Our Derivative Rules So Far

Constant Function Rule: $\frac{d}{dx} c = 0$ constant function

Power Rule: $\frac{d}{dx} x^n = n x^{n-1}$

Sum + Constant Multiple Rule: $\frac{d}{dx}\left(a f(x) + b g(x)\right) = a \frac{d}{dx} f(x) + b \frac{d}{dx} g(x)$

a, b constants

Product Rule: $\frac{d}{dx}\left(f(x) \cdot g(x)\right) = \left(\frac{d}{dx} f(x)\right) \cdot g(x) + f(x) \cdot \left(\frac{d}{dx} g(x)\right)$

Quotient Rule: $\frac{d}{dx}\left(\frac{top(x)}{bottom(x)}\right) = \frac{\left(\frac{d}{dx} top(x)\right) \cdot bottom(x) - top(x) \cdot \left(\frac{d}{dx} bottom(x)\right)}{\left(bottom(x)\right)^2}$

Today: Section 2.5 The Chain Rule

The Chain Rule is used for finding the derivative of a composition of functions. "nested functions". That is, functions of the form

$$\text{outer}(\text{inner}(x))$$

The Chain Rule

$$\frac{d}{dx}\,\text{outer}(\text{inner}(x)) = \text{outer}'(\text{inner}(x)) \cdot \text{inner}'(x)$$

Part 1

Examples with Power Function outer function

[Example #1] Let $f(x) = (4x^5 - 17x^3 + x - 13)^{19}$ Find $f'(x)$

Solution

$$f'(x) = \frac{d}{dx}\left((4x^5 - 17x^3 + x - 13)^{19} \right)$$

$$= \frac{d}{dx}\, outer(inner(x))$$

Chain rule

$$= outer'(inner(x)) \cdot inner'(x)$$

Chain Rule Details

$inner(x) = 4x^5 - 17x^3 + x - 13$

$inner'(x) = 20x^4 - 51x^2 + 1$

$outer(\;) = (\;)^{19} \leftarrow n = 19$ empty version

$outer'(\;) = 19(\;)^{18}$ empty version

used power rule

$$= 19\,(4x^5 - 17x^3 + x - 13)^{18} \cdot (20x^4 - 51x^2 + 1)$$

↑ V.I.P.

Example #2 let $f(x) = \dfrac{1}{(4x^5 - 17x^3 + x - 13)^{19}}$ Find $f'(x)$

Solution

$$f'(x) = \frac{d}{dx} \frac{1}{(4x^5 - 17x^3 + x - 13)^{19}}$$

$$= \frac{d}{dx} (4x^5 - 17x^3 + x - 13)^{-19}$$

$$= \frac{d}{dx} \text{outer}(\text{inner}(x))$$

chainrule

$$= \text{outer}'(\text{inner}(x)) \cdot \text{inner}'(x)$$

$$= \frac{-19}{(4x^5 - 17x^3 + x - 13)^{20}} \cdot (20x^4 - 51x^2 + 1)$$

$$= -\frac{19(20x^4 - 51x^2 + 1)}{(4x^5 - 17x^3 + x - 13)^{20}}$$

Chain Rule Details

$\text{inner}(x) = 4x^5 - 17x^3 + x - 13$

$\text{inner}'(x) = 20x^4 - 51x^2 + 1$

$\text{outer}(\;) = (\quad)^{-19} \leftarrow n = -19$

empty version

$\text{outer}'(\;) = -19(\quad)^{-19-1}$ power rule

$= -19(\;)^{-20}$

$= -19 \cdot \frac{1}{(\;)^{20}}$

$= -\frac{19}{(\;)^{20}}$

Example 3 Let $f(x) = \sqrt{4x^5 - 17x^3 + x - 13}$ Find $f'(x)$

Solution

$$f'(x) = \frac{d}{dx}\sqrt{4x^5 - 17x^3 + x - 13}$$

$$= \frac{d}{dx}\,\text{outer}(\text{inner}(x))$$

chain rule

$$= \text{outer}'(\text{inner}(x)) \cdot \text{inner}'(x)$$

$$= \frac{1}{2\sqrt{(4x^5 - 17x^3 + x - 13)}} \cdot (20x^4 - 51x^2 + 1)$$

$$= \frac{20x^4 - 51x^2 + 1}{2\sqrt{4x^5 - 17x^3 + x - 13}}$$

Chain Rule Details

$$\text{inner}(x) = 4x^5 - 17x^3 + x - 13$$

$$\text{inner}'(x) = 20x^4 - 51x^2 + 1$$

$n = \frac{1}{2}$

$$\text{outer}(\ ) = \sqrt{(\quad)} = (\quad)^{1/2}$$

$$\text{outer}'(\ ) = \frac{1}{2}(\quad)^{\frac{1}{2}-1}$$

used power rule

$$= \frac{1}{2}(\quad)^{-\frac{1}{2}}$$

$$a^{-b} = \frac{1}{a^b}$$

$$= \frac{1}{2}\frac{1}{(\quad)^{1/2}}$$

$$= \frac{1}{2\sqrt{(\quad)}}$$

[Example 4] $f(x) = (\cos(x))^2 = \cos^2(x) = \cos^2 x$

more common notation

Find $f'(x)$

Solution

$f'(x) = \frac{d}{dx}(\cos(x))^2$

$= \frac{d}{dx}\, outer(inner(x))$

Chain rule

$= outer'(inner(x)) \cdot inner'(x)$

$= 2(\cos(x)\ ) \cdot (-\sin(x))$

$= -2\cos(x)\sin(x)$

Chain Rule details

$inner(x) = \cos(x)$

$inner'(x) = -\sin(x)$

$outer(\ ) = (\quad)^2$ empty version

$outer'(\ ) = 2(\quad)$

Example 5 let $f(x) = \cos x^2 = \cos(x^2)$ (a) Find $f'(x)$

Solution

$$f'(x) = \frac{d}{dx}\cos(x^2)$$

$$= \frac{d}{dx}\,\text{outer}(\text{inner}(x))$$

chain Rule

$$= \text{outer}'(\text{inner}(x)) \cdot \text{inner}'(x)$$

$$= -\sin(x^2) \cdot 2x$$

$$= -2x\sin(x^2)$$

Chain Rule Details

$\text{inner}(x) = x^2$

$\text{inner}'(x) = 2x$

$\text{outer}(\ ) = \cos(\ )$

$\text{outer}'(\ ) = -\sin(\ )$

(b) Find $f''(x)$ The second derivative

Solution

$$f''(x) = \frac{d}{dx} f'(x)$$

$$= \frac{d}{dx} \underbrace{-2}_{\text{constant multiple}} x\sin(x^2)$$

$$= -2 \frac{d}{dx} x\sin(x^2)$$

product rule

$$= -2\left[\left(\frac{d}{dx} x\right)\sin(x^2) + x\left(\frac{d}{dx} \sin(x^2)\right)\right]$$

chain rule

$$= -2\left[(1)\sin(x^2) + x\left(\cos(x^2)\cdot 2x\right)\right]$$

$$= -2\left[\sin(x^2) + 2x^2\cos(x^2)\right]$$