MATH 2301 Section 110 (Barsamian) Day 28 (Mon Mar 6)

Today Sections 3.2 & 3.3

Tomorrow Recitation: Section 3.3

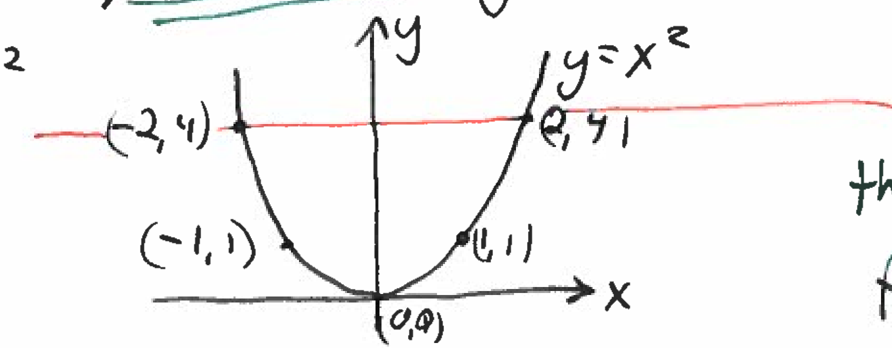
Wednesday: Section 3.4

Friday Exam X2 Covering Sections 2.3 through Chapter 3

Section 3.2 Inverse Functions and Logarithms

Consider starting with a given function $y =$ some stuff involving x

Example $y = x^2$



this is a function

Observe this graph fails the horizontal line test

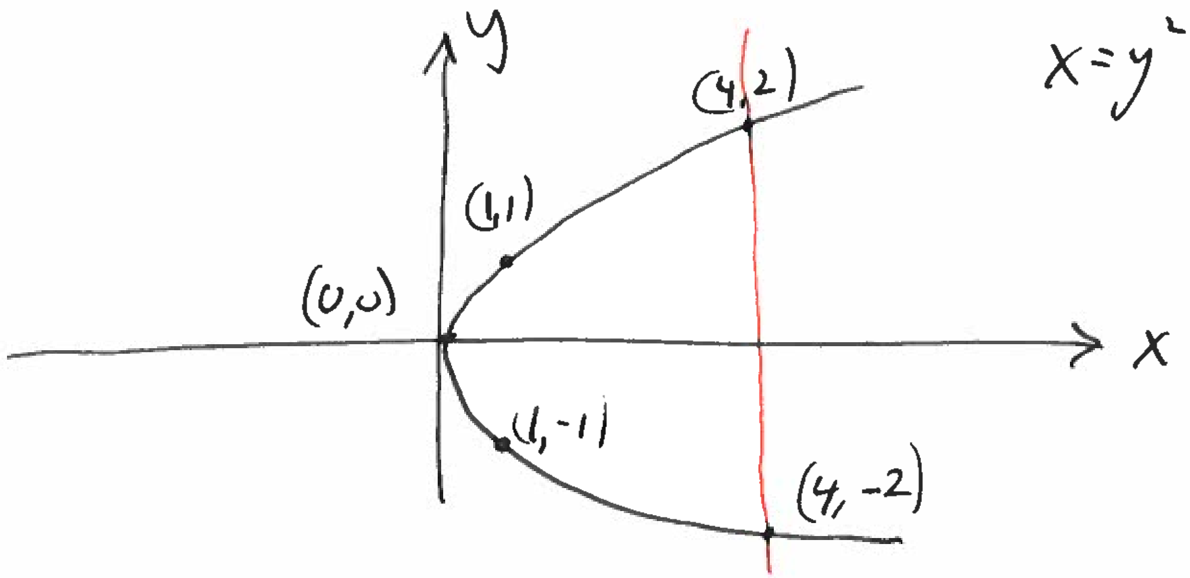
for some y values, there is more than one x value.

The function $y = x^2$ fails to be one-to-one.

What if we interchange $x \longleftrightarrow y$ in the equation

$$x = y^2$$

Then we would interchange $x \longleftrightarrow y$ for all the points on graph



Not the graph of a function because it

fails the Vertical Line Test

(for $x=4$, there are two y-values! Not allowed for a function)

flipped graph fails to be a function

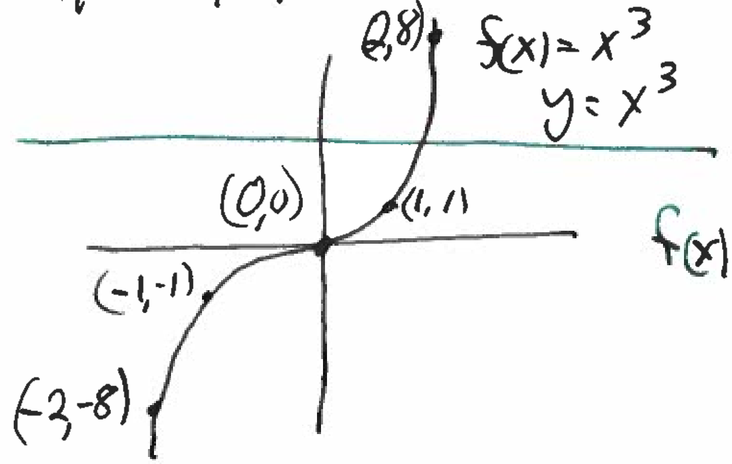
When a given function $f(x)$ is one-to-one (it passes the horizontal line test), the flipped graph will pass the vertical line test

(flipped graph will be the graph of a function).

Example

$f(x) = x^3$

$y = x^3$



passes horiz line test

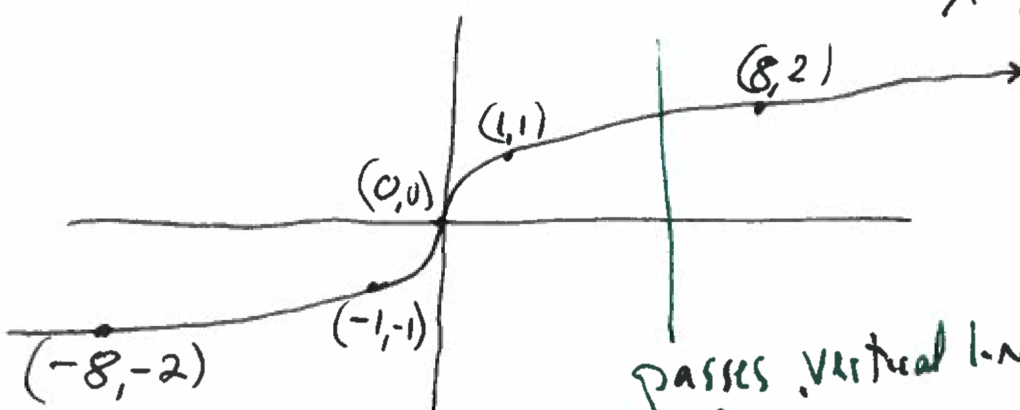
$f(x) = x^3$ is one-to-one

interchange x, y coordinates

$x = y^3$

can solve for y

$y = x^{1/3}$



passes vertical line test

the flipped graph is a function

The Inverse function

~~when~~ If a function $f(x)$ is one-to-one, then it is possible to define a new function, denoted by f^{-1}, called the inverse function for f, defined in the following way

$$\cancel{f^{-1}(y) = x \text{, means } f(x) = y}$$

$$f^{-1}(b) = a \quad \text{means} \quad f(a) = b$$

(b, a) is a point on graph of f^{-1}

(a, b) is a point on graph of f

Example for $f(x) = x^3$

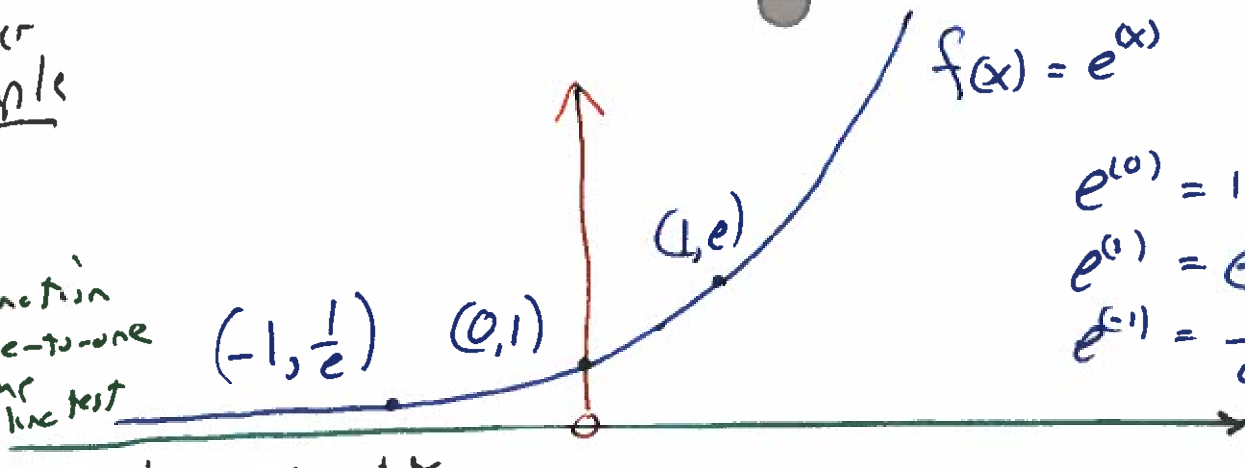
$f(2) = 2^3 = 8$ (2,8) is on graph of f

$f^{-1}(8) = 2$ (8, 2) is on graph of f^{-1}

$(8)^{1/3} = 2$

Another Example

$f(x) = e^{(x)}$



this function is one-to-one passes the horiz line test

horiz asymptote $y=0$

$e^{(0)} = 1$

$e^{(1)} = e$

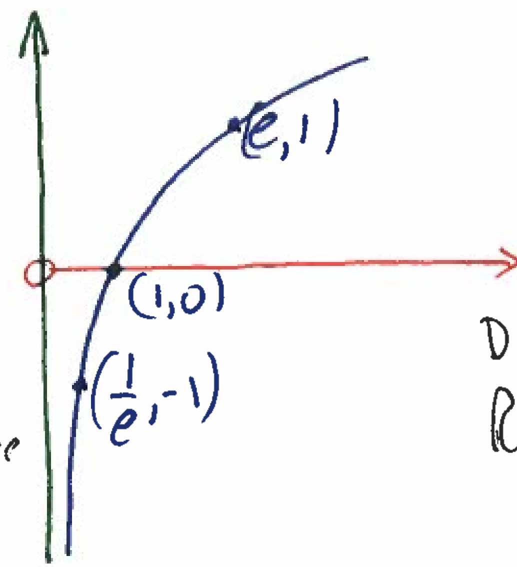
$e^{(-1)} = \frac{1}{e^1} = e$

Domain all real numbers

Range all positive numbers >0

~~So the graph of the inverse function ...~~

So if we interchange all x, y, the new graph will qualify to be called a function, called the inverse function



$y = \ln(x)$

$f^{-1}(x) = \ln(x)$

$\ln(e) = 1$

$\ln(1) = 0$

$\ln(\frac{1}{e}) = -1$

vertical asymptote $x=0$

Domain all positive numbers

Range: all real numbers

Cancellation equations

If f is a one-to-one function with domain A and range B then f^{-1} will be a function with domain B and range A

and these equations will be true ("cancellation equations")

$$f^{-1}(f(a)) = a$$ for every a in A the domain of f

and $$f(f^{-1}(b)) = b$$ for ever b in B the domain of f^{-1}

Example for $f(x) = x^3$ and its inverse function $f^{-1}(x) = x^{1/3}$

observe $f^{-1}(f(x)) = (x^3)^{1/3} = x^{3 \cdot \frac{1}{3}} = x^1 = x$

rule of exponents $(a^b)^c = a^{b \cdot c}$

$f^{-1}(f(2)) = (2^3)^{1/3} = (8)^{1/3} = 2$

Example $e^{(\ln(5))} = 5$

$e^{(\ln(137\pi))} = 137\pi$

$e^{\ln(-3)}$ = Does not exist! Because $\ln(-3)$ does not exist because domain of $\ln(x)$ is positive numbers

But the other order always works

$$\ln\left(e^{(5)}\right) = 5$$

$$\ln\left(e^{(137\pi)}\right) = 137\pi$$

$$\ln\left(e^{(-3)}\right) = -3 \quad \text{this time it works!}$$

<u>Conclusion</u>

$$e^{(\ln(x))} = x \quad \text{for all } x > 0$$

$$\ln\left(e^{(x)}\right) = x \quad \text{for all } x$$

Derivatives of logarithmic & exponential functions

Amazing fact Proven in book

$$\frac{d}{dx} \ln|x| = \frac{1}{x} \quad (\text{restricted to } x>0)$$

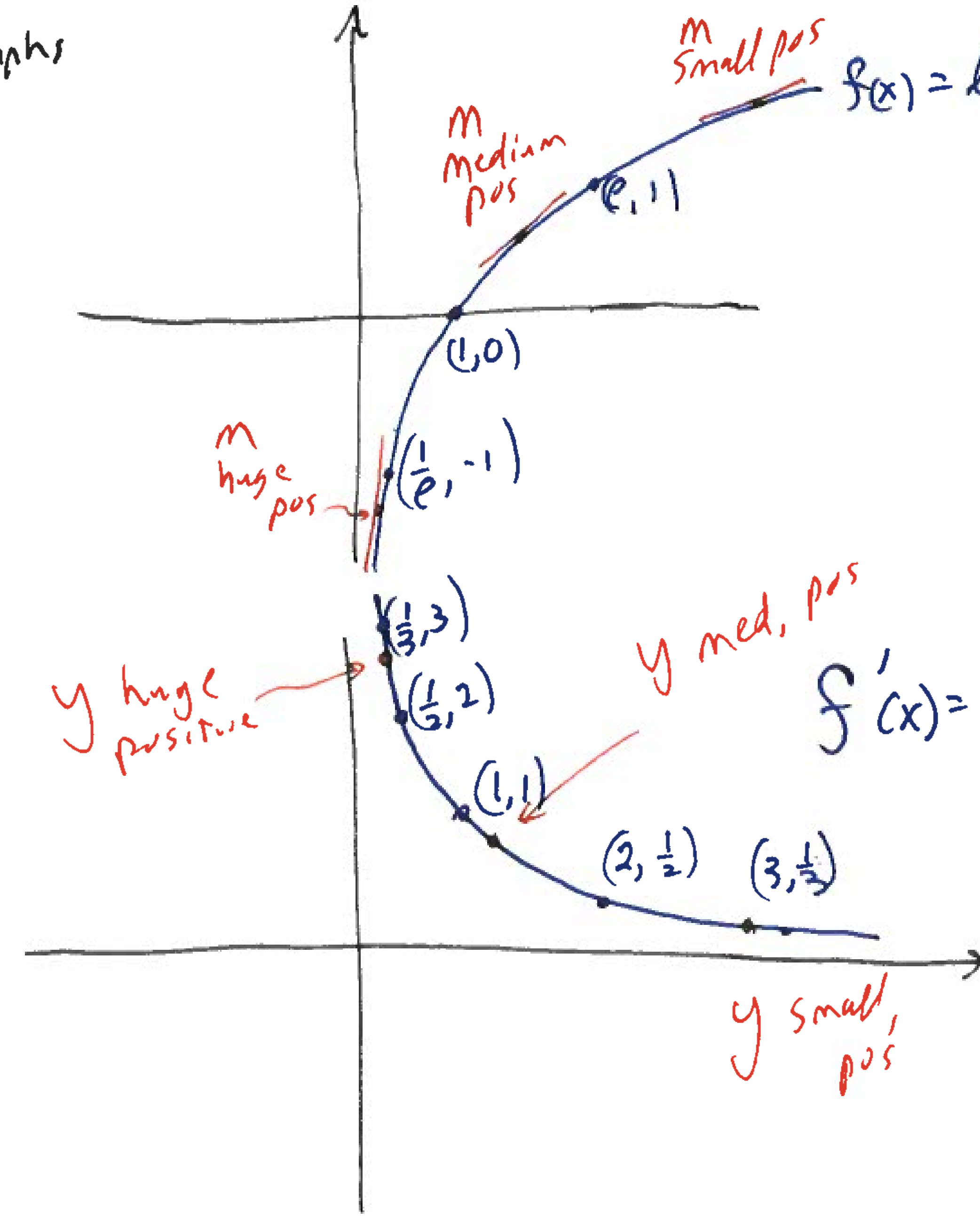
more generally

$$\frac{d}{dx} \log_b(x) = \frac{1}{x \cdot \ln(b)}$$

where here $b>0$ and $b \neq 1$

I won't prove this. But why is it believable?

Consider the graphs



$f'(x) = \frac{1}{x}$ restricted to $x>0$

Example

let $f(x) = 5\ln(x)$ find $f'(x)$

Solution $f'(x) = \frac{d}{dx} 5\ln(x) = 5\frac{d}{dx}\ln(x) = 5 \cdot \frac{1}{x} = \frac{5}{x}$

let $f(x) = 5\log_3(x)$ find $f'(x)$

Solution $f'(x) = \dots = 5\frac{d}{dx}\log_3(x) = 5 \cdot \frac{1}{x\ln(3)} = \frac{5}{x\ln(3)}$

Example

$f(x) = \ln(x^7)$ find $f'(x)$

Two possible solutions

method 1 rewrite $f(x) = \ln(x^7) = 7\ln(x)$

then find $f'(x) = \frac{d}{dx} 7\ln(x) = 7\frac{d}{dx}\ln(x) = \frac{7}{x}$

method 2 Chain rule

$$\frac{d}{dx}\ln(x^7) = \frac{d}{dx}\,\text{outer}(\text{inner}(x))$$

$$= \text{outer}'(\text{inner}(x)) \cdot \text{inner}'(x)$$

$$= \frac{1}{(x^7)} \cdot 7x^6$$

$$= \frac{7x^6}{x^7} = \frac{7}{x}$$

Chain rule details

$\text{inner}(x) = x^7$

$\text{inner}'(x) = 7x^6$

$\text{outer}(\ ) = \ln(\ )$

$\text{outer}'(\ ) = \frac{1}{(\ )}$

Another fact from book

$$\frac{d}{dx} e^{(x)} = e^{(x)}$$

$$\frac{d}{dx} b^{(x)} = b^{(x)} \cdot \ln(b) \qquad \text{when } b>0,\ b \neq 1$$

Example

$$f(x) = e^{(7x^2-3x+5)} \quad \text{find } f'(x)$$

Solution

$$f'(x) = \frac{d}{dx} e^{(7x^2-3x+5)}$$

$$= \frac{d}{dx} \text{outer}(\text{inner}(x))$$

$$= \text{outer}'(\text{inner}(x)) \cdot \text{inner}'(x)$$

$$= e^{(7x^2-3x+5)} \cdot (14x-3)$$

Chain Rule Details

$\text{inner}(x) = 7x^2 - 3x + 5$

$\text{inner}'(x) = 14x - 3$

$\text{outer}(\;) = e^{(\;)}$

$\text{outer}'(\;) = e^{(\;)}$