MATH 2301 Section 110 (Bassamian) Day 26 (Wed March 1, 2023)

Today: 2.8 Differentials
3.1 Exponential Functions

Fri: 3.2 Inverse Functions, Logarithms Quiz Q5

Fri March 10 Exam X2

Continuing Discussion of Section 2.8 Linearizations + Differentials

Recall Linearizations

Given function $f(x)$ and a real number "a".

The equation for the line tangent to the graph of f at $x=a$

$$(y-f(a)) = f'(a)(x-a)$$

The Linearization of f at a

$$L(x) = f(a) + f'(a)(x-a)$$

Describes y values on tangent line at a as a function of x.

The Linearization is useful for approximations

[Example] 2.8 #11 (a) Use linearization to estimate $(1.999)^4$

Notice: we could find $(1.999)^4$ exactly by hand. It would be a number with 12 decimal places.

To use a linearization to approximate the value

- Identify a function $f(x)$ that is implicated.
- Identify a convenient nearby x value. Call it a.
- Find the linearization of f at a.

  $$L(x) = f(a) + f'(a)(x-a)$$

- Find the value of ~~f(x) for~~ $L(x)$ for the given value of x.

$f(x) = x^4$

$a = 2$

convenient because $f(2) = 2^4$ would be easy

$L(x) = 16 + 32(x-2)$

Get $L(x)$

Get Parts $f(x) = x^4$

$a = 2$ = convenient nearby x value

$f(a) = f(2) = 2^4 = 16$

$f'(x) = 4 \cdot x^3$

$f'(a) = f'(2) = 4 \cdot (2)^3 = 4 \cdot 8 = 32$

Build $L(x)$

$L(x) = 16 + 32(x-2)$ oddly, don't simplify

Given value of x is $x = 1.999$

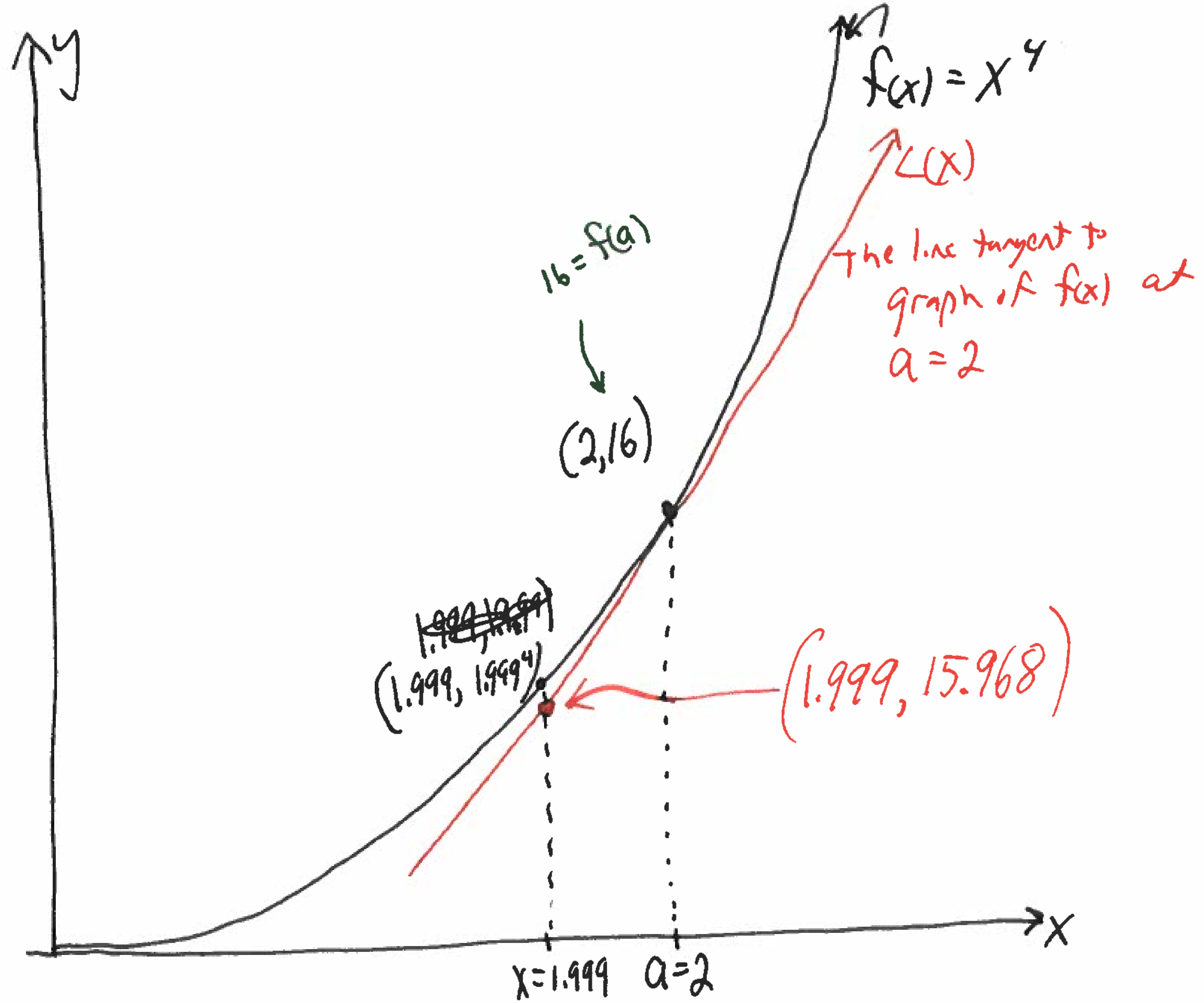
$L(1.999) = 16 + 32(1.999 - 2)$

$= 16 + 32(-0.001)$

$= 16 - 0.032$

$= 15.968$

(b) Illustrate the result of (a) on a graph



The big idea of Linearizations:

for x near a,

$$\underbrace{f(x)}_{\substack{\uparrow \\ \text{exact value}}} \approx \underbrace{L(x)}_{\substack{\uparrow \\ \text{approximate value}}}$$

So $\underbrace{f(1.999)}_{\text{exact}} \approx \underbrace{L(1.999)}_{\text{approx}} = 15.968$

From graph, we see that $L(1.999) < f(1.999)$

$$15.968 \approx (1.999)^4 = 15.968023992001$$

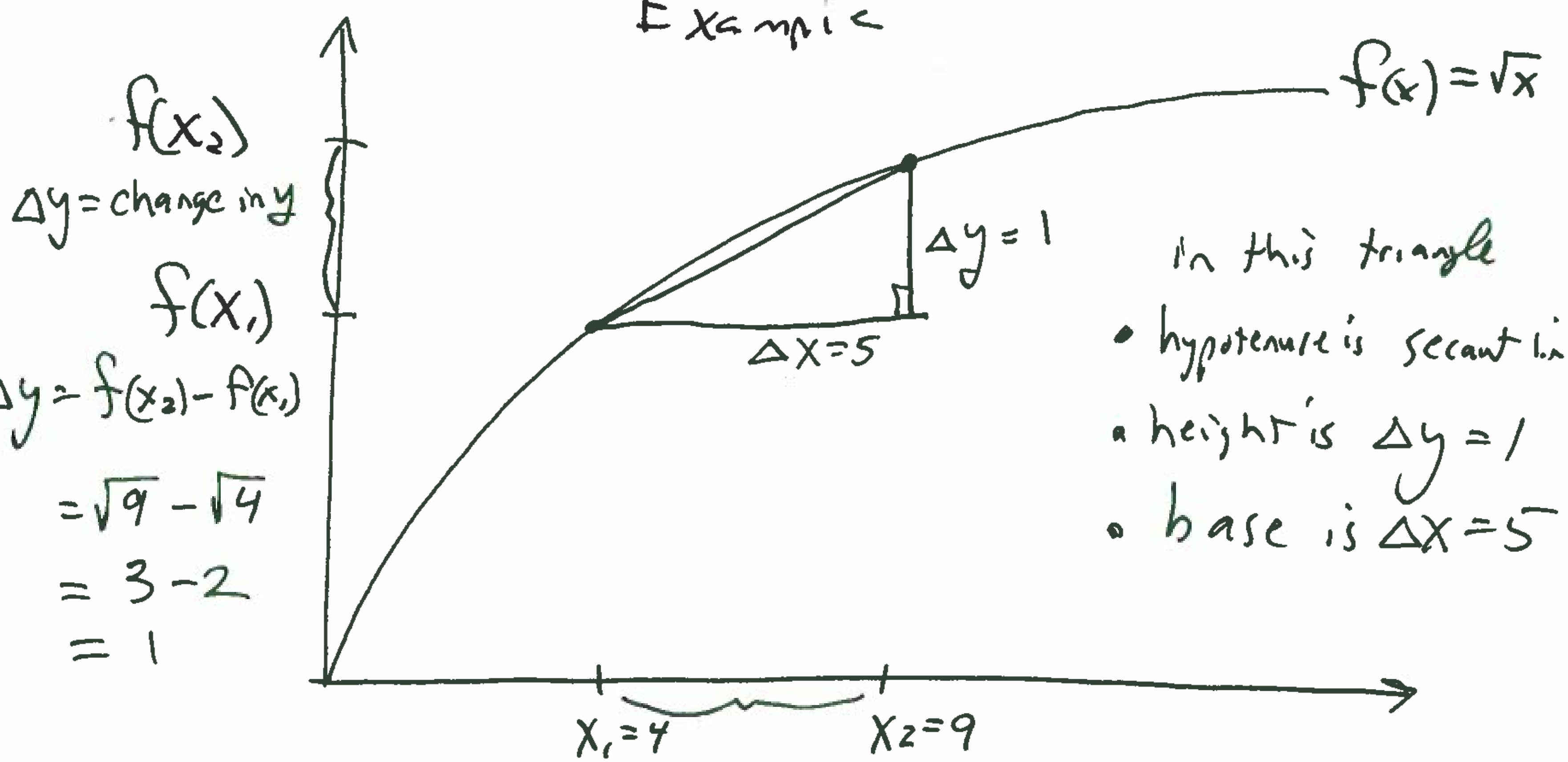
New topic: Differentials (Section 2.8)

Consider function $f(x)$ and known x values x_1, x_2.

Question: How much does $f(x)$ change when x changes from x_1 to x_2?

We'll discuss exact and approximate answers to this question.

Example



$\Delta y = f(x_2) - f(x_1)$

$= \sqrt{9} - \sqrt{4}$

$= 3 - 2$

$= 1$

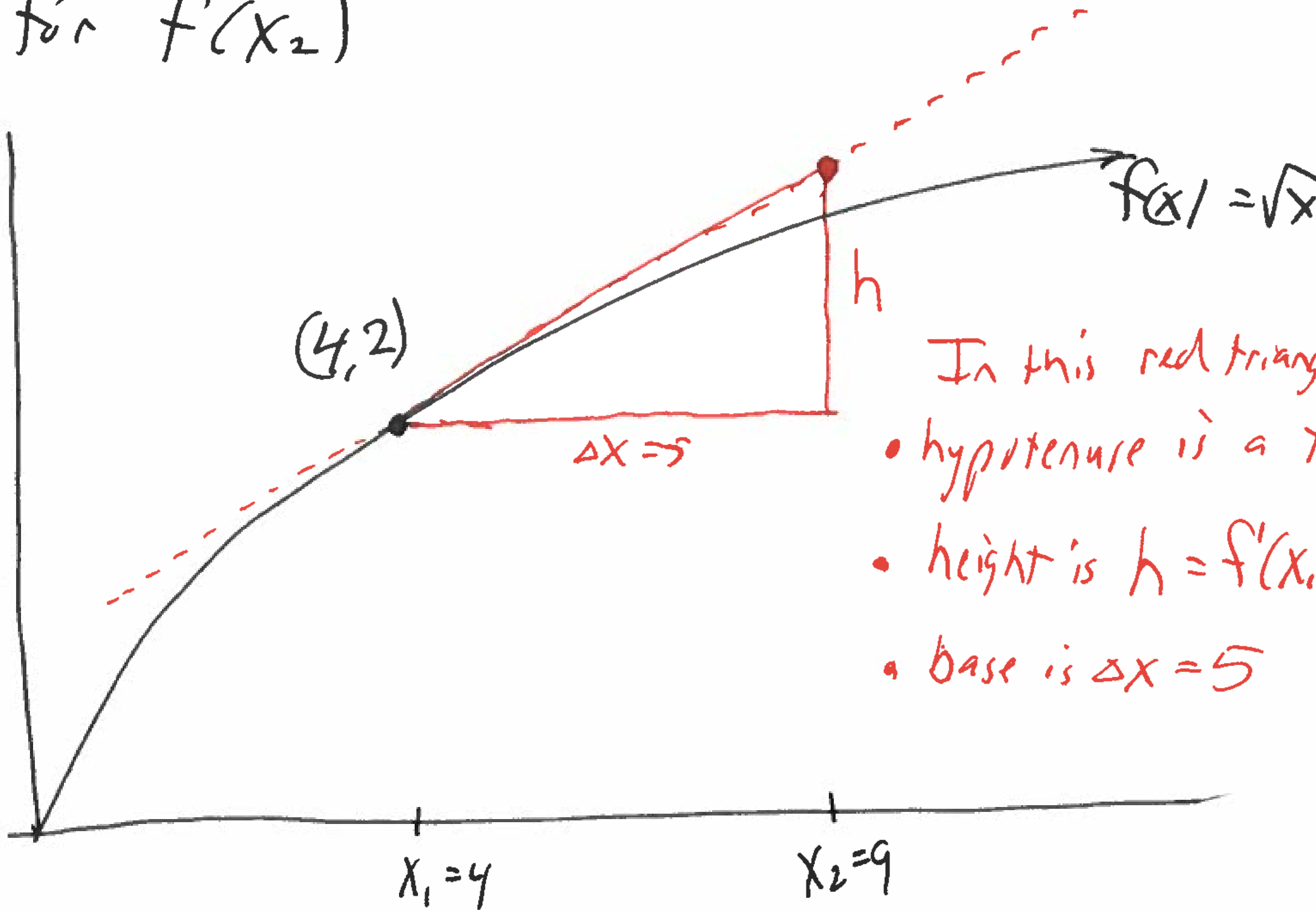
In this triangle

- hypotenuse is secant line
- height is $\Delta y = 1$
- base is $\Delta x = 5$

Δx = change in x $= x_2 - x_1 = 9 - 4 = 5$

There is another way to get a height.

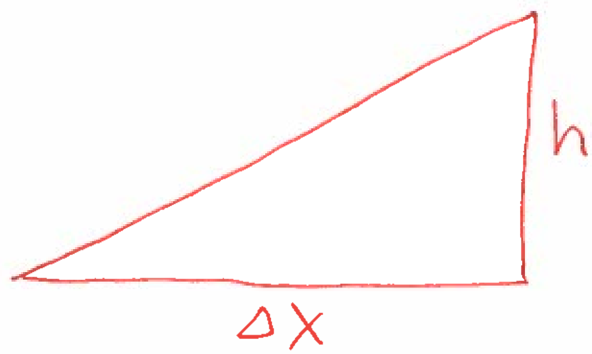
Use tangent line at x_1 to get an approximation for $f(x_2)$



In this red triangle

- hypotenuse is a tangent line
- height is $h = f'(x_1) \cdot \Delta x$
- base is $\Delta x = 5$

Figure out height h



Notice: Slope of red hypotenuse is $m = \frac{\text{rise}}{\text{run}} = \frac{h}{\Delta x}$

but the red line is the line tangent to graph of $f(x)$ at x_1, So its slope is $m = f'(x_1)$

So we get an equation

$$\frac{h}{\Delta x} = f'(x_1)$$

Solve for h by multiplying both sides by Δx

$$h = f'(x_1) \cdot \Delta x$$

This quantity has a name:

- the differential of $f(x)$ at $x_1 = 4$ when $\Delta x = 5$
- symbol $dy = f'(x_1) \cdot \Delta x$

$$= f'(4) \cdot 5$$

$$\boxed{\begin{array}{l}\text{the differential of } f(x) \text{ at } x_1 \text{ is} \\ \qquad dy = f'(x_1) \cdot \Delta x\end{array}}$$

For our example

$$dy = f'(4)\cdot \Delta x = f'(4)\cdot 5$$

our function $f(x) = \sqrt{x} = x^{1/2}$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{2\cdot 2} = \frac{1}{4}$$

$$dy = f'(4)\cdot \Delta x = \frac{1}{4}\cdot 5 = \frac{5}{4}$$

Notice!

Exact change	$\approx$	Approximate change
Δy	$\approx$	dy
$f(5) - f(4)$	$\approx$	$f'(4)\cdot\Delta x$
$\sqrt{9} - \sqrt{4}$	$\approx$	$\frac{1}{4}\cdot 5$
$3 - 2$	$\approx$	$\frac{5}{4}$
1	$\approx$	$\frac{5}{4}$
exact change		approx change