Exam X1 Average: 74 (C) Median: 77 (C+)

Question from end of meeting on Monday

Regarding the sum & constant multiple rule

$$\frac{d}{dx}\left(\underline{a}\,f(x) + \underline{b}\,g(x)\right) = \underline{a}\frac{d}{dx}f(x) + \underline{b}\frac{d}{dx}g(x)$$

constants

Question: Why don't you have to take the derivative of the constant?

That is, why not do this? $\cancel{\frac{d}{dx}\left(a f(x)\right) = \left(\frac{d}{dx}a\right)\cdot\left(\frac{d}{dx}f(x)\right)}$

~~derivative of a product = product of the derivatives~~

not the way it works!

Why is $\frac{d}{dx}\, a\cdot f(x) = a \frac{d}{dx} f(x)$

when a is a constant?

Explanation #1 Build the derivative using definition of the derivative

$$\frac{d}{dx}\, af(x) \underset{\text{definition of the derivative}}{=} \lim_{h\to 0} \frac{af(x+h) - af(x)}{h}$$

factor out a in numerator

$$= \lim_{h\to 0} \frac{a\left(f(x+h) - f(x)\right)}{h}$$

bring a in front of fraction

$$= \lim_{h\to 0} \left(a \cdot \frac{f(x+h) - f(x)}{h} \right)$$

use property of limits

$$= a \cdot \lim_{h\to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$

use definition of derivative again

$$= a \cdot f'(x)$$

Example #1

Let $f(x) = x^2$

(a) find $f'(x)$

(b) find $f'(3)$

(c) Illustrate on a graph

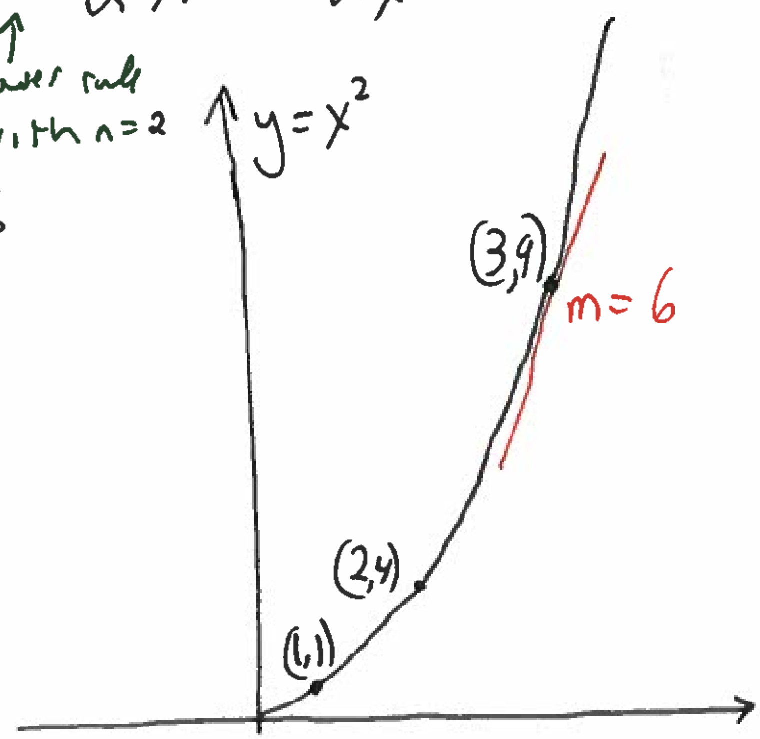
Solution (a) $f'(x) = \frac{d}{dx} x^2 = 2x^{2-1} = 2x$

n = 2

power rule with n=2

(b) $f'(3) = 2 \cdot (3) = 6$

(c) Illustrate



Example #2 Let $g(x) = 5X^2$

ⓐ find $g'(x)$

ⓑ find $g'(3)$

ⓒ Illustrate

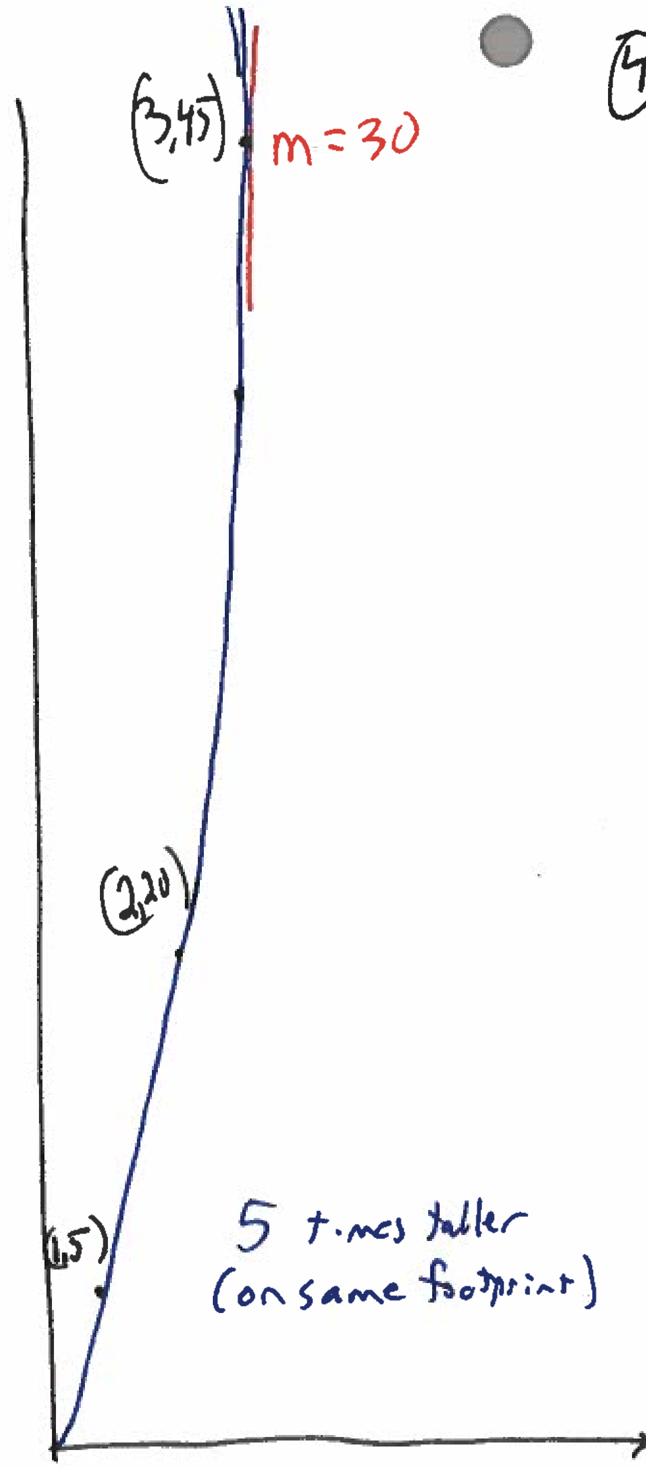
Solution.

ⓐ $g'(x) = \frac{d}{dx}\left(5X^2\right) = 5\frac{d}{dx}X^2 = 5\cdot 2X = 10X$

constant multiple rule

ⓑ $g'(3) = 10(3) = 30$

ⓒ Illustrate



What if we do the bad thing?

$$\frac{d}{dx}(5x^2) = \left(\frac{d}{d}5\right)\left(\frac{d}{dx}x^2\right)$$

constant function rule

$$= \cancel{0 \cdot 2x}$$

$$= \cancel{0}$$

can't be right. The method would always give a result of 0.

Tangent line & Normal Line Examples

2.3 #30 Let $f(x) = x - \sqrt{x}$

(a) Find equation of the line tangent to graph of f at $x = 1$.
(b) Find equation of the line normal to graph of f at $x=1$

Solution

(a) Recall: The tangent line at $x=a$ has two defining properties

- contains the point $(a, f(a))$ point of tangency
- has slope $m = f'(a)$

So the point slope form of the equation is

$$y - f(a) = f'(a)(x-a)$$

we need to build this

Get Parts

$a = 1$ x coordinate of point of tangency

$f(a) = f(1) = 1 - \sqrt{1} = 1 - 1 = 0$ y coordinate of point of tangency

$f'(x) = \frac{d}{dx}(x - \sqrt{x}) = \frac{d}{dx} \overset{n=1}{x} - \frac{d}{dx} x^{\overset{n=1/2}{1/2}} = 1 \cdot x^{1-1} - \frac{1}{2} x^{\frac{1}{2}-1} = 1 \cdot x^0 - \frac{1}{2} x^{-\frac{1}{2}}$

$= 1 \cdot 1 - \frac{1}{2}\left(\frac{1}{x^{1/2}}\right) = 1 - \frac{1}{2\sqrt{x}}$

$f'(a) = f'(1) = 1 - \frac{1}{2\sqrt{1}} = 1 - \frac{1}{2} = \frac{1}{2}$ Slope of tangent line

Substitute into equation

$(y - 0) = \frac{1}{2}(x - 1)$

$y = \frac{1}{2}x - \frac{1}{2}$ tangent line

~~don't write $\frac{1}{2x}$~~

(b) What we know about the normal line at $x=a$

- line contains the point $(a, f(a))$ (same as point of tangency)
- line is perpendicular to the tangent line.
  - If tangent line slope is m_T
    then normal line slope $m_N = \dfrac{-1}{m_T}$

    because $m_T \cdot m_N = -1$ for perpendicular lines

  - If tangent line is horizontal, so $m_T = 0$,
    then the normal line is vertical

In our case we already know

$(a, f(a))$ is the point $(1, 0)$

we know $m_T = \frac{1}{2}$

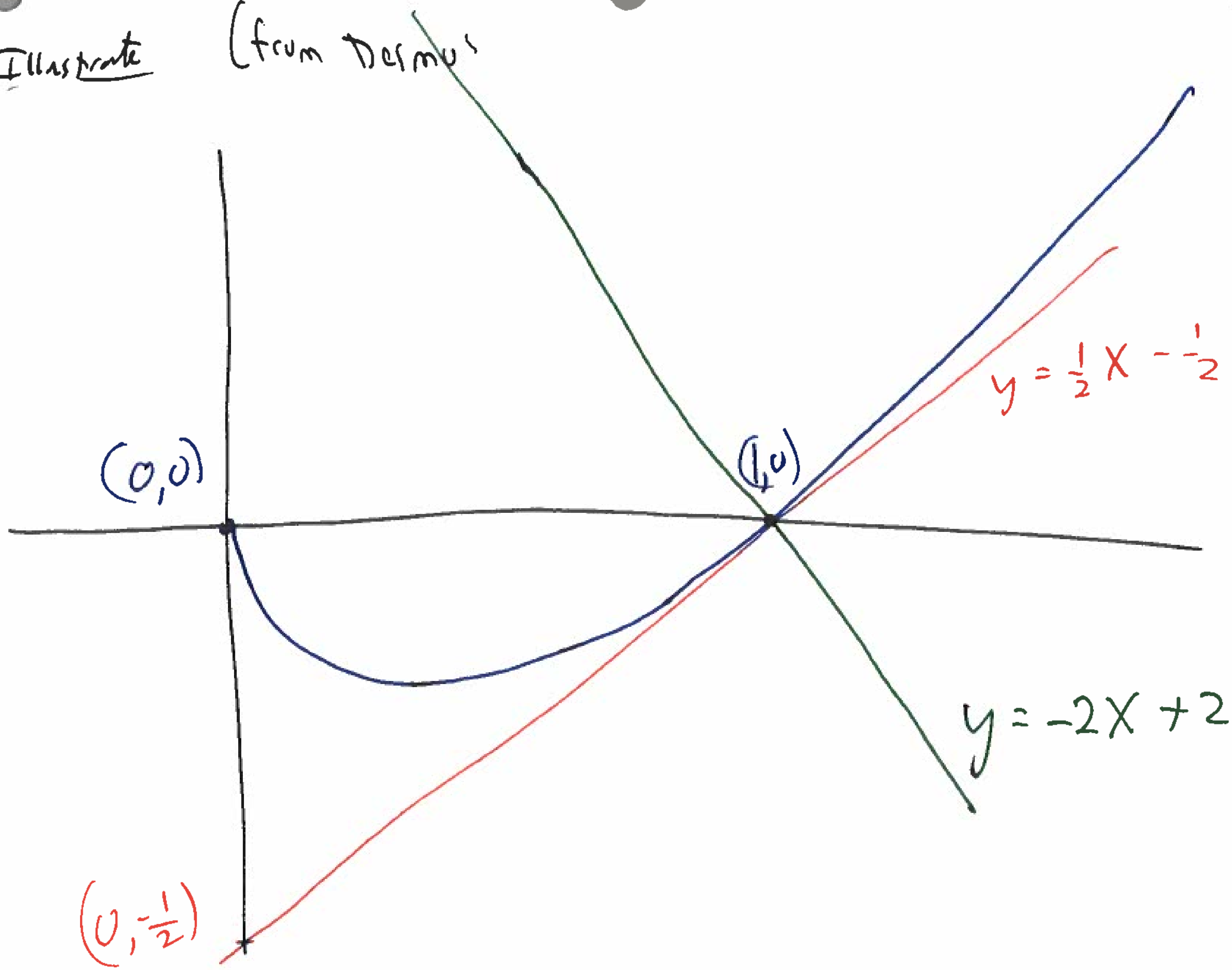
So $m_N = \frac{-1}{1/2} = -2$

So normal line equation

$$(y - 0) = -2(x - 1)$$

$$y = -2x + 2$$

(c) Illustrate (from Desmos)



Example

For $f(x) = \sin(x)$

(a) Find equation for tangent line & (b) normal line at $x = \frac{\pi}{2}$

Solution

(a) we need to build tangent line equation

$$y - f(a) = f'(a)(x-a)$$

Parts

$a = \frac{\pi}{2}$ = x coord of point of tangency

$$f(a) = f(\pi/2) = \sin\left(\frac{\pi}{2}\right) = 1$$

$$f'(x) = \frac{d}{dx}\sin(x) = \cos(x)$$

$$f'(a) = \cos\left(\frac{\pi}{2}\right) = 0$$ ← slope of the tangent line

Substitute parts into equation

$$y - 1 = 0\cdot\left(x - \frac{\pi}{2}\right) = 0$$

$$y = 1$$

(b) The normal line

contains the point $(a, f(a)) = \left(\frac{\pi}{2}, f\left(\frac{\pi}{2}\right)\right) =$

$= \left(\frac{\pi}{2}, \sin\left(\frac{\pi}{2}\right)\right) = \left(\frac{\pi}{2}, 1\right)$

Tangent line is horizontal (so $m_T = 0$)

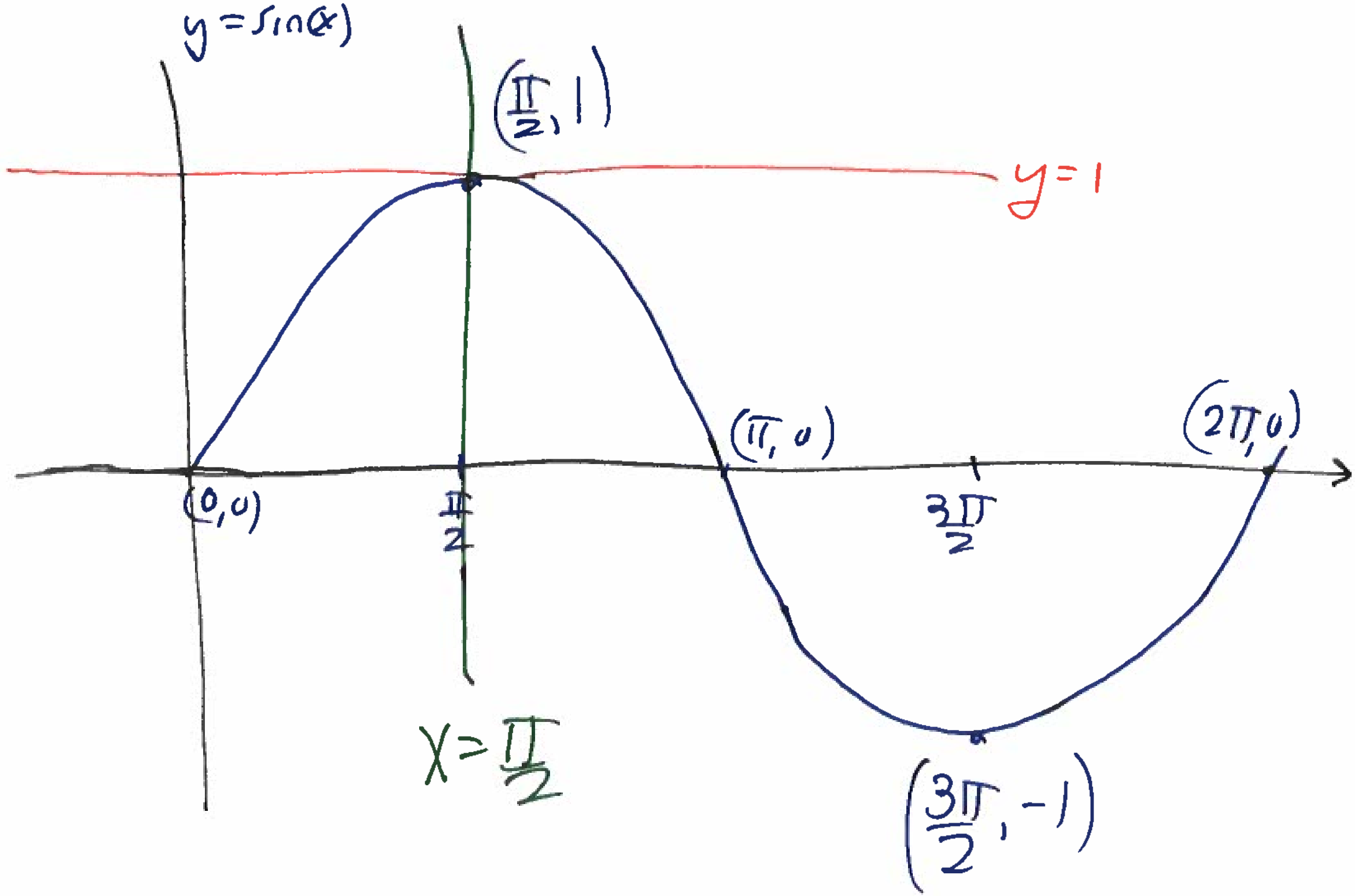
Normal line is vertical

So the equation for the normal line is

$$x = \frac{\pi}{2}$$

(13)

(c) Illustrate



Trick Problem

Find the value of $\lim_{h\to 0} \frac{(5+h)^4 - 625}{h}$

Solution

Recognize that this is computing the value of $f'(x)$ at some $x=a$

It is computing $f'(a)$

recall $f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$

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Figure out the function and the value of a.

key is $f(a+h)$ is $(5+h)^4$

So a must be $a=5$

the function must be

$$f(\quad) = (\quad)^4$$ empty version

so $f(x) = x^4$

Think about this question (different question)

for $f(x) = x^4$, find $f'(5)$

build definition of derivative

find $f'(x)$

$$f'(5) = \lim_{h \to 0} \frac{(5+h)^4 - 5^4}{h}$$

$$f'(x) = 4x^3$$

really hard!

Substitute in $x = a = 5$

$$f'(5) = 4(5)^3 = 4(125) = 500$$