$

(b) Illustrate R using a *directed graph*.



(c) Find the *equivalence classes* [0], [1], [2], [3], [4]

$[0] = \{0,4\}$ $[1] = \{1,3\}$ $[2] = \{2\}$ $[3] = \{1,3\}$ $[4] = \{1,4\}$

(d) What are the *distinct equivalence classes*?

. $[0] = [4] = \{0,4\}$ $[1] = [3] = \{1,3\}$ $[2] = \{2\}$

[Example 2] Let $X = \{-1,0,1\}$

(a) List the elements of $\mathcal{P}(x)$ The Power Set of X. The set of all subsets of X.

$$P(X) = \{\phi, \{1\}, \{0\}, \{0,1\}, \{-1\}, \{-1,1\}, \{-1,0\}, \{-1,0,1\}\}$$

Define relation R on ~~$\mathcal{P}(x)$~~ $P(X)$ by saying that

ARB means that the product of the elements in A equals the product of the elements in B

Then relation R is an *equivalence relation*.

Set	Product
ϕ	Product does not exist
$\{1\}$	1
$\{0\}$	0
$\{0,1\}$	$0 \cdot 1 = 0$
$\{-1\}$	-1
$\{-1,1\}$	$-1 \cdot 1 = -1$
$\{-1,0\}$	$-1 \cdot 0 = 0$
$\{-1,0,1\}$	$-1 \cdot 0 \cdot 1 = 0$

(b) Find the *distinct equivalence classes*.

$[\phi] = \{\phi\}$

$[\{1\}] = \{\{1\}\}$

$[\{0\}] = [\{0,1\}] = [\{-1,0\}] = [\{-1,0,1\}] = \{\{0\}, \{0,1\}, \{-1,0\}, \{-1,0,1\}\}$

$[\{-1\}] = [\{-1,1\}] = \{\{-1\}, \{-1,1\}\}$

[Example 3] Let $A = \{1,2,3,\dots,20\}$. Define relation R on A by

$$mRn \Leftrightarrow 4|(m-n)$$

Then relation R is an *equivalence relation*. (We will prove this in the next example.)

(a) Find the *equivalence classes* [1], [2], [3], [4], [5]

$[1] = \{m \in A \mid mR1\} = \{m \in A \mid 4 \mid (m-1)\} =$

$= \{m \in A \mid m-1 = 4k \text{ for some integer } k\}$

$= \{m \in A \mid m = 4k+1 \text{ for some integer } k\}$

(b) Find the *distinct equivalence classes*.

$[1] = \{1,5,9,13,17\}$

$[2] = \{m \in A \mid m = 4k+2 \text{ for some integer } k\} = \{2,6,10,14,18\}$

$[3] = \{m \in A \mid m = 4k+3 \text{ for some integer } k\} = \{3,7,11,15,19\}$

$[4] = \{4,8,12,16,20\}$

$[5] = \{1,5,9,13,17\}$

The distinct equivalence classes will be $[1], [2], [3], [4]$

[Example 4] Congruence Modulo 4

Define relation R on $\boldsymbol{Z}$ by

$$mRn \Leftrightarrow \cancel{d}|(m - n) \qquad 4\,|\,(m-n)$$

(This relation is called the *Congruence Modulo 4 relation*.)

(a) Prove that R is an *equivalence relation*.

(b) Describe the *equivalence class* $[a]$ for $a \in \boldsymbol{Z}$

(c) Describe the *distinct equivalence classes* of R.

Prove that R is Reflexive

$\forall n \in \mathbb{Z}\ (nRn)$

(1) Let n be an integer (generic particular element)

(2) Then $n-n = 0 = 4 \cdot 0$

(3) Let $k=0$. Notice that k is an integer

(4) $n-n = 4k$ for some integer k

(5) $4\,|\,(n-n)$ (by (4) and definition of divides)

(6) Therefore nRn (By (5) and definition of R)

End of Proof

Proof That R is Symmetric $\forall a,b \in \mathbb{Z}$ If aRb then bRa

(Direct Proof)

(1) Suppose $a, b \in \mathbb{Z}$ and aRb (generic particular element)

(2) $4 \mid (a-b)$ by (1) and definition of R

(3) $a-b=4k$ for some integer k (by (2) and def of divides)

(4) Then $b-a=-4k$ (multiplied by (-1))

$= 4\cdot(-k)$

(5) Let $j=-k$ observe that j is an integer

(6) $b-a=4j$ for some integer j

(7) $4 \mid (b-a)$ (by (6) and definition of divides)

(8) Therefore bRa (by (7) and definition of R)

End of Proof

Proof That R is Transitive $\forall a,b,c \in \mathbb{Z}$ (If aRb and bRc then aRc)

(1) Suppose that $a,b,c \in \mathbb{Z}$ and aRb and bRc (generic particular element)

(2) Then $a-b=4k$ for some integer k and $b-c=4m$ for some integer m

(3) $a=b+4k$ $b=c+4m$

(4) $a=(c+4m)+4k$ (substituted)

$= c+4(m+k)$

(5) let $n=m+k$ observe that n is an integer

(6) $a=c+4n$ for some integer n

(7) Then $a-c=4n$ for some integer n.

(8) Therefore aRc

End of Proof

Conclude that R is an equivalence relation because it is Reflexive and Symmetric and Transitive

(b) Equivalence Classes

$$[a] = \{b \in \mathbb{Z} \mid aRb\}$$
$$= \{b \in \mathbb{Z} \mid 4 \mid (a-b)\}$$
$$= \{b \in \mathbb{Z} \mid (a-b) = 4k \text{ for some integer } k\}$$
$$= \{b \in \mathbb{Z} \mid a = 4k + b \text{ for some integer } k\}$$
$$= \{b \in \mathbb{Z} \mid b = 4j + a \text{ for some integer } j\}$$

$[0] = \{\ldots, -8, -4, 0, 4, 8, 12, \ldots\} = [4] = [8] = [-4]$ etc

$[1] = \{\ldots, -7, -3, 1, 5, 9, 13, \ldots\} = [5] = [9] = [-3]$ etc

$[2] = \{\ldots, -6, -2, 2, 6, \ldots\} = [-2] = [6]$ etc

$[3] = \{\ldots, -5, -1, 3, 7, 11, \ldots\}$

Based on our observations of equivalence classes, the following definition of the representative of an equivalence class makes sense:

> **Definition**
>
> Suppose R is an equivalence relation on a set A and S is an equivalence class of R. A **representative** of the class S is any element a such that $[a] = S$.

for example, for the equivalence class

$S = \{ \ldots, -8, -4, 0, 4, 8, \ldots \}$

The number 8 is a representative because $[8] = S$

The number -4 is also a representative.

These Lemmas & Theorems from pages 513 - 514 also make sense. I won't discuss the proofs.

Lemma 8.3.2

Suppose A is a set, R is an equivalence relation on A, and a and b are elements of A. If $a \, R \, b$, then $[a] = [b]$.

Lemma 8.3.2 is used in the proof of Lemma 8.3.3

Lemma 8.3.3

If A is a set, R is an equivalence relation on A, and a and b are elements of A, then

$$\text{either} \quad [a] \cap [b] = \varnothing \quad \text{or} \quad [a] = [b].$$

Lemma 8.3.3 is used in the proof of Theorem 8.3.4.

Theorem 8.3.4 The Partition Induced by an Equivalence Relation

If A is a set and R is an equivalence relation on A, then the distinct equivalence classes of R form a partition of A; that is, the union of the equivalence classes is all of A, and the intersection of any two distinct classes is empty.

Observe that the equivalence classes of the *Congruence Modulo* 4 relation form a partition of the integers.

$$[0] \cup [1] \cup [2] \cup [3] = \mathbb{Z}$$

$$[0] \cap [1] = \emptyset$$

$$[1] \cap [2] = \emptyset$$

etc

Definition of Congruence Modulo $\boldsymbol{d}$

Definition of *Congruence Modulo* $\mathbf{d}$

Words: the ***Congruence Modulo*** $\mathbf{d}$ ***relation***

Usage: d is a positive integer

Meaning: the relation R on Z defined by

$$mRn \Leftrightarrow d|(m-n)$$

Additional terminology and notation

Words: $\boldsymbol{m}$ ***is congruent to*** $\mathbf{n}$ ***modulo*** $\boldsymbol{d}$

Symbol: $m \equiv n(\text{mod } d)$

Meaning: mRn is true. That is, $d|(m-n)$

Remarks

- The *Congruence Modulo* d relation is an *equivalence relation* on $\boldsymbol{Z}$.
- The *distinct equivalence classes* are $[0], [1], \ldots, [d-1]$

Consider the equivalence classes of the $Congruence\ Modulo$ 2 relation.

$d = 2$

equivalence classes

$[0] = \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$ even integers

$[1] = \{\ldots, -5, -3, -1, 1, 3, 5, 7, \ldots\}$ odd integers

[Example 5] Expressions involving the Congruence Modulo d symbol

Explain why each expression is true or false.

(a) $13 \equiv -2 \pmod{5}$ true because $13-(-2) = 15 = 5 \cdot 3$ so $5 \mid (13-(-2))$

integer

(b) $-17 \equiv 7 \pmod{10}$ false $(-17)-7 = -24$, which is not a multiple of 5

(c) $13 \equiv -2(\text{mod }(-5))$ There is an error in the video in problem (c).

FALSE! Can't have negative d. (Even though $13-(-2)=15$, which is a multiple of -5.)

(d) $0 \equiv 15 \pmod{5}$

True because $0-15 = -15 = 5 \cdot (-3)$ $0-15$ is a multiple of 5.

integer

(e) $15 \equiv 0 \pmod{5}$ True because $15-0$ is a multiple of 5

(f) $0 \equiv 0 \pmod{5}$ True because $0-0 = 0 = 5 \cdot 0$

$0-0$ is a multiple of 5.

integer

(g) $\cancel{0 \equiv 0 \pmod{0}}$ even though $0-0=0=0 \cdot 0$ is true

False Can't use this symbol with $d=0$