MATH 2301 Section 110 (Barsamian) Day 22 (Wed Feb 22) (1)

Today: Section 2.6 Implicit Differentiation

Friday: Section 2.7 Related Rates

Quiz Q4 Over sections 2.4, 2.5, 2.6

Section 2.6 Implicit Differentiation

Equations, Functions, Describing something explicitly, or implicitly

Consider this equation involving x, y

$$x^3 + y^3 = 7$$

expresses a relationship between x and y.

Describes y ~~in~~ as a function implicitly

(you have to do some arranging, but it can ~~be done~~ be solved for y in terms of x.)

Solve that equation for y

$$x^3 + y^3 = 7$$

$$y^3 = 7 - x^3$$

$$(y^3)^{1/3} = (7 - x^3)^{1/3}$$

$$y = (7 - x^3)^{1/3}$$

equation involving x, y

expresses a relationship between x and y

Solved for y in terms of x

Describes y as a <u>function</u> of x.

Describes y <u>explicitly</u>. That is y = some stuff not involving y.

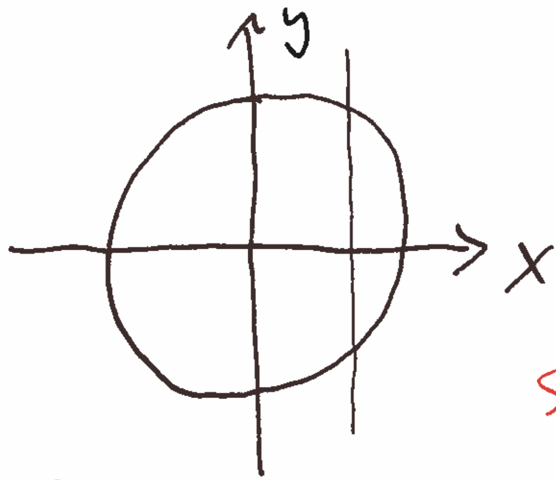
Now consider this equation involving x and y

$$x^2 + y^2 = 1$$

expresses a relationship between x and y.

Cannot be solved for y as a function of x.

The graph of the equation $x^2+y^2=1$ is a circle



fails the vertical line test

for some values of x, there is more than one value of y.

So this is <u>not</u> the graph of a function.

So the equation $x^2+y^2=1$ <u>does not</u> describe y <u>explicitly</u>, and it also <u>does not</u> describe y implicitly.

~~#~~ **Conclusion**

Some equations involving x, y describe y implicitly, but some do not.

Big Fact All equations involving x, y do ~~describing~~ describe $\frac{dy}{dx}$ implicitly.

That is

given equation involving x, y $\xrightarrow[\text{called Implicit differentiation}]{\text{process}}$ equation describing $\frac{dy}{dx}$ explicitly

$$\frac{dy}{dx} = \underbrace{\text{Blah}}_{}$$

stuff on right does not have $\frac{dy}{dx}$ in it.

Method of Implicit Differentiation

Used for finding an explicit description of $\frac{dy}{dx}$

when an equation is given involving x, y.

Step 1 In the equation, replace all y with $y(x)$, indicating that y is (sort of) a function of x.

Result is a new equation involving x and $y(x)$

Step 2 Take $\frac{d}{dx}$ of both sides of this equation.

This may require the chain rule

The inner function will be $y(x)$

So the inner'(x) will be $\frac{dy(x)}{dx}$

Result will be a new equation involving x, $y(x)$, $\frac{dy(x)}{dx}$

Step 3 Replace all the $y(x)$ with y. Result will be a new equation involving $x, y, \frac{dy}{dx}$

Step 4 Solve that equation for $\frac{dy}{dx}$

Result will be an equation of the form $\frac{dy}{dx}$ = Stuff involving x, y

This equation describes dy/dx explicitly

Example 1 Suppose $x^3 + y^3 = 7$

(a) Solve for y explicitly and use ordinary differentiation to find $\frac{dy}{dx}$

Solution

$$x^3 + y^3 = 7$$

$$y^3 = 7 - x^3$$

$$y = (7 - x^3)^{1/3}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left((7-x^3)^{1/3}\right)$$

$$= \frac{d}{dx}\, \text{outer}(\text{inner}(x))$$

chain rule

$$= \text{outer}'(\text{inner}(x)) \cdot \text{inner}'(x)$$

$$= \frac{1}{3(7-x^3)^{2/3}}(-3x^2)$$

$$= -\frac{x^2}{(7-x^3)^{2/3}}$$

Chain rule details

$$\text{inner}(x) = 7 - x^3$$

$$\text{inner}'(x) = -3x^2$$

$n = 1/3$

$$\text{outer}(\ ) = (\quad)^{1/3}$$

empty version

$$\text{outer}'(\ ) = \frac{1}{3}(\quad)^{1/3 - 1} = \frac{1}{3}(\ )^{-2/3}$$

$$= \frac{1}{3} \cdot \frac{1}{(\ )^{2/3}} = \frac{1}{3(\ )^{2/3}}$$

(h) Start over Find $\frac{dy}{dx}$ using implicit differentiation

$$x^3 + y^3 = 7$$

Step 1 Replace y with $y(x)$

Result $x^3 + (y(x))^3 = 7$ equation involving x, $y(x)$

Step 2 Find $\frac{d}{dx}$ of both sides

$$\frac{d}{dx}\left(x^3 + (y(x))^3\right) = \frac{d}{dx} 7$$

$$\frac{d}{dx}x^3 + \underbrace{\frac{d}{dx}\left((y(x))^3\right)}_{\text{chain rule}} = 0$$

$$3x^2 + 3(y(x))^2 \cdot \frac{dy(x)}{dx} = 0$$

equation involving x, $y(x)$, $\frac{dy(x)}{dx}$

Chain rule Details

$\text{inner}(x) = y(x)$

$\text{inner}'(x) = \frac{d}{dx} y(x)$

$\text{outer}(\ ) = (\ )^3$

$\text{outer}'(\ ) = 3(\ )^2$

Step 3 Replace all $y(x)$ with just y

$$3x^2 + 3y^2\frac{dy}{dx} = 0$$

equation involving $x, y, \frac{dy}{dx}$

Step 4 Solve for $\frac{dy}{dx}$

$$3y^2\frac{dy}{dx} = -3x^2$$

$$\frac{dy}{dx} = \frac{-3x^2}{3y^2}$$

$$\frac{dy}{dx} = \frac{-x^2}{y^2}$$

equation ~~~~ describing ~~~~ $\frac{dy}{dx}$ = stuff involving x, y

Compare results

(a) $$\frac{dy}{dx} = -\frac{x^2}{(7-x^3)^{2/3}}$$

(b) $$\frac{dy}{dx} = -\frac{x^2}{y^2}$$

Why don't these match?!?

Remember from part (a) that we could solve original equation for y

$$y = (7-x^3)^{1/3}$$

Substitute that expression into the result of (b)

result of (b): $$\frac{dy}{dx} = -\frac{x^2}{y^2} = \frac{-x^2}{\left((7-x^3)^{1/3}\right)^2} = -\frac{x^2}{(7-x^3)^{2/3}}$$ = result of (a)

Moral If original equation can be solved for y in terms of x, that is the best route to finding $\frac{dy}{dx}$

(11)

Example 2.6#7

Suppose $y\cos(x) = x^2 + y^2$

Find $\frac{dy}{dx}$

Solution Cant solve for y in terms of x
so must use implicit differentiation

Step 1 replace y with $y(x)$

$$y(x)\cdot\cos(x) = x^2 + (y(x))^2$$

Step 2 Find $\frac{d}{dx}$ of both sides

$$\underbrace{\frac{d}{dx}\left(y(x)\cdot\cos(x)\right)}_{\text{product rule}} = \frac{d}{dx}\left(x^2 + \underbrace{(y(x))^2}_{\text{Chain rule}}\right)$$

Chain Details
inner(x) = y(x)
inner'(x) = $\frac{d}{dx}y(x)$
Outer( ) = $(\ )^2$
Outer'( ) = $2(\ )'$

$$\left(\frac{d}{dx}y(x)\right)\cdot\cos(x) + y(x)\frac{d}{dx}\cos(x) = 2x + 2\cdot y(x)\cdot\frac{d}{dx}y(x)$$

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$$\left(\frac{dy(x)}{dx}\right)\cdot \cos(x) + y(x)(-\sin(x)) = 2x + 2y(x)\frac{dy(x)}{dx}$$

$$\frac{dy(x)}{dx}\cdot \cos x - y(x)\sin(x) = 2x + 2y(x)\frac{dy(x)}{dx}$$

Step 3 replace $y(x)$ with just y

$$\frac{dy}{dx}\cdot \cos(x) - y\sin(x) = 2x + 2y\frac{dy}{dx}$$

Step 4 Solve for $\frac{dy}{dx}$

$$\frac{dy}{dx}\cos x - 2y\frac{dy}{dx} = 2x + y\sin(x)$$

factor out dy/dx

$$\frac{dy}{dx}\cdot(\cos(x) - 2y) = 2x + y\sin(x)$$

Divide both sides by $\cos(x) - 2y$

$$\frac{dy}{dx} = \frac{2x + y\sin(x)}{\cos(x) - 2y}$$