MATH 2301 Section 110 (Barsamian) Meeting # 54 (Wed Apr 26, 2023)

Today & Friday: Section 5.5 Substitution

Tuesday May 2: Final Exam, 4:40-6:40 pm, here

Meeting Part 1 More Examples of finding Indefinite Integrals using Substitution

Example 5.5 #6 Find $\int \frac{\cos(\pi/x)}{x^2}\,dx$

Solution

Identify inner function $u = \pi/x$

Build expression for dx

$$u = \frac{\pi}{x} = \pi \cdot x^{-1}$$

so $u' = \frac{d}{dx}\pi \cdot x^{-1} = \pi \frac{d}{dx} x^{-1} = \pi \cdot (-1) x^{-1-1}$

$= -\pi x^{-2} = -\frac{\pi}{x^2}$

build $dx = \frac{1}{u'}\,du = \frac{1}{-\pi/x^2}\,du = -\frac{x^2}{\pi}\,du = dx$

Substitute, Cancel, Simplify

$$\int \frac{\cos(\pi/x)}{x^2}\,dx = \int \frac{\cos(u)}{x^2}\left(-\frac{x^2}{\pi}\,du\right) = \int \cos(u)\left(-\frac{1}{\pi}\,du\right) = \left(-\frac{1}{\pi}\right)\int \cos(u)\,du$$

Substitute

cancel the x^2

Simplify

Integrate

$$-\frac{1}{\pi}\int \cos(u)\,du = -\frac{1}{\pi}\left(\sin(u) + c\right) = \left(-\frac{1}{\pi}\right)\sin(u) - \left(\frac{1}{\pi}\right)c = \left(-\frac{1}{\pi}\right)\sin(u) + K$$

c can be any real number

this could be any real number

K can be any real number

Substitute Back

$$\left(-\frac{1}{\pi}\right)\sin(u) + K = -\frac{1}{\pi}\sin\left(\pi/x\right) + K$$

Conclusion: $\displaystyle \int \frac{\cos(\pi/x)}{x^2}\,dx = -\frac{1}{\pi}\sin(\pi/x) + K$

End of example

More streamlined solution to same example

$$\int \frac{\cos(\pi/x)}{x^2}\,dx = \int \frac{\cos(u)}{x^2}\left(-\frac{x^2}{\pi}\,du\right) = -\frac{1}{\pi}\int \cos(u)\,du = -\frac{1}{\pi}\left(\sin(u)+C\right) = -\frac{1}{\pi}\sin(u)+k = -\frac{1}{\pi}\sin\left(\frac{\pi}{x}\right)+k$$

$\frac{\pi}{x} = u$

$dx = -\frac{x^2}{\pi}\,du$

substitute

Cancel + Simplify

integrate

Substitute back

Recall some derivative + indefinite integral pairs

$$\frac{d}{dx} e^{(x)} = e^{(x)}$$

$$\int e^{(x)} dx = e^{(x)} + C$$

$$\frac{d}{dx} b^{(x)} = b^{(x)} \ln(b)$$

$$\int b^{(x)} dx = \frac{b^{(x)}}{\ln(b)} + C$$

example: $\frac{d}{dx} 10^{(x)} = 10^{(x)} \cdot \ln(10)$

example $\int 10^{x} dx = \frac{10^{x}}{\ln(10)} + C$

constant

$$\begin{cases} \dfrac{d}{dx} e^{(kx)} = k e^{(kx)} \\ \displaystyle\int e^{(kx)} dx = \dfrac{e^{(kx)}}{k} + D \end{cases}$$

Substitution Rule

(5)

Chain rule details

inner$(x) = kx$
inner$'(x) = k$
outer$(\ ) = e^{(\ )}$
outer$'(\ ) = e^{(\ )}$

Substitution Rule Details

inner function $\quad u = kx$

build expression for dx $\quad u' = \dfrac{du}{dx} = \dfrac{d\,kx}{dx} = k$

$$dx = \frac{1}{u'} du = \frac{1}{k} du$$

Substitute cancel simplify

$$\int e^{(kx)} dx = \int e^{(u)} \frac{1}{k} du = \frac{1}{k} \int e^u du$$

$$= \frac{1}{k}\left(e^u + C\right) =$$

$$= \frac{1}{k} \cdot e^u + \frac{1}{k} \cdot C$$

$$= \frac{e^u}{k} + D$$

Problem Similar to 5.5 #8 find $\int x e^{(x^2)} dx$

Solution

Inner Function $\quad x^2 = u$

Build expression for dx $\quad dx = \frac{1}{u'} du$

$$dx = \frac{1}{2x} du$$

Sub, cancel, simplify

$$\int x e^{(x^2)} dx \underset{\text{sub}}{=} \int x e^{(u)} \left(\frac{1}{2x} du\right) \underset{\text{cancel}}{=} \int \frac{e^u}{2} du \underset{\text{simplify}}{=} \frac{1}{2} \int e^u du$$

Integrate

$$\frac{1}{2} \int e^u du = \frac{1}{2} e^u + K$$

we know how to handle the + c

Substitute Back $\quad \frac{1}{2} e^{u} + K = \frac{1}{2} e^{x^2} + K$

Conclusion

$$\int x e^{x^2} dx = \frac{1}{2} e^{x^2} + c$$

end of example

Question what is $\int e^{(x^2)}dx$?

Alice says since $\int e^{(x)}dx = e^{(x)} + C$, it is clear that $\int e^{x^2}dx = e^{(x^2)} + c$

Bob says since $\int x^2 dx$ is $\frac{x^3}{3} + c$, it is clear that $\int e^{x^2}dx = e^{\frac{x^3}{3}} + C$

Charlie says need <u>substitution</u> to find $\int e^{(x^2)}dx$ because of the nested function.

<u>Check Alice's result by differentiating</u>

$$\frac{d}{dx}\left(e^{(x^2)}+c\right) = \underset{\text{chain rule}}{e^{(x^2)}\cdot 2x} + 0 = 2xe^{(x^2)} \neq e^{(x^2)}$$

So $\int e^{(x^2)}dx \neq e^{(x^2)}+c$

<u>Check Bob's Result by Differentiating</u>

$$\frac{d}{dx}e^{\left(\frac{x^3}{3}+c\right)} = e^{\left(\frac{x^3}{3}+c\right)}\cdot x^2 \neq e^{x^2}$$

So $\int e^{(x^2)}dx \neq e^{\left(\frac{x^3}{3}+c\right)}$

Charlie Says use Substitution to find $\int e^{(x^2)}\,dx$

Inner ~~Ex~~ Function $u = x^2$

Build expression for dx

$$dx = \frac{1}{u'}\,du$$

$$dx = \frac{1}{2x}\,du$$

Substitute, Cancel, Simplify

$$\int e^{(x^2)}\,dx \underset{\uparrow\ \text{substitute}}{=} \int e^{(u)}\,\frac{1}{2x}\,du \underset{\uparrow\ \text{simplify}}{=} \frac{1}{2}\int \frac{e^u}{x}\,du$$

no cancelling available

integral involves both u and x

can't integrate this

Substitution method fails!!

So what __is__ $\int e^{(x^2)}\,dx$? That is what is an antiderivative for e^{x^2}?

__Good News__ e^{x^2} __does__ have an antiderivative

__Bad News__ (Important fact from higher math)

~~the~~ antiderivative __cannot__ be written as

a finite combination of ~~?~~ basic functions.

Remember the Fundamental Theorem of Calculus

$$\frac{d}{dx}\left(\int_0^x e^{(t^2)}\,dt\right) = e^{x^2}$$

So $\int_0^x e^{(t^2)}\,dt$ is an antiderivative of $e^{(x^2)}$

End of Example

End of Lecture