Day 10 (Wed Feb 1 Lecture)

Section 2.1 Rates of Change and Derivatives

Definition of Average Rate of Change

Words: Average rate of change of f from $x=a$ to $x=b$

Usage: a, b are real numbers with $a<b$,
and f is a function that is continuous on the interval $[a, b]$

meaning The number $m = \frac{f(b)-f(a)}{b-a}$ $\quad m = \frac{f(b)-f(a)}{b-a}$

graphical significance the number m is the slope of the secant line that passes through the points $(a, f(a))$ and $(b, f(b))$ on the graph of f.

Additional Terminology

when the variable is x or t representing time and the function f is a position function for a moving object moving in 1 dimension, so $f(t)$ is the position of the object at time t

the quantity $m = \frac{f(b)-f(a)}{b-a}$ is called the average velocity from time a to time b

Example An object moves with position function

$$f(t) = 8t - t^2 = 8t - t^2$$

Find average velocity from $t=1$ to $t=4$ and illustrate using a graph of $f(t)$.

Solution

we need to build $m = \dfrac{f(4) - f(1)}{4 - 1}$

$$f(1) = 8(1) - (1)^2 = 8 - 1 = 7$$
$$f(4) = 8(4) - (4)^2 = 32 - 16 = 16$$

$$m = \frac{f(4) - f(1)}{4 - 1} = \frac{16 - 7}{3} = \frac{9}{3} = 3$$

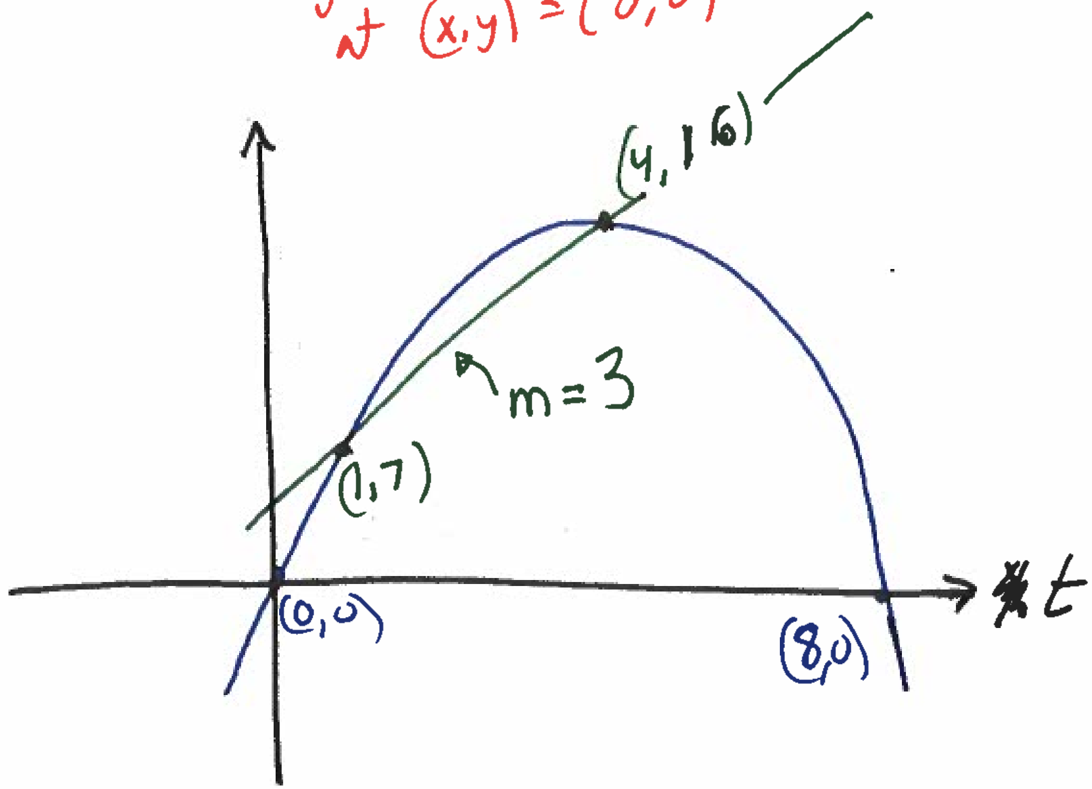
graph $f(x) = \underbrace{8x - x^2}_{\text{standard form}} = \underbrace{x(8-x)}_{\text{factored form}}$

Should have been t, not x

facing down parabola

no constant term so y intercept is at $(x,y) = (0,0)$

x intercepts $(x,y) = (0,0)$, $(x,y) = (8,0)$



Alternate presentation of Average Rate of change

words: average rate of change of f from $x=a$ to $x=a+h$

usage: a is a real number

h is a positive real number

f is a function that is continuous on interval $[a, a+h]$

meaning: The number $m = \dfrac{f(a+h) - f(a)}{(a+h) - a} = \dfrac{f(a+h) - f(a)}{h}$

Example for object with position function $f(t) = 8t - t^2$

Find the average velocity over the interval $[a, a+h]$

for $a=1$ and $h=2$, $h=1$, $h=0.5$, $h=0.1$

and illustrate with a graph

Solution

when $h=2$ $\quad m = \frac{f(1+2)-f(1)}{2} = \frac{f(3)-f(1)}{2}$

$f(1) = 7$ from before

$f(3) = 8\cdot(3) - (3)^2 = 24 - 9 = 15$

$m = \frac{15-7}{2} = \frac{8}{2} = 4$

when $h=1$ $\quad$ we need to compute $m = \frac{f(1+1)-f(1)}{1} = \frac{f(2)-f(1)}{1}$

we know $f(1) = 7$

compute $f(2) = 8\cdot(2) - (2)^2 = 16 - 4 = 12$

$m = \frac{12-7}{1} = 5$

when $h = 0.5$ we need to compute

$$m = \frac{f(1+0.5) - f(1)}{0.5} = \frac{f(1.5) - f(1)}{0.5}$$

we know $f(1) = 7$

compute $f(1.5) = 8(1.5) - (1.5)^2 = 12 - 2.25 = 9.75$

$$m = \frac{f(1.5) - f(1)}{0.5} = \frac{9.75 - 7}{0.5} = \frac{2.75}{0.5} = (2.75)(2) = 5.5$$

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when $h = 0.1$ we need to comput

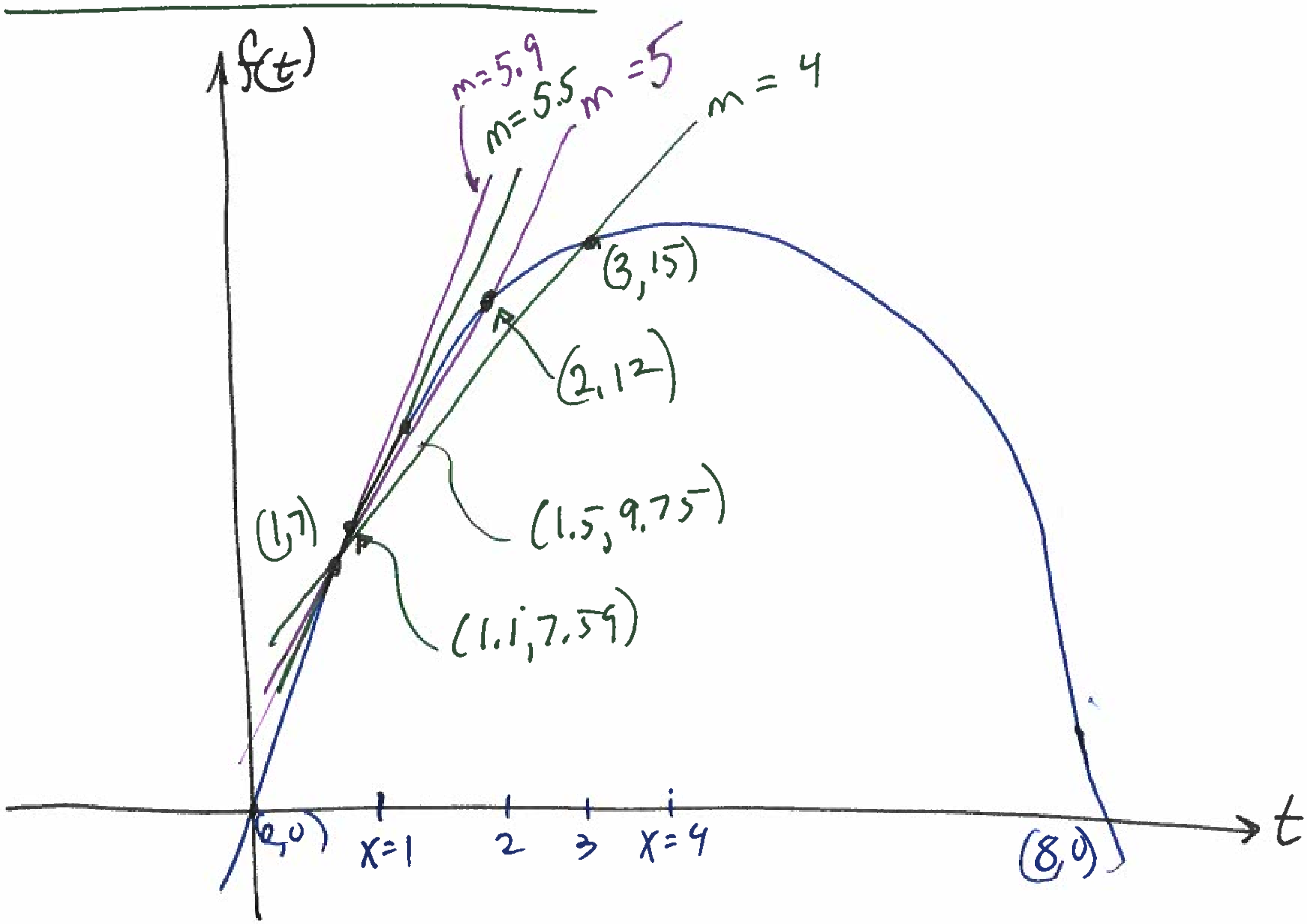
$$m = \frac{f(1+0.1) - f(1)}{0.1} = \frac{f(1.1) - f(1)}{0.1}$$

we know $f(1) = 7$

compute $f(1.1) = 8(1.1) - (1.1)^2 = 8.8 - 1.21 = 7.59$

So $$m = \frac{f(1.1) - f(1)}{0.1} = \frac{7.59 - 7}{0.1} = \frac{0.59}{0.1} = 5.9$$

Illustrate with a graph



Definition of Instantaneous Rate of Change

words: The instantaneous Rate of change of f at $x=a$

usage: a is a number

f is a function that is continuous on an interval around $x=a$

~~Meaning~~

alternate words: The derivative of f at a.

alternate symbol: $f'(a)$

meaning the number $m = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

average rate of change

instantaneous rate of change

additional terminology

when f is a position function, the quantity $f'(a)$ is called the instantaneous velocity at $x=a$.

Example for object with position function

$$f(t) = 8t - t^2$$

Find the instantaneous velocity at $t=1$

Solution we need to build $m = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$

we know $f(1) = 7$ from before

compute ~~f(1)~~ $f(1+h) = 8(1+h) - (1+h)^2$

$$= 8 + 8h - (1 + 2h + h^2)$$

$$= 8 + 8h - 1 - 2h - h^2$$

$$= 7 + 6h - h^2$$

use parts to build the limit

$$m = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0} \frac{(7+6h-h^2) - (7)}{h}$$

indeterminate form

$$= \lim_{h\to 0} \frac{6h-h^2}{h}$$

$$= \lim_{h\to 0} \frac{h(6-h)}{h}$$

Since $h\to 0$, we know $h\neq 0$, so we can cancel

$$= \lim_{h\to 0} 6-h$$

no longer indeterminate

$$= 6-(0)$$

$$= 6$$