MATH 2301 Section 110 (Barsamian) Day 30 (Wed Mar 8)

Today Section 3.4 Exponential Growth & Decay

Review stuff

- one more example involving differentials
- Quiz Q5 problems

Friday Exam X2

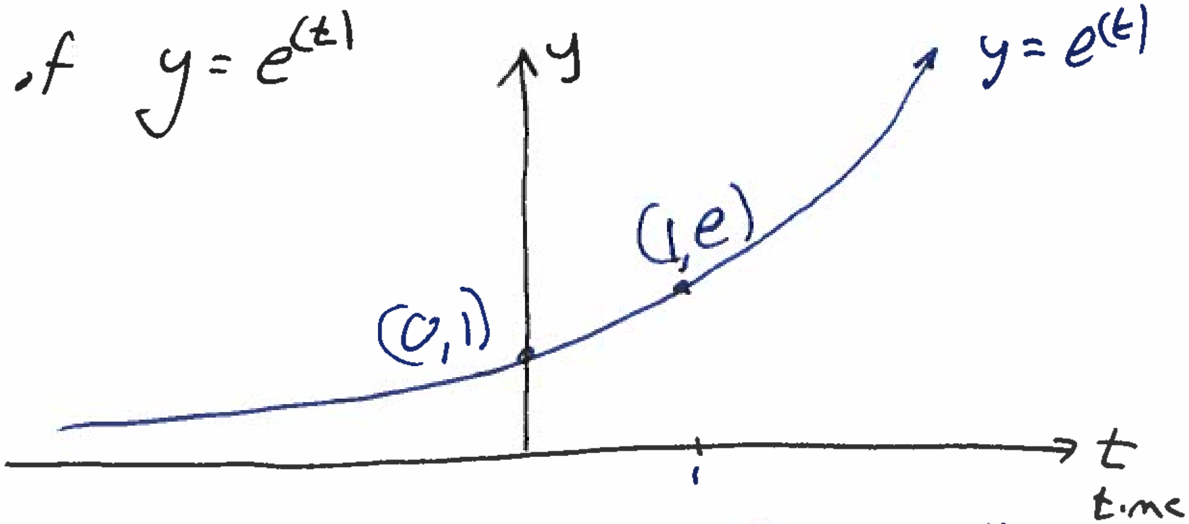
Covers 2.3, 2.4, 2.5, 2.6, 2.7, 2.8

3.1, 3.2, 3.3, 3.4

- I'll be posting X2 Info on Calendar on Web Page Today.

Section 3.4 Exponential Growth and Decay

Recall graph of $y = e^{(t)}$

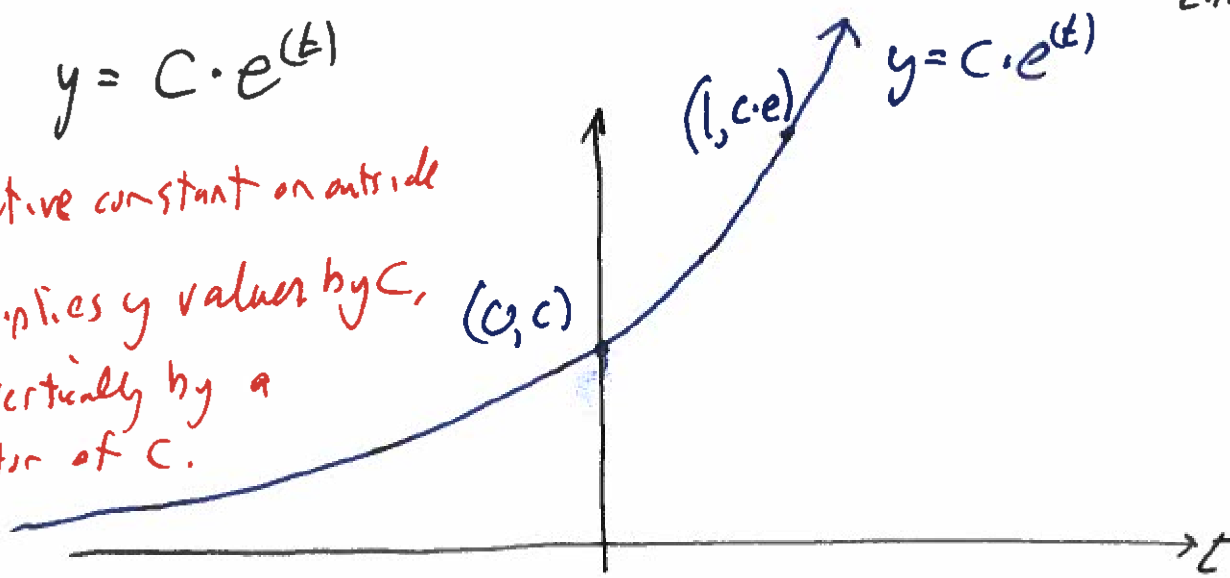


graph of $y = C \cdot e^{(t)}$

multiplicative constant on outside

Multiplies y values by C,

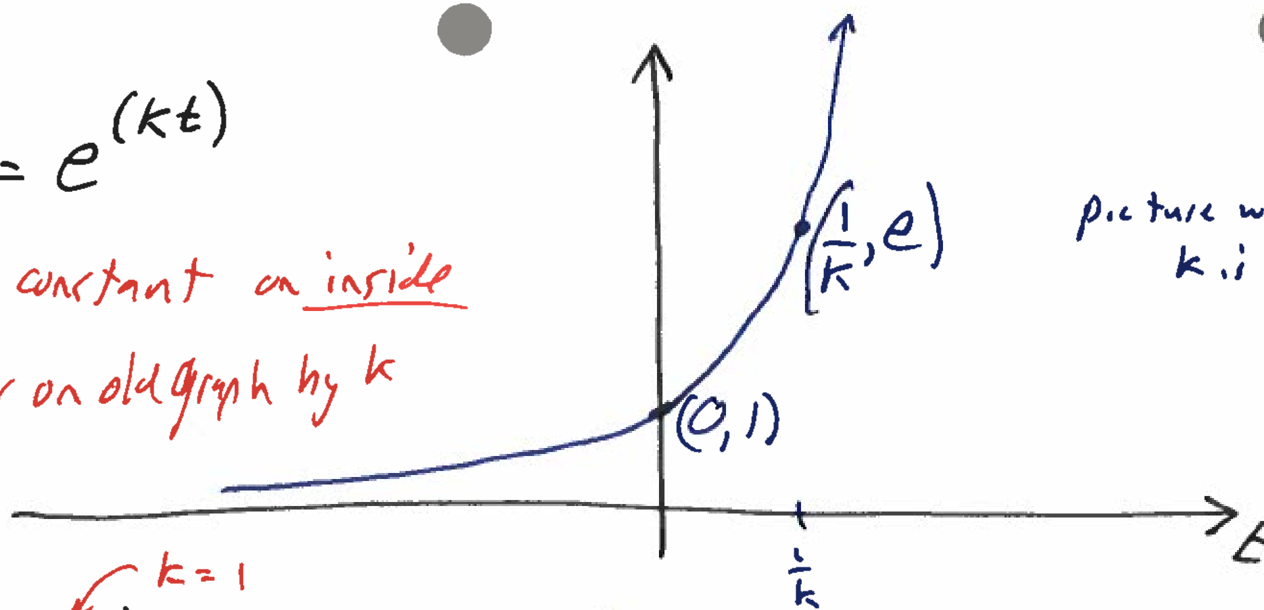
Stretches vertically by a factor of C.



graph of $y = e^{(kt)}$

multiplicative constant on inside

divide x values on old graph by k



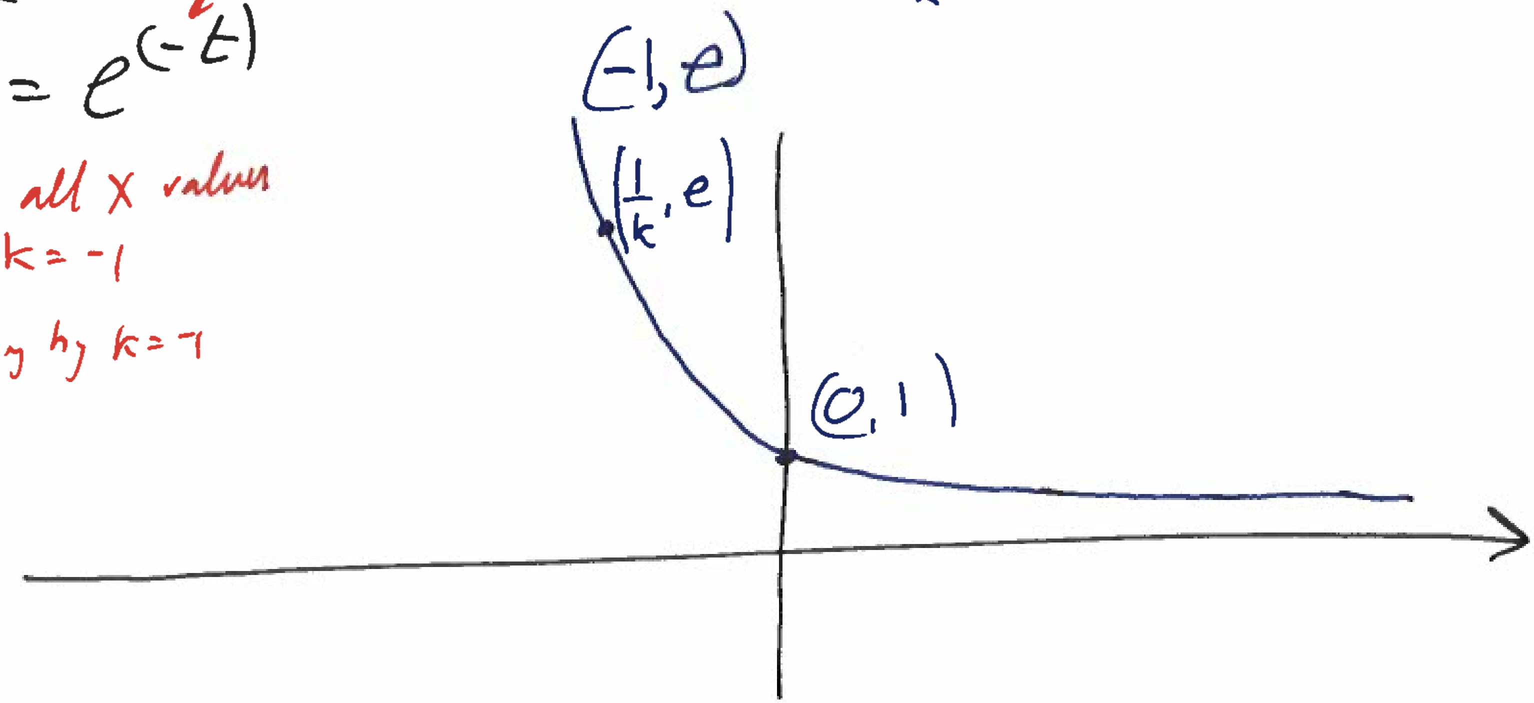
picture when k is positive

graph of ~~graph~~ $y = e^{(-t)}$

k = 1

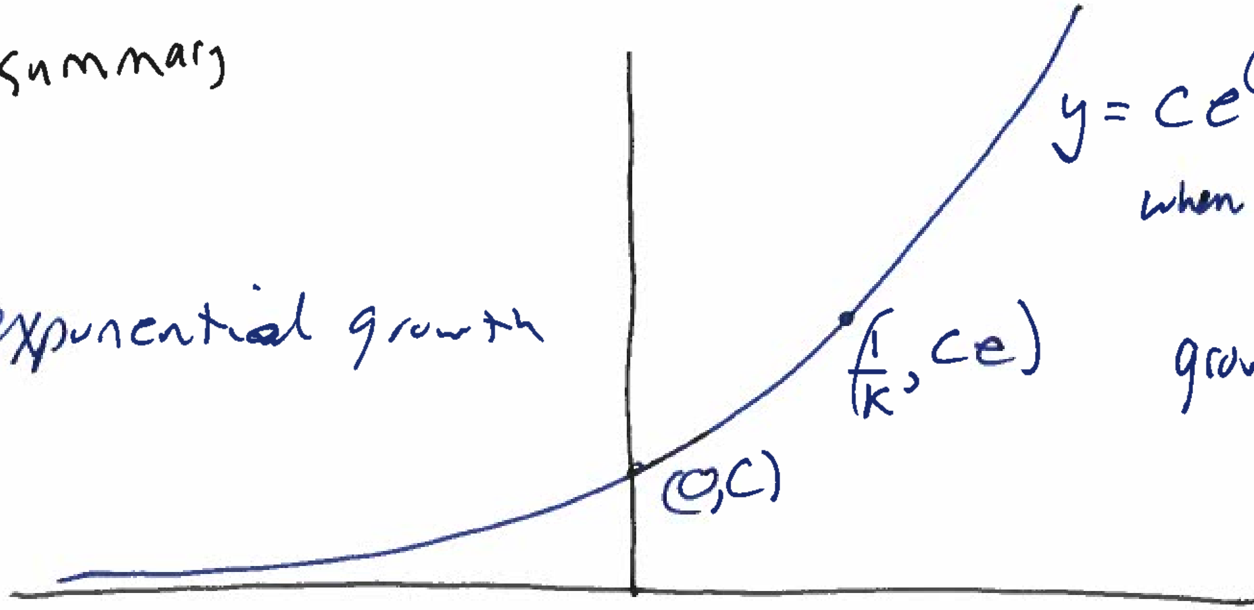
divide all x values by k = -1

Same as multiplying by k = -1



In summary

exponential growth

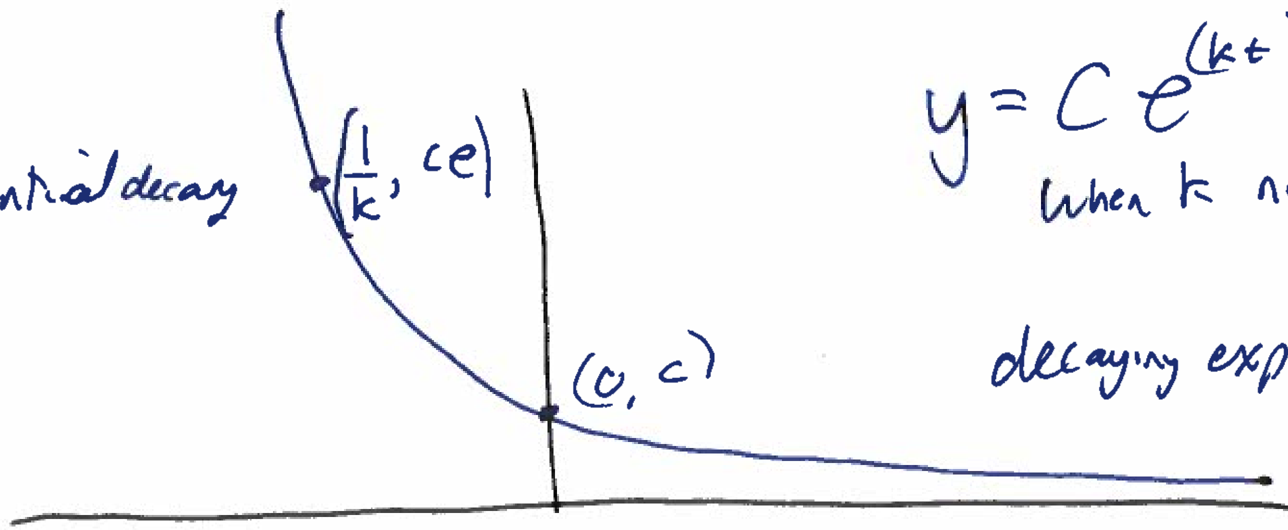


$y = Ce^{(kt)}$

when k is positive

growing exponential

exponential decay



$y = Ce^{(kt)}$

when k negative

decaying exponential

Question Suppose ~~$f(t) = Ce^{k}$~~

$$f(t) = Ce^{(kt)}$$

How do we find $f'(t)$?

$$f'(t) = \frac{d}{dt}\left(\underline{c}\,e^{(kt)}\right)$$

constant multiple rule

$$= \underline{c}\,\frac{d}{dt} e^{(kt)}$$

$$= c \cdot \frac{d}{dt}\, outer(inner(t))$$

chain rule

$$= C \cdot outer'(inner(t)) \cdot inner'(t)$$

$$= c \cdot e^{(kt)} \cdot k$$

$$= k \cdot \underbrace{c \cdot e^{(kt)}}_{\text{this is just } f(t)}$$

$$= k \cdot f(t)$$

chain rule details

$inner(t) = kt$

$inner'(t) = k$

$outer(\ ) = e^{(\ )}$

$outer'(\ ) = e^{(\ )}$

We did two important things here

$$\boxed{\text{New Derivative Rule: } \quad \frac{d}{dx} e^{(kx)} = k \cdot e^{(kx)}}$$

~~When~~ If $f(t) = c\,e^{(kt)}$

then $f'(t) = k \cdot c\,e^{(kt)} = k \cdot f(t)$

the rate at which f is growing $= k \cdot f$

~~to~~ the rate at which f is growing is proportional to f

the constant k is the constant of proportionality.

Relative Growth Rates

or consider this ratio

$$\frac{\text{growth rate}}{\text{size of the function}}$$

$$\frac{f'(t)}{f(t)} = \frac{k \cdot f(t)}{f(t)} = k$$

relative growth rate $= k$

The significance of the constant k

~~6~~ 7(a)

<u>Big fact</u> the <u>only functions</u> $f(t)$ that satisfy

$$\frac{d}{dt} f(t) = k \cdot f(t)$$

are functions of the form

$$f(t) = C e^{(kt)}$$

These functions have constant relative growth rate k

What is the significance of the number C in the formula $f(t) = Ce^{(kt)}$?

Notice:

Value of f at time $t=0$ $= f(0) = Ce^{(k\cdot 0)} = Ce^0 = C\cdot 1 = C$

$$\boxed{\text{initial value} = C}$$

Other symbols are sometimes used, because of this

$$f(t) = f_0 e^{(kt)}$$

↑ symbol for the constant that is the initial value

Examples

3.4 #1 A population develops with a constant relative growth rate of 0.9 per member per day.

On day zero, the population is 10 members.

Find the population size after 7 days.

Solution

The red underlined phrase tells us that the population has this formula

$$f(t) = f_0 e^{(kt)}$$

with $k = 0.9$

The green underlined phrase tells us that

$$f_0 = 10$$

So population is described by

$$f(t) = 10\, e^{(0.9t)}$$

Find population after 7 days:

Solution

$$f(7) = 10\, e^{((0.9)7)} \approx 5446$$

exact

approximate

3.4 #3

Bacteria culture ~~grow~~ has 500 cells and grows at a rate proportional to its size

After an hour, the population is 600.

(a) Find an expression for number of bacteria after t hours.

Solution

we know
$$f(t) = f_0 e^{(kt)}$$

We need to figure out values for f_0 and for k.

We know initial value $= 500 = f_0$

$$\text{So } f(t) = 500\, e^{(kt)}$$

We still need to figure out k

We know population after 1 hour is 600

$$f(1) = 600$$

turn this around

$$600 = f(1) = 500\,e^{(k\cdot 1)} = 500e^{k}$$

We need to find k

Solve for k

$$\left(\frac{600}{500}\right) = e^{(k)}$$

$$\ln\left(\frac{600}{500}\right) = \ln\left(e^{(k)}\right) = k$$

$$\text{So } f(t) = 500\cdot e^{\ln\left(\frac{600}{500}\right)\cdot t}$$

Observe this can be simplified

$$f(t) = 500\, e^{\ln\left(\frac{600}{500}\right)\cdot t}$$

$$= 500\left(e^{\ln\left(\frac{600}{500}\right)}\right)^{t}$$

$a^{b\cdot c} = (a^b)^c$

$$= 500\cdot\left(\frac{600}{500}\right)^{t} = 500\left(\frac{6}{5}\right)^{t}$$

(b) Find population ~~#~~ after 7 hours

Solution $f(7) = 500\cdot\left(\frac{6}{5}\right)^7 = 500\cdot\frac{6}{5}\cdot\frac{6}{5}\cdot\frac{6}{5}\cdot\frac{6}{5}\cdot\frac{6}{5}\cdot\frac{6}{5}\cdot\frac{6}{5}$