Today: The Product and Quotient Rules, Section 2.4

The Product Rule

Question:

How do we find the derivative of a Product of functions?

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = ?$$

Good News! there is an obvious thing!

$$\cancel{\frac{d}{dx}\left(f(x)\cdot g(x)\right) = \frac{d}{dx}f(x)\cdot\frac{d}{dx}g(x)}$$ WRONG

derivative of a product = product of the derivatives

Bad News! The obvious thing is wrong!!

~~Unfortunat~~
Unfortunately the correct method is not as easy to remember!

The Product Rule

used for finding the derivative of a product of functions.

$$\frac{d}{dx}\left(f(x) \cdot g(x) \right) = \left(\frac{d}{dx} f(x) \right) \cdot g(x) + f(x) \cdot \left(\frac{d}{dx} g(x) \right)$$

$$\left(f(x) \cdot g(x) \right)' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Example Find the derivative of $y = \sqrt{x} \cdot \sin(x)$

Solution

$\sqrt{x} = x^{1/2}$ $n = \frac{1}{2}$

$$\frac{d}{dx}\left(\sqrt{x} \cdot \sin(x)\right) \underset{\text{product rule}}{=} \left(\frac{d}{dx}\sqrt{x}\right) \cdot \sin(x) + \sqrt{x} \cdot \left(\frac{d}{dx}\sin(x)\right)$$

power rule with $n = \frac{1}{2}$

$$= \left(\frac{1}{2} \cdot x^{\frac{1}{2}-1}\right) \cdot \sin(x) + \sqrt{x} \cdot \left(\cos(x)\right)$$

simplify

$$= \frac{1}{2} x^{-\frac{1}{2}} \cdot \sin(x) + \sqrt{x} \cdot \cos(x)$$

eliminate negative exponent

$$= \frac{1}{2}\left(\frac{1}{x^{1/2}}\right) \cdot \sin(x) + \sqrt{x} \cos(x)$$

Simplify some more

$$= \frac{\sin(x)}{2\sqrt{x}} + \sqrt{x} \cos(x)$$

The Quotient Rule

Question: How to find the derivative of a quotient?

$$\frac{d}{dx}\left(\frac{\mathrm{top}(x)}{\mathrm{bottom}(x)}\right) = ?$$

Good news! There's an obvious way

$$\cancel{\frac{d}{dx}\left(\frac{\text{top}(x)}{\text{bottom}(x)}\right) = \frac{\frac{d}{dx}\text{top}(x)}{\frac{d}{dx}\text{bottom}(x)}} \quad \text{WRONG}$$

derivative of a quotient = quotient of the derivatives

Bad News! This obvious way is wrong!

The Quotient Rule

Used to find the Derivative of a quotient

$$\frac{d}{dx}\left(\frac{top(x)}{bottom(x)}\right) = \frac{top'(x)\cdot bottom(x) - \cancel{bottom'(x)\cdot top(x)}\ top(x)\cdot bottom'(x)}{(bottom(x))^2}$$

Example Find derivative of $\dfrac{5t}{5+\sqrt{t}}$

Solution

$$\frac{d}{dt}\left(\frac{5t}{5+\sqrt{t}}\right) = \frac{\left(\frac{d}{dt}5t\right)\cdot(5+\sqrt{t}) - \cancel{\frac{d}{dt}}(5t)\cdot\left(\frac{d}{dt}5+\sqrt{t}\right)}{(5+\sqrt{t})^2}$$

Warning cannot cancel $5+\sqrt{t}$!!

$$= \frac{(5)(5+\sqrt{t}) - 5t\cdot\left(\frac{1}{2\sqrt{t}}\right)}{(5+\sqrt{t})^2}$$

$n=1$

$$\frac{d}{dt}5t = 5\frac{d}{dt}t = 5\cdot1\cdot t^{1-1}$$
$$= 5\cdot1\cdot t^0$$
$$= 5\cdot1\cdot1$$
$$= 5$$

factor out 5

$$= \frac{5\left((5+\sqrt{t}) - \frac{t}{2\sqrt{t}}\right)}{(5+\sqrt{t})^2}$$

Quotient Rule can be really messy!

constant $\quad n=\frac{1}{2}$

$$\frac{d}{dt}(5+\sqrt{t}) = 0+\frac{1}{2\sqrt{t}}$$

$$= \frac{5\left(5+\sqrt{t} - \frac{\sqrt{t}}{2}\right)}{(5+\sqrt{t})^2}$$

$$\sqrt{t}\cdot\sqrt{t} = t$$
$$\sqrt{t} = \frac{t}{\sqrt{t}}$$

$$= \frac{5\left(5+\frac{\sqrt{t}}{2}\right)}{(5+\sqrt{t})^2}$$

<u>Similar Example</u> ~~Ex~~ Find derivative of $\dfrac{t-\sqrt{t}}{t^{1/3}}$

<u>Looks Like a Quotient</u> Try Quotient Rule

$$\frac{d}{dt}\left(\frac{t-\sqrt{t}}{t^{1/3}}\right) = \frac{\frac{d}{dt}\left(t-\sqrt{t}\right)\cdot t^{1/3} - \left(t-\sqrt{t}\right)\cdot\left(\frac{d}{dt}\, t^{1/3}\right)}{\left(t^{1/3}\right)^2}$$

quotient rule

ugh! Too hard!

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Much Better Solution

Part 1 Rewrite function in power function form

$$y = \frac{t - \sqrt{t}}{t^{1/3}} = \frac{t}{t^{1/3}} - \frac{\sqrt{t}}{t^{1/3}} =$$

$$= \frac{t^1}{t^{1/3}} - \frac{t^{1/2}}{t^{1/3}} = t^{1-\frac{1}{3}} - t^{\frac{1}{2}-\frac{1}{3}}$$

Part 2 Differentiate

$$y = t^{\frac{2}{3}} - t^{1/6}$$

$n = 2/3$ $\quad$ $n = 1/6$

$$y' = \frac{d}{dt}\left(t^{2/3} - t^{1/6}\right) = \frac{2}{3}t^{\frac{2}{3}-1} - \frac{1}{6}t^{\frac{1}{6}-1} = \frac{2}{3}t^{-1/3} - \frac{1}{6}t^{-5/6}$$

$$= \frac{2}{3t^{1/3}} - \frac{1}{6t^{5/6}}$$