MATH 2301 Section 110 (Barsamian) Meeting #35 (Fri Mar 24, 2023) ①

Today: Section 4.2 The Mean Value Theorem (MVT)

First: Rolle's theorem

If f satisfies these three requirements (the hypotheses)

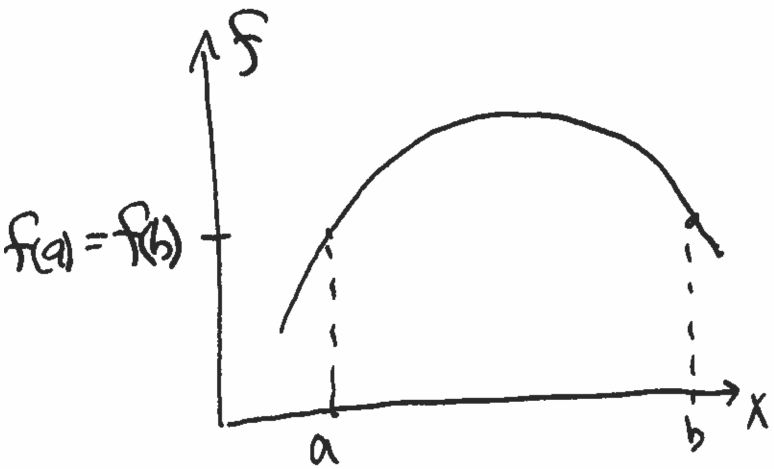
(1) f is continuous on $[a, b]$

(2) f is differentiable on the open interval (a, b)

(3) $f(a) = f(b)$

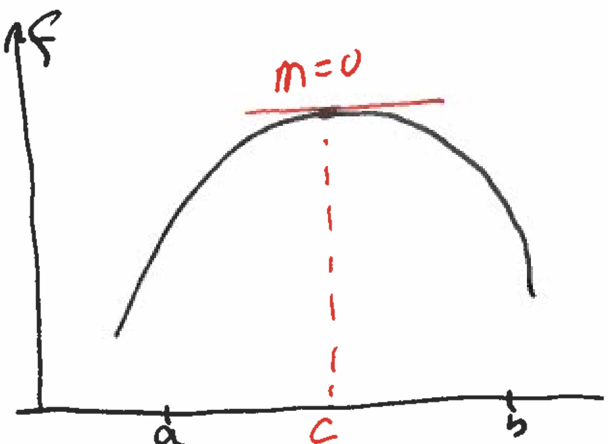
Then there is a number $x = c$ with $a < c < b$ such that $f'(c) = 0$

the conclusion



Picture of the hypotheses

$\xrightarrow{\text{Rolle's Theorem}}$



Picture of the conclusion

Example ~~4.3#3~~ 4.2#3 Let $f(x) = \sqrt{x} - \frac{1}{3}x$ on the interval $[0,9]$ ②

$a=0$ $b=9$

ⓐ Verify that f satisfies the hypotheses of Rolle's theorem

Solution

✓ f is continuous on $[0,9]$ (f is actually continuous on whole interval $[0,\infty)$)

Find $f'(x) = \frac{d}{dx}\left(\sqrt{x} - \frac{1}{3}x\right) = \frac{d}{dx}\left(x^{1/2} - \frac{1}{3}x\right) = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{3} =$

✓ $f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{3}$ f' is continuous on $(0,\infty)$

So it is certainly continuous on $(0,9)$

✓ $f(a) = f(0) = \sqrt{0} - \frac{1}{3}\cdot 0 = 0$ these match

$f(b) = f(9) = \sqrt{9} - \frac{1}{3}\cdot 9 = 3 - 3 = 0$

So Rolle's theorem tells us that there exists $x=c$ with

$0 < c < 9$ such that $f'(c) = 0$

(But Rolle's theorem doesn't tell us where c is!)

(3)

(b) Find ~~the value~~ a value $x=c$ that satisfies $f'(c)=0$

Solution

$$0 = f'(c) = \frac{1}{2\sqrt{x}} - \frac{1}{3}$$

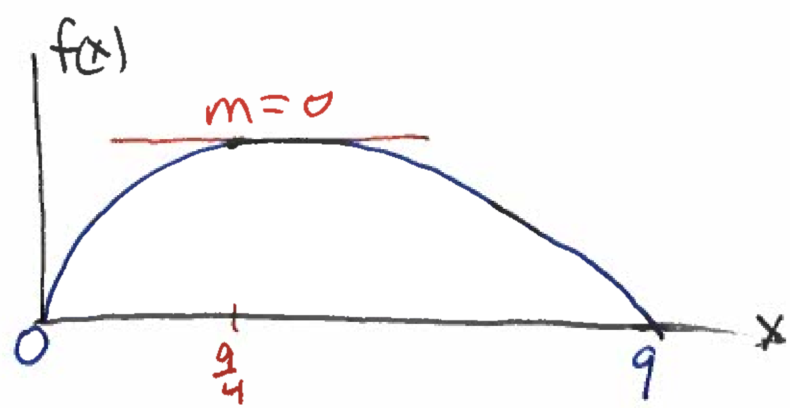
$$\frac{1}{3} = \frac{1}{2\sqrt{x}}$$

$$3 = 2\sqrt{x}$$

$$\frac{3}{2} = \sqrt{x}$$

$$\frac{9}{4} = x$$

$$x = 2.25$$



Example 2 Illustrating that <u>all</u> <u>three</u> hypotheses must be satisfied in order to guarantee the the conclusion is true.

Let $f(x) = x^{2/3}$ on interval $[-8, 8]$

Observe: ✓ f is continuous on $[-8, 8]$

✗ f is <u>not</u> differentiable on $[-8, 8)$

$$f'(x) = \frac{d}{dx} x^{2/3} = \frac{2}{3} x^{-1/3} = \frac{2}{3x^{1/3}}$$

~~Be~~ $f'(0)$ does not exist

✓ $f(-8) = (-8)^{2/3} = \left((-8)^{1/3}\right)^2 = (-2)^2 = 4$ these match

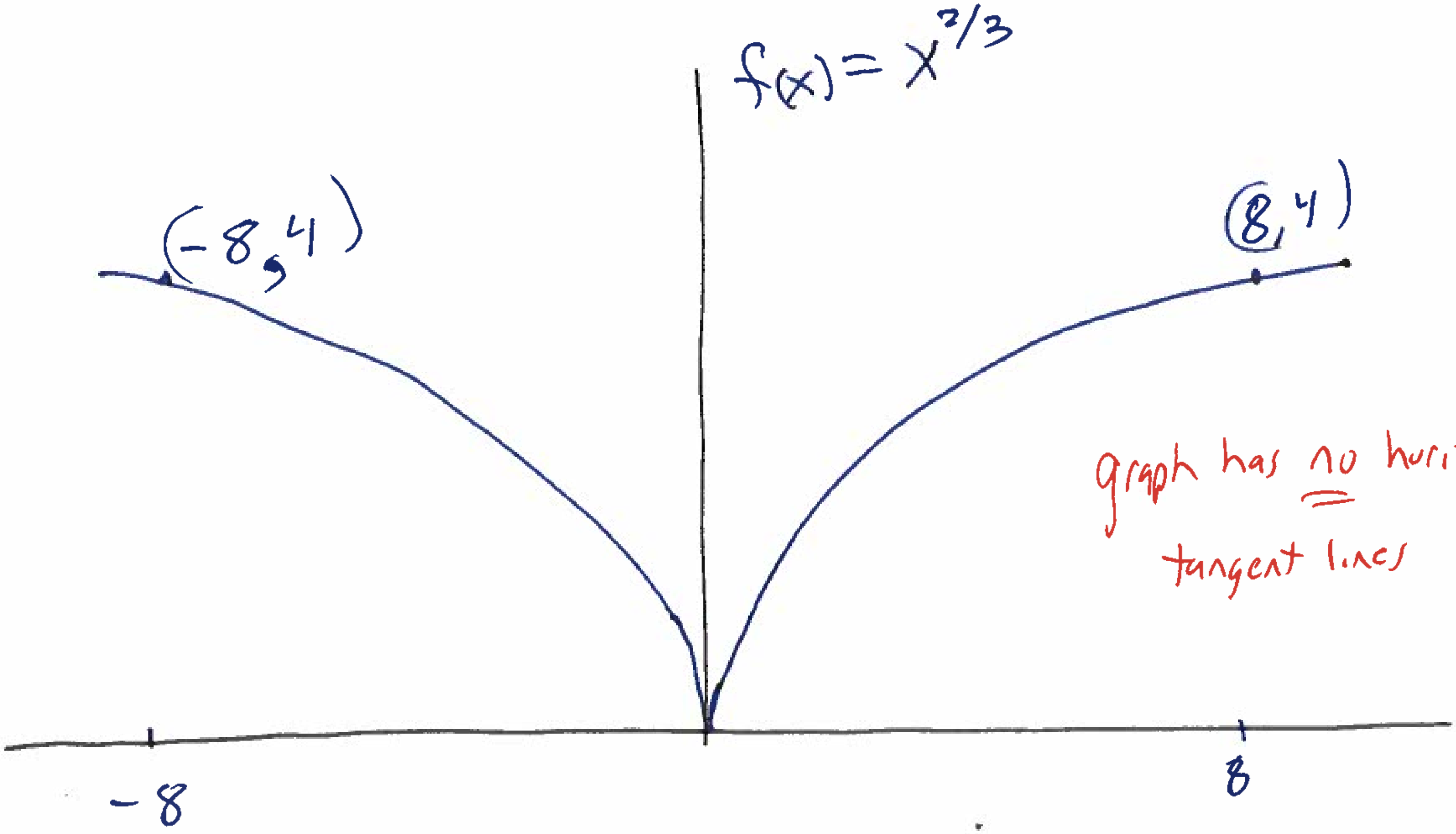
$f(8) = (8)^{2/3} = 4$

Not all hypotheses are satisfied

~~Observe~~ Also observe that the conclusion is <u>not</u> true, because $f'(x) = \frac{2}{3x^{1/3}}$ will <u>never</u> equal zero!

because its numerator will never be zero

(5)



graph has no horizontal tangent lines

The Mean Value Theorem (MVT)

If function f satisfies these hypotheses

f continuous on $[a,b]$

f differentiable on (a,b)

Then there exists an $x=c$ such that $a<c<b$

and

$$\underbrace{f'(c)}_{\text{slope } m \text{ of line tangent at } x=c} = \underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{secant line slope}}$$

Example 4.2 #13 Consider $f(x)=\sqrt{x}$ on interval $[0,4]$

(a) Show that hypotheses of MVT are satisfied

✓ $f(x)=\sqrt{x}$ is actually continuous on interval $[0,\infty)$,

So it is continuous on interval $[0,4]$

✓ $f'(x)=\frac{d}{dx}\sqrt{x}=\frac{d}{dx}x^{1/2}=\cdots=\frac{1}{2\sqrt{x}}$ is continuous on $(0,\infty)$

So it is continuous on $(0,4)$

So MVT says there exists an $x=c$ with $0<c<4$

Such that $f'(c)=\frac{f(4)-f(0)}{4-0}=\frac{\sqrt{4}-\sqrt{0}}{4-0}=\frac{2}{4}=\frac{1}{2}$

(b) Find the $x=c$ that works

$$f'(c) = \frac{1}{2}$$

$$\frac{1}{2\sqrt{c}} = \frac{1}{2}$$

$$2\sqrt{c} = 2$$

$$\sqrt{c} = 1$$

$$c = 1$$

Check: $f'(c) = \frac{1}{2\sqrt{1}} = \frac{1}{2}$ ✓