MATH 2301 Section 110 (Barsamian) Day 24 (Mon Feb 27) ①

Today: Section 2.8 Linear Approximations and Differentials

Tomorrow in Recitation: Section 2.7 (Related Rates) and Section 2.8

Wednesday: Section 3.1 Exponential Functions

Fri: Quiz Q5 and Section 3.2

One More Section 2.7 Example (related rates) (Similar to 2.2 #27)

Gravel being dumped at a rate of 50 ft^3/min
forms a conical pile whose base diameter and height are always equal.

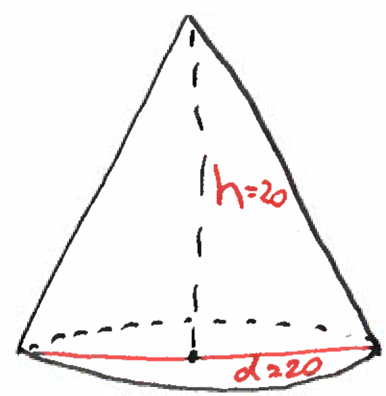
How fast is the height growing when the height is 20 ft?

Solution Follow Procedure from wed

Step 1 Draw Picture

Step 2 Label picture with important quantities (d, h, V)

Step 3 Label known values or known rates



$d = h = 20$ ft
h' unknown ← find h'
V unknown
$V' = 50\ ft^3/minute$

Step 4 Find h'

Step 5 Find an equation that relates the quantities (d, h, V)

$$V = \frac{1}{3} \cdot \text{area of base} \cdot h = \frac{1}{3} \cdot \pi r^2 \cdot h = \frac{1}{3} \cdot \pi \left(\frac{d}{2}\right)^2 \cdot h = \frac{1}{3} \pi \left(\frac{h}{2}\right)^2 \cdot h = \frac{\pi h^3}{12}$$

Step 6 Find $\frac{d}{dt}$ of both sides using implicit differentiation

$$V = \frac{\pi h^3}{12}$$ equation relating V, h

$$\frac{d}{dt}\left(V\right) = \frac{d}{dt}\left(\frac{\pi h^3}{12}\right)$$

$$V' = \frac{\pi}{12}\frac{d}{dt}\left(h(t)\right)^3$$

$$= \frac{\pi}{12}\frac{d}{dt}\,\text{outer}(\text{inner}(t))$$

Chain Rule

$$= \frac{\pi}{12}\cdot \text{Outer}'(\text{inner}(t))\cdot \text{inner}'(t)$$

$$= \frac{\pi}{12}\cdot 3(h(t))^2\cdot h'(t)$$

$$V' = \frac{\pi h^2\cdot h'}{4}$$ equation relating V', h', V, h

Chain Rule Details

inner$(t) = h(t)$

inner$'(t) = h'(t)$ unknown find it!!

Outer$(\ ) = (\ )^3$

Outer$'(\ ) = 3(\ )^2$

Step 7 Solve for the unknown rate

Solve for h'

$$V' = \frac{\pi h^2 \cdot h'}{4}$$

multiply by 4, divide by πh^2

$$h' = \frac{V' \cdot 4}{\pi h^2}$$

Step 8 Substitute in known values and simplify

$$h' = \frac{(50\ ft^3/min) \cdot 4}{\pi \cdot (20\ ft)^2} = \left(\frac{50\ ft^3}{min}\right)\frac{4}{\pi (20\ ft)^2} = \frac{50\ ft^3 \cdot 4}{min\ \pi \cdot 400\ ft^2}$$

$$= \frac{50 \cdot 4}{\pi \cdot 400} \frac{ft^{\cancel{3}\,1}}{min \cdot \cancel{ft^2}} = \frac{200}{\pi \cdot 400} \cdot \frac{ft}{min} = \frac{1}{2\pi} \frac{ft}{min}$$

End of Example

2.8 Linear Approximations and Differentials

The Linearization of a Function

Given a function f,

we know that the equation for the line tangent to $f(x)$ at $x=a$.

(Can somebody say it?!?)

$$y - f(a) = f'(a)(x-a)$$

$a, f(a), f'(a)$ are all constants. Real numbers.

This equation expresses relationship between variables x, y for all points on the tangent line.

Solve this equation for y by adding $f(a)$ to both sides

$$y = f(a) + f'(a)(x-a)$$

constants $a, f(a), f'(a)$

This equation gives y as a function of x
for all points on the tangent line

Invent a name for this function

name: The Linearization of f at a.

symbol: $L(x)$

formula: $L(x) = f(a) + f'(a)(x-a)$

Example (2.8#1) $f(x) = x^4 + 3x^2$

Find the linearization of f at $a=-1$.

Solution We need to build

$$L(x) = f(a) + f'(a)(x-a)$$

Get parts

$a=-1$

$$f(a) = f(-1) = (-1)^4 + 3(-1)^2 = 1 + 3\cdot 1 = 4$$

$$f'(x) = 4x^3 + 6x$$

$$f'(a) = f(-1) = 4(-1)^3 + 6(-1) = 4(-1) - 6 = -4-6 = -10$$

Assemble the function oddly, Don't simplify all the way

$$L(x) = 4 + (-10)(x-(-1)) = 4 - 10(x+1)$$

Example 2 $f(x) = x^{1/3}$

Find the linearization of f at $a = 1000$

Solution We need to build

$$L(x) = f(a) + f'(a)(x-a)$$

Get parts

$a = 1000$

$$f(a) = f(1000) = (1000)^{1/3} = 10$$

$n = 1/3$

$$f'(x) = \frac{d}{dx} x^{1/3} = \left(\frac{1}{3}\right) x^{\frac{1}{3}-1} = \left(\frac{1}{3}\right) x^{-\frac{2}{3}} = \left(\frac{1}{3}\right) \frac{1}{x^{2/3}} = \frac{1}{3x^{2/3}}$$

$$f'(a) = f'(1000) = \frac{1}{3(1000)^{2/3}} = \frac{1}{3((1000^{\frac{1}{3}})^2)}$$

$(a^{b \cdot c}) = (a^b)^c$

$$= \frac{1}{3 \cdot (10)^2} = \frac{1}{3 \cdot 100} = \frac{1}{300}$$

Build the Linearization

$$L(x) = 10 + \left(\frac{1}{300}\right)(x - 1000)$$

Use the linearization to get an approximate value for $\sqrt[3]{1001}$

Solution ~~exact value~~

function involved: $f(x) = \sqrt[3]{x} = x^{1/3}$

Convenient nearby x value $= a = 1000$

~~Actual exact value of $f(a) = (1000)^{1/3}$~~

Our x value $x = 1001$

exact value of $f(x)$ would be $f(1001) = (1001)^{1/3}$

But it is easy to get $L(1001)$

~~f(1001)~~

$$\sqrt[3]{1001} = (1001)^{1/3} = f(1001) \approx L(1001) = 10 + \frac{1}{300}(1001-1000)$$

$$= 10 + \frac{1}{300}(1)$$

$$= 10 + \frac{1}{300}$$

Observe

$$f(1001) = (1001)^{1/3} \approx 10.0033322$$

$$L(1001) = 10 + \frac{1}{300} \approx 10.00\overline{333}$$