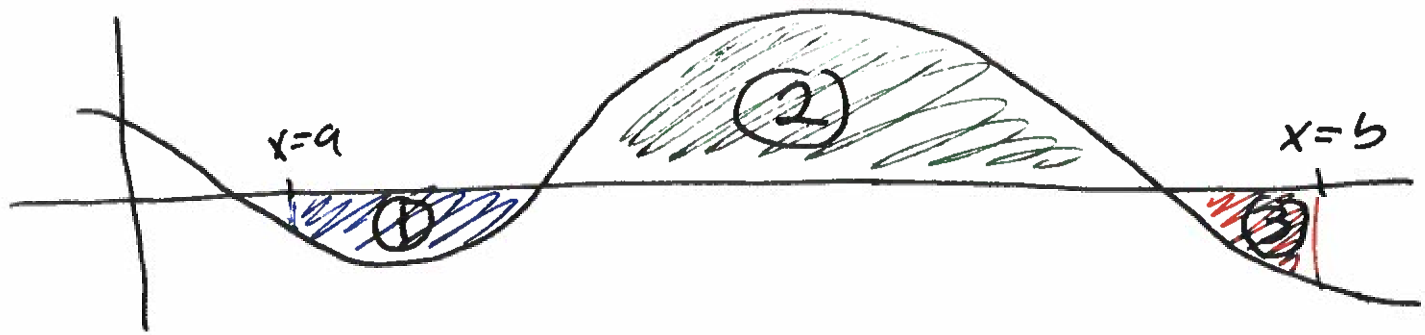
MATH 2301 Section 110 (Barsamian) Meeting #44 (Mon Apr 10, 2023)

Section 5.1 Areas

Area between graph of a function and the x axis from $x=a$ to $x=b$



unsigned area (USA) = 1 + 2 + 3

Signed area (SA) = −1 + 2 − 3

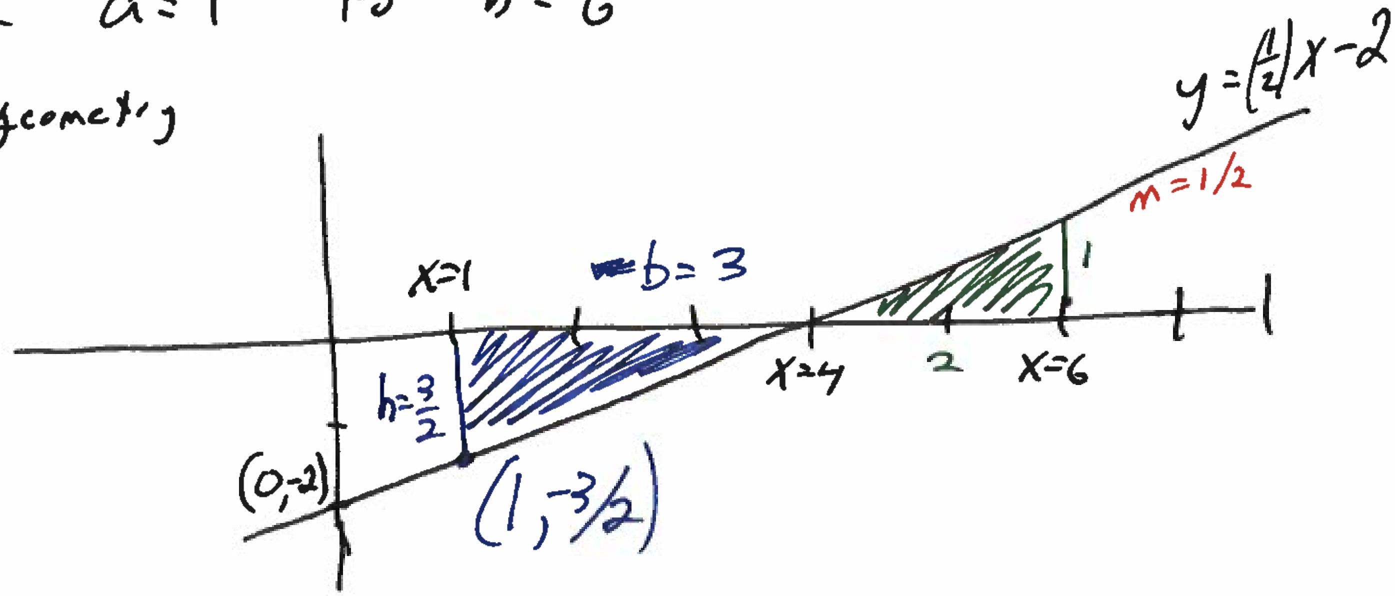
For signed area, regions below the x axis get a minus sign.

Example Let $f(x) = \left(\frac{1}{2}\right)x - 2$

$$f(x) = \left(\frac{1}{2}\right)x - 2$$

Find signed area between graph of f and x axis from $a = 1$ to $b = 6$

Solution: Use geometry



$f(1) = \left(\frac{1}{2}\right)1 - 2 = -\frac{3}{2}$

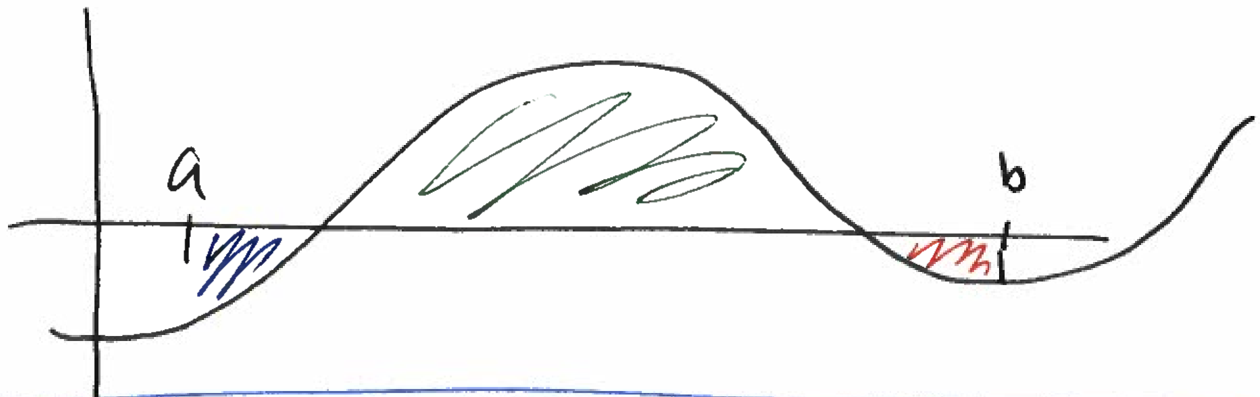
Using Geometry

Blue area $= A = \frac{1}{2} b \cdot h = \frac{1}{2}(3) \cdot \left(\frac{3}{2}\right) = \frac{9}{4}$

Green area $= A = \frac{1}{2} b \cdot h = \frac{1}{2}(2)(1) = 1$

Signed Area $= SA = -\frac{9}{4} + 1 = -\frac{5}{4}$

What do we do to compute signed area for a <u>curvy region</u>, not made up of basic ~~geo~~ geometric shapes?



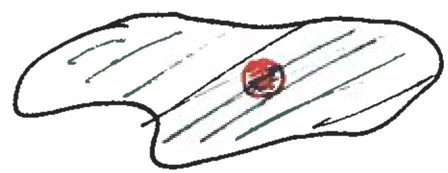
The <u>area problem</u> Two Questions

- How do we even <u>define</u> what is meant by area of a curvy region?
- How do we compute the <u>value</u> of that area?

Rather than attacking the area problem directly, we'll attack a different, simpler problem.

First, though, articulate two requirements that "area", whatever that means, ought to satisfy.

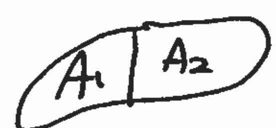
- If the region contains a disc, the area of the region must be a positive number



Green area > 0

a line has zero area

- Area is "additive"   Area = Sum of parts



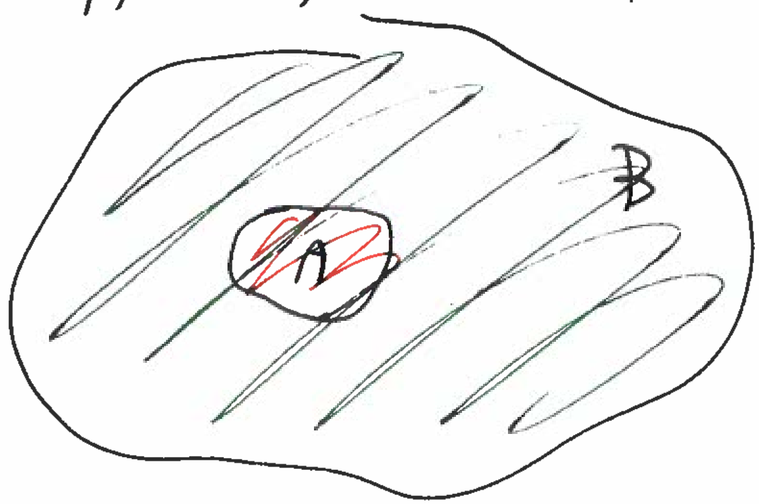
Total area = $A_1 + A_2$

As a consequence

If Region A is a proper subset of Region B

the

Area A < Area B

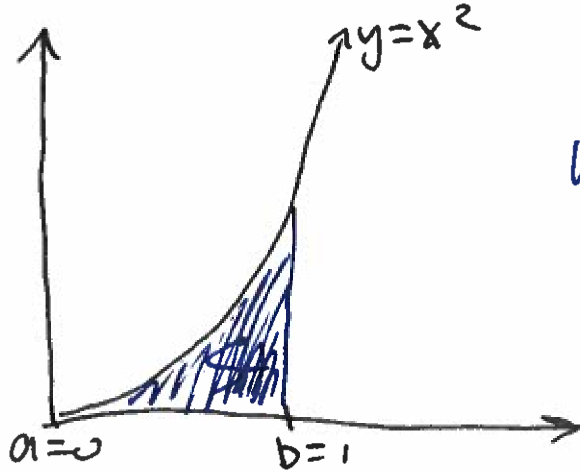


Simpler Area Problem that we will attack first

Finding ~~___~~ approximations for unknown, curvy areas by using simpler regions made up of geometric shapes.

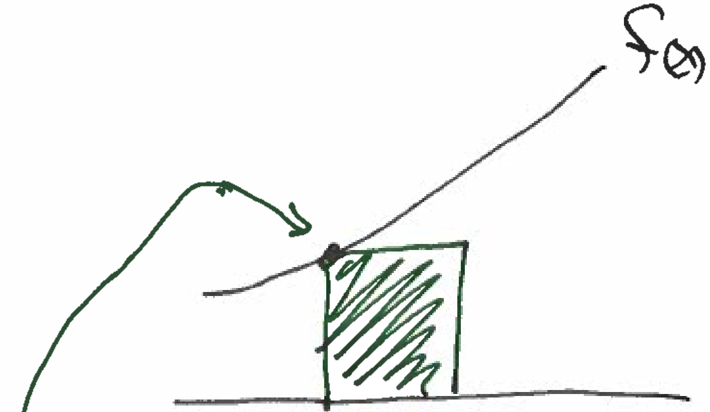
Example Let $f(x) = x^2$

Find approximations for the signed area SA of the region between graph of $f(x)$ and x axis on interval from $x=0$ to $x=1$



unknown (and <u>undefined</u>) SA

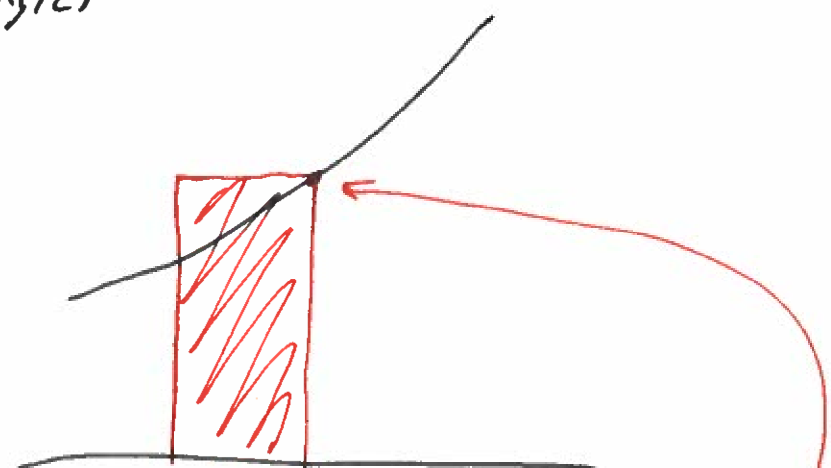
will use "Left rectangles"



sits on x axis

touches graph of f(x) at rectangle's <u>left</u> <u>edge</u>

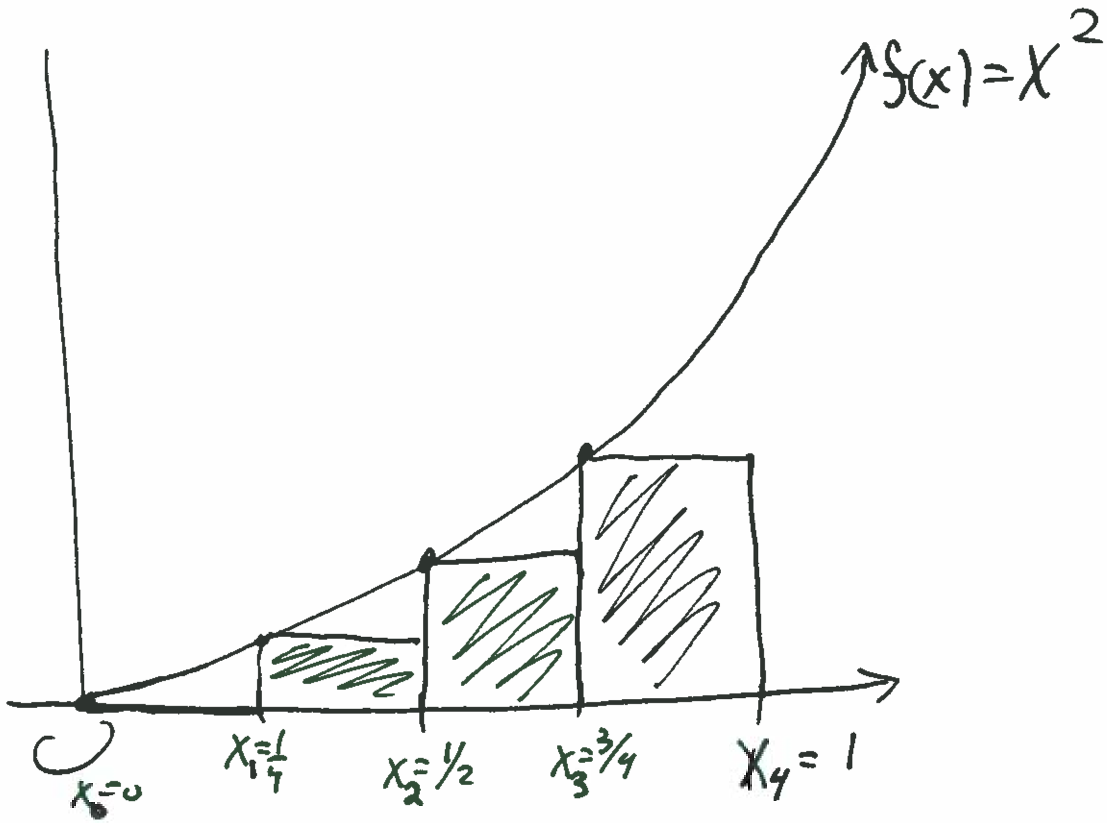
we'll also use "right rectangles"



sits on x axis

touches graph of f(x) at rectangle's right edge

L_4 = Left Riemann Sum ~~w~~ with 4 rectangles
"Ree-mon"
= total area of 4 equal width left rectangles
that sit on the same interval $x=0$ to $x=1$



L_4 = sum of those four areas

$= w_1 \cdot h_1 + w_2 h_2 + w_3 \cdot h_3 + w_4 h_4$

all of the widths are $\Delta x = \frac{1-0}{4} = \frac{b-a}{n} = \frac{1}{4}$

$L_4 = \Delta x h_1 + \Delta x h_2 + \Delta x \cdot h_3 + \Delta x h_4$

$= (h_1 + h_2 + h_3 + h_4) \cdot \Delta x$

$= (f(x_0) + f(x_1) + f(x_2) + f(x_3)) \Delta x$

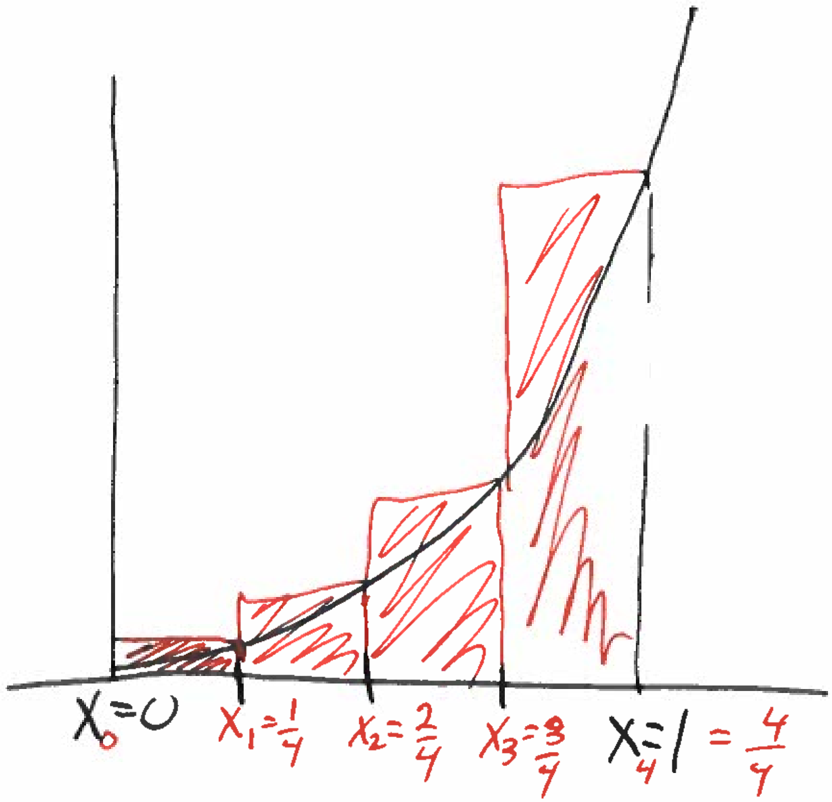
$= \left((0)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{2}{4}\right)^2 + \left(\frac{3}{4}\right)^2\right) \Delta x$

use $\Delta x = 1/4$

$= \left(0 + \frac{1}{16} + \frac{4}{16} + \frac{9}{16}\right) \frac{1}{4}$

$= \cancel{\frac{14}{64}} \left(\frac{14}{16}\right) \cdot \frac{1}{4} = \frac{14}{64} = \cancel{\frac{14}{32}} \frac{7}{32} = .21875$

R_4 = Riemann Sum with 4 Right Rectangles



R_4 = sum of areas of those four red rectangles

all rectangles have same width $\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4}$

the heights of the rectangles are

$$f(x_1),\ f(x_2),\ f(x_3),\ f(x_4)$$

$$R_4 = \left(f(x_1) + f(x_2) + f(x_3) + f(x_4) \right) \cdot \Delta x$$

$$= \left(\left(\tfrac{1}{4}\right)^2 + \left(\tfrac{2}{4}\right)^2 + \left(\tfrac{3}{4}\right)^2 + \left(\tfrac{4}{4}\right)^2 \right) \cdot \frac{1}{4}$$

$$= \left(\frac{1}{16} + \frac{4}{16} + \frac{9}{16} + \frac{16}{16} \right) \frac{1}{4}$$

$$= \left(\frac{\overset{30}{\cancel{29}}}{16} \right) \frac{1}{4} = \frac{30}{64} = \frac{15}{32} = .46875$$

So we get this area sandwich

$$L_4 < \begin{matrix}\text{unknown}\\\text{blue}\\\text{"area"}\end{matrix} < R_4$$

$$.21875 < SA < .46875$$

use computer

n	L_n		R_n
4	.2188	SA	.4688
10	.285	SA	.385
100	.3284	SA	.3384
1000	.3328	SA	.3338

Observe: L_n + R_n seem to be getting closer & closer together and they seem to be approaching around .333

That is it seems like

$$\lim_{n\to\infty} L_n \approx .333 \approx \lim_{n\to\infty} R_n$$

Inspired by this we define

Signed area of the blue Region $= SA \overset{\text{def}}{=} \lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n$

It is a huge fact from higher math that these limits do in fact exist and are equal

This answers the area question.