Axioms, Definitions, and Theorems for 2022-2023 Spring Semester MATH 3110/5110

Relations and Functions (Section 1.2)

Definition of a Reflexive Relation

Words: Binary relation $\sim$ on set S is ***reflexive***.

Meaning: For all $a \in S$, $a \sim a$.

Definition of a Symmetric Relation

Words: Binary relation $\sim$ on set S is ***symmetric***.

Meaning: For all $a, b \in S$, if $a \sim b$, then $b \sim a$.

Definition of a Transitive Relation

Words: Binary relation $\sim$ on set S is ***transitive***.

Meaning: For all $a, b, c \in S$, if $a \sim b$ and $b \sim c$, then $a \sim c$.

Functions (Section 1.3)

Definition of a Surjective Function

Words: Function $f: S \to T$ is ***surjective***.

Meaning: For every $t \in T$, there exists an $s \in S$ such that $f(s) = t$.

Definition of an Injective Function

Words: Function $f: S \to T$ is ***injective***.

Meaning: For every $s_1, s_2 \in S$, if $f(s_1) = f(s_2)$ then $s_1 = s_2$.

Theorem 1.3.7 If $f: S \to T$ and $g: T \to V$ are both surjections, then $g \circ f: S \to V$ is a surjection.

Theorem 1.3.8 If $f: S \to T$ and $g: T \to V$ are both injections, then $g \circ f: S \to V$ is an injection.

Theorem 1.3.7 If $f: S \to T$ and $g: T \to V$ are both bijections, then $g \circ f: S \to V$ is a bijection.

Definition of Inverse Function

Symbol: f^{-1}

Usage: There is a bijective function $f: X \to Y$ in the discussion.

Spoken: The inverse function for f

Meaning: f^{-1} is the function $f^{-1}: Y \to X$ defined by

$$f^{-1}(y) \underset{\text{def}}{=} \textit{the unique } x \in X \textit{ such that } f(x) = y$$

Equivalently, f^{-1} can be defined by saying that the symbol $f^{-1}(y) = x$ means $y = f(x)$.

Theorem D: (Exercise 1.3#10) Given $f: S \to T$ and $g: T \to V$

If $g \circ f$ is surjective, then g is surjective

Theorem E: (Exercise 1.3#11) Given $f: S \to T$ and $g: T \to V$

If $g \circ f$ is injective, then f is injective

Theorem 1.3.10

If $f: X \to Y$ then the following two statements are equivalent. (That is, they are either both true, or they are both false.) (The book says (1) if and only if (2).)

(1) f is a bijection

(2) There exists some function $g: Y \to X$ such that $g \circ f = id_X$ and $f \circ g = id_Y$

Furthermore, the inverse function of f is the function g in this case.

Corollary: (Exercse 1.3#9) If function f is bijective, then its inverse function f^{-1} is also bijective.

Theorem 1.3.12 If $f: S \to T$ and $h: T \to V$ are bijections, then $(h \circ f)^{-1} = f^{-1} \circ h^{-1}$

Abstract Geometry (Section 2.1)

Definition of Abstract Geometry (Barsamian's version, correcting error in book definition)

An *abstract geometry* $\mathcal{A}$ is an ordered pair $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ where $\mathcal{P}$ denotes a set whose elements are called *points* and $\mathcal{L}$ denotes a ***non-empty*** set whose elements are called *lines*, which are *sets of points* satisfying the following two requirements, called *axioms*:

(i) For any two distinct points $A, B \in \mathcal{P}$, there exists at least one line that both points lie on.

(ii) For every line $l \in \mathcal{L}$ there exist at least two distinct points that lie on the line.

Definition of Collinear Points in an Abstract Geometry

Words: The set of points S is **collinear**.

Usage: $S \subset \mathcal{P}$ is a set of points in an abstract geometry.

Meaning: There exists a single line $L \in \mathcal{L}$ that all the points in set S lie on.

Meaning written formally: $\exists L \in \mathcal{L}\big(\forall P \in S(P \in L)\big)$

Definition of Concurrent Lines in an Abstract Geometry

Words: The set of lines T is **concurrent**.

Usage: $T \subset \mathcal{L}$ is a set of lines in an abstract geometry.

Meaning: There exists a single point $P \in \mathcal{P}$ that all the lines in set T pass through.

Meaning written formally: $\exists P \in \mathcal{P}\big(\forall L \in T(P \in L)\big)$

Definition of Parallel Lines in an Abstract Geometry

Symbol: $l_1 \| l_2$

Spoken: Lines l_1 and l_2 are **parallel**.

Usage: l_1 and l_2 are lines in an abstract geometry.

Meaning: Either l_1 and l_2 do not intersect or l_1 and l_2 are the same line.

Meaning in Symbols: Either $l_1 \cap l_2 = \phi$ or $l_1 = l_2$.

The BIG QUESTIONS. (two common questions about geometries)

- **BIG QUESTION #1:** *Do parallel lines exist?*
- **BIG QUESTION #2:** *Given a line L and a point P not on L, how many lines exist that contain P and are parallel to L?*

Challenge Theorem CT.2.1.1 (Not Presented as a Theorem in the book. Can you prove it?)

In an Abstract Geometry, there exist two distinct points.

Challenge Theorem CT.2.1.2 (Not Presented as a Theorem in the book. Can you prove it?)

In an Abstract Geometry, given any point $A \in \mathcal{P}$, there exists a line that passes through A.

The *Cartesian Plane* (Section 2.1)

Definition of the *Cartesian Plane* $\mathcal{C} = (\mathbb{R}^2, \mathcal{L}_E)$

- **Points:** The set of points, denoted $\mathbb{R}^2$, is the Cartesian product, $\mathbb{R} \times \mathbb{R}$. That is, a point is an ordered pair of real numbers.
- **Lines:** The set of lines, denoted $\mathcal{L}_E$, consists of *vertical lines* and *non-vertical lines*
    - *vertical lines* are subsets of $\mathbb{R}^2$ of the form $L_a = \{(x, y) \in \mathbb{R}^2 | x = a\}$, where a is a fixed real number.
    - *non-vertical lines* are subsets of $\mathbb{R}^2$ of the form $L_{m,b} = \{(x, y) \in \mathbb{R}^2 | y = mx + b\}$, where m and b are fixed real numbers.

Proposition 2.1.1 The Cartesian Plane $\mathcal{C} = (\mathbb{R}^2, \mathcal{L}_E)$ is an example (a model) of Abstract Geometry.

Procedure for Finding the *Cartesian* Line Passing through Two Distinct Points in $\mathbb{R}^2$

Suppose $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are any two distinct points of $\mathbb{R}^2$.

If $x_1 = x_2$ then let $a = x_1 = x_2$. In this case, $L_a \in \mathcal{L}_H$ and $P, Q \in L_a$.

If $x_1 \neq x_2$ then define constants m, b by the following formulas:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$b = y_2 - mx_2$$

Then $P, \in L_{m,b}$ and $Q \in L_{m,b}$.

The *Poincaré Plane* (Section 2.1)

Definition of the *Poincaré Plane* $\mathcal{H} = (\mathbb{H}, \mathcal{L}_{\mathbb{H}})$

- **Points:** The set of points, denoted $\mathbb{H}$, is the Cartesian product, $\mathbb{R} \times \mathbb{R}^{+}$. That is, a point is an ordered pair of real numbers with a positive y coordinate. That is, $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 | y > 0\}$
- **Lines:** The set of lines, denoted $\mathcal{L}_{\mathbb{H}}$, consists of *type I lines* and *type II lines*
  - *type I lines* are subsets of $\mathbb{H}$ of the form ${}_aL = \{(a, y) \in \mathbb{H}\}$, where a is a fixed real number.
  - *type II lines* are subsets of $\mathbb{H}$ of the form ${}_cL_r = \{(x, y) \in \mathbb{H} | (x - c)^2 + y^2 = r^2\}$, where c and r are fixed real numbers and $r > 0$.

Proposition 2.1.2 The *Poincaré* Plane $\mathcal{H} = (\mathbb{H}, \mathcal{L}_{\mathbb{H}})$ is an example (a model) of Abstract Geometry.

Procedure for Finding the *Poincaré Line* Passing Through Two Distinct Points in $\mathbb{H}$

Suppose $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are any two distinct points of $\mathbb{H}$.

If $x_1 = x_2$ then let $a = x_1 = x_2$. In this case, ${}_aL \in \mathcal{L}_H$ and $P, Q \in {}_aL$.

If $x_1 \neq x_2$ then define constants c, r by the following formulas:

$$c = \frac{x_2^2 - x_1^2 + y_2^2 - y_1^2}{2(x_2 - x_1)}$$

$$r = \sqrt{(x_1 - c)^2 + y_1^2}$$

In this case, ${}_cL_r \in \mathcal{L}_H$ and $P, Q \in {}_cL_r$.

Incidence Geometry (Section 2.1)

Definition of Incidence Geometry

An *incidence geometry* is an ordered pair $(\mathcal{P}, \mathcal{L})$ that is satisfies all the requirements of an *abstract geometry* and also satisfies the following two additional *axioms*:

(i) For any two distinct points $A, B \in \mathcal{P}$, there exists **exactly one** line that both points lie on.

(ii) There exist three non-collinear points.

Definition of Notation for the Unique Line Containing Two Given Distinct Points in an Incidence Geometry

Symbol: $\overleftrightarrow{AB}$

Spoken: *line A B*

Usage: There is an *incidence geometry* in the discussion and A, B are distinct points in it.

Meaning: the unique line $l \in \mathcal{L}$ that both points lie on.

Theorem 2.1.6 Given two lines l_1 and l_2 in an *incidence geometry*,
If $l_1 \cap l_2$ has two or more distinct points, then l_1 and l_2 are the same line. That is, $l_1 = l_2$.

Corollary 2.1.7 (contrapositive of Theorem 2.1.6)
Given two lines l_1 and l_2 in an *incidence geometry*,
If lines l_1 and l_2 are known to be distinct lines (that is, $l_1 \neq l_2$),
then either lines l_1 and l_2 do not intersect or they intersect in exactly one point.

Challenge Theorem CT.2.1.3 (Not Presented as a Theorem in the book. Can you prove it?)
In an Incidence Geometry, there exist three distinct lines.

Challenge Theorem CT.2.1.4 (Not Presented as a Theorem in the book. Can you prove it?)
In an Incidence Geometry, there exist three distinct lines that are not concurrent.

Challenge Theorem CT.2.1.5 (Not Presented as a Theorem in the book. Can you prove it?)
In an Incidence Geometry, given any point P, there exist two lines that pass through P.

Challenge Theorem CT.2.1.6 (Not Presented as a Theorem in the book. Can you prove it?)
In an Incidence Geometry, given any point P, there exists a line that does not pass through P.

Proposition 2.1.4 The Cartesian Plane $\mathcal{C} = (\mathbb{R}^2, \mathcal{L}_E)$ is an example (a model) of Incidence Geometry.

Proposition 2.1.5 The *Poincaré* Plane $\mathcal{H} = (\mathbb{H}, \mathcal{L}_{\mathbb{H}})$ is an example (a model) of Incidence Geometry.

Distance Functions (Section 2.2)

Definition of Distance Function

words: d is a distance function on set S

meaning: d is a function $d: S \times S \to \mathbb{R}$ that satisfies these requirements

(i) $\forall P, Q \in S(d(P, Q) \geq 0)$

(ii) $d(P, Q) = 0$ if and only if $P = Q$

(iii) $d(P, Q) = d(Q, P)$

Definition of the Absolute Value Distance Function on $\mathbb{R}$

symbol: $d_{\mathbb{R}}$

meaning: the function $d_{\mathbb{R}}: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $d_{\mathbb{R}}(x, y) = |x - y|$

Definition of the Euclidean Distance Function on $\mathbb{R}^2$

symbol: d_E

meaning: the function $d_E: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$d_E\big((x_1, y_1), (x_2, y_2)\big) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Definition of the Taxicab Distance Function on $\mathbb{R}^2$

symbol: d_T

meaning: the function $d_T: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$d_T\big((x_1, y_1), (x_2, y_2)\big) = |x_1 - x_2| + |y_1 - y_2|$$

Definition of the Max (or Supremum) Distance Function on $\mathbb{R}^2$

symbol: d_S

meaning: the function $d_S: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$d_S\big((x_1, y_1), (x_2, y_2)\big) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$$

Definition of the Poincaré Distance Function on $\mathbb{H}$

symbol: d_H

meaning: the function $d_H: \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ defined in the following way

Suppose $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are any two points of $\mathbb{H}$.

If $x_1 = x_2$ then compute the distance between them using the formula

$$d_H(P,Q) = \left|\ln\left(\frac{y_2}{y_1}\right)\right|$$

If $x_1 \neq x_2$ then compute the distance between them using the formula

$$d_H(P,Q) = \left|\ln\left(\frac{\frac{x_1 - c + r}{y_1}}{\frac{x_2 - c + r}{y_2}}\right)\right|$$

where c, r are the constants describing the *type II* line that passes through P and Q.

Remark on the Distance Functions just presented: Five functions were just defined and declared to be *distance functions*. It is fine to define a function. But to be precise, one cannot just *declare* that a particular function is a distance function. One must first verify that the function does in fact satisfy the requirements of a distance function that are prescribed by the **Definition of Distance Function**, above.

The fact that the **Absolute Value Distance Function** on $\mathbb{R}$ really satisfies the requirements of a distance function would be straightforward to prove by using the definition of the absolute value function. The fact that the remaining four functions—the **Euclidean Distance Function**, the **Taxicab Distance Function**, the **Max Distance Function**, and the **Poincaré Distance Function**—really do satisfy the requirements of a distance function is proven in various propositions and exercises in Section 2.2.

Definition of Circle

symbol: circle(P, r)

words: the circle with center P and radius r

usage: There is some set S with distance function d in the discussion, and $P \in S$, and $r > 0$.

meaning: the set

$$\text{circle}(P,r) = \{Q \in S | d(P,Q) = r\}$$

Rulers (Section 2.2)

Definition of a Ruler for a Line

words: f is a ruler for line l. (**alternate words:** f is a coordinate function for line l.)

usage: There is an incidence geometry $(\mathcal{P}, \mathcal{L})$ in the discussion, and there is a distance function d on the set of points $\mathcal{P}$ in the discussion, and $l \in \mathcal{L}$.

meaning: f is a function $f: l \to \mathbb{R}$ that satisfies these requirements

(i) f is a *bijection*.

(ii) f "agrees with" the distance function d in the following way:

For each pair of points P and Q (not necessarily distinct) on line l, this equation is true:

$$|f(P) - f(Q)| = d(P, Q)$$

Additional Terminology: The equation above is called the ***Ruler Equation***.

The number $f(P)$ is called the ***coordinate of P with respect to f***.

Lemma 2.2.3 Suppose that there is an incidence geometry $(\mathcal{P}, \mathcal{L})$ in the discussion, and there is a distance function d on the set of points $\mathcal{P}$ in the discussion, and $l \in \mathcal{L}$.

If $f: l \to \mathbb{R}$ is *surjective* and also satisfies the *Ruler Equation*,

then f is also *injective*. (So f is bijective, and qualifies to be called a ruler for line l.)

Metric Geometry (Section 2.2)

Definition of Metric Geometry

A *metric geometry* $\mathcal{M}$ is an ordered triple $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$ that satisfies the following:

- $(\mathcal{P}, \mathcal{L})$ is an *incidence geometry*.
- d is a *distance function* on the set of points $\mathcal{P}$.
- Every line $L \in \mathcal{L}$ has a *ruler*. (This requirement is called the ***Ruler Postulate***.)

Definition of Notation for the Distance Between Two Given Points

Symbol: AB

Spoken: *the distance between* A and B

Usage: There is a *metric geometry* in the discussion and A, B are (not necessarily distinct) points.

Meaning: the number $d(A, B)$. (Notice that this number could be zero!)

Lemma 2.2.3 If $(\mathcal{P}, \mathcal{L})$ is an *incidence geometry*, and there is a distance function d on the set of points $\mathcal{P}$, and $l \in \mathcal{L}$.

If $f: l \to \mathbb{R}$ is *surjective* and also satisfies the *Ruler Equation*,

then f is also *injective*. (So f is bijective, and qualifies to be called a ruler for line l.)

Theorem 2.2.8 If $(\mathcal{P}, \mathcal{L})$ is an *incidence geometry* with the following additional property:

For each line $l \in \mathcal{L}$, there exists a bijective function $f_l: l \to \mathbb{R}$

then there exists a distance function d for the set of points $\mathcal{P}$ such that each $f_l: l \to \mathbb{R}$ is a ruler. This means that the triple $(\mathcal{P}, \mathcal{L}, d)$ qualifies to be called a *metric geometry*.

Challenge Theorem CT.2.2.1 (Not Presented as a Theorem in the book.)

In a Metric Geometry, given any line $l \in \mathcal{L}$, the set of points that lie on l is an infinite set.

Challenge Theorem CT.2.2.1b (New on this list. Not Presented as a Theorem in the book.) (Renumber this list the next time this list is edited)

In an Incidence Geometry, given any point P, there is an infinite set of lines that pass through P.

Challenge Theorem CT.2.2.2 (Exercise 2.2#17) (Not Presented as a Theorem in book.)

If $(\mathcal{P}, \mathcal{L}, d)$ is a Metric Geometry, given any point $P \in \mathcal{P}$ and any number $r > 0$,

There exists a point $Q \in \mathcal{P}$ such that $d(P, Q) = r$.

Challenge Theorem CT2.2.3 (Exercise 2.2#20) (Not Presented as a Theorem in book.)

Given a metric geometry $(\mathcal{P}, \mathcal{L}, d)$ and point $P \in \mathcal{P}$ and line $l \in \mathcal{L}$ such that $P \in l$ and circle(P, r), line l intersects circle(P, r) in exactly two points.

Remark on the Metric Geometries to be Presented: Three geometries will be introduced below and declared to be *metric geometries*. But one cannot just *declare* that something is a metric geometry. One must first verify that the function does in fact satisfy the requirements of a metric geometry that are prescribed by the **Definition of Metric Geometry**, above. The fact that the three geometries presented—the **Euclidean Plane Model**, the **Taxicab Plane Model**, and the **Poincaré Plane Model**—really do satisfy the requirements of a metric geometry is proven in various propositions and exercises in Section 2.2.

Definition of The *Euclidean Plane model* $\mathcal{E} = (\mathbb{R}^2, \mathcal{L}_E, d_E)$ of *Metric Geometry*

- **Points and Lines:** $(\mathbb{R}^2, \mathcal{L}_E)$ is the Cartesian plane model of incidence geometry.

  $\mathcal{L}_E$ consists of *vertical lines* and *non-vertical lines*
    - *vertical lines* are lines of the form $L_a = \{(x,y) \in \mathbb{R}^2 | x = a\}$
    - *non-vertical lines* are lines of the form $L_{m,b} = \{(x,y) \in \mathbb{R}^2 | y = mx + b\}$
- **Distance Function:** $d_E\big((x_1,y_1),(x_2,y_2)\big) = \sqrt{(x_2 - x_1)^2 + (y_2 - 1_1)^2}$
- **Standard Ruler:**
    - for vertical lines: $f(x,y) = y$
    - for non-vertical lines: $f(x,y) = x\sqrt{1 + m^2}$

Definition of The *Taxicab Plane model* $\mathcal{T} = (\mathbb{R}^2, \mathcal{L}_E, d_T)$ of *Metric Geometry*

- **Points and Lines:** $(\mathbb{R}^2, \mathcal{L}_E)$ is the Cartesian plane model of incidence geometry.

  $\mathcal{L}_E$ consists of *vertical lines* and *non-vertical lines*
    - *vertical lines* are lines of the form $L_a = \{(x,y) \in \mathbb{R}^2 | x = a\}$
    - *non-vertical lines* are lines of the form $L_{m,b} = \{(x,y) \in \mathbb{R}^2 | y = mx + b\}$
- **Distance Function:** $d_T\big((x_1,y_1),(x_2,y_2)\big) = |x_2 - x_1| + |y_2 - 1_1|$
- **Standard Ruler:**
    - for *vertical lines*: $f(x,y) = y$
    - for *non-vertical lines*: $f(x,y) = x(1 + |m|)$

Definition of The *Poincaré Plane model* $\mathcal{H} = (\mathbb{H}, \mathcal{L}_H, d_H)$ of *Metric Geometry*

- **Points and Lines:** $(\mathbb{H}, \mathcal{L}_H)$ is the *Poincaré plane* model of *incidence geometry.*
- **Points:** $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 | y > 0\}$
- **Lines:** $\mathcal{L}_h$ consists of *type I lines* and *type II lines*
  - *type I lines* are lines of the form $L_a = \{(a, y) \in \mathbb{H}\}$
  - *type II lines* are lines of the form ${}_cL_r = \{(x, y) \in \mathbb{H} | (x - c)^2 + y^2 = r^2\}$
- **Distance Function:** the function $d_H: \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ defined in the following way

Suppose $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are any two points of $\mathbb{H}$.

If $x_1 = x_2$ then compute the distance between them using the formula

$$d_H(P, Q) = \left| \ln\left(\frac{y_2}{y_1}\right) \right|$$

If $x_1 \neq x_2$ then compute the distance between them using the formula

$$d_H(P, Q) = \left| \ln\left(\frac{\frac{x_1 - c + r}{y_1}}{\frac{x_2 - c + r}{y_2}} \right) \right|$$

where c, r are the constants describing the *type II line* that passes through P and Q.

$$c = \frac{x_2^2 - x_1^2 + y_2^2 - y_1^2}{2(x_2 - x_1)}$$

$$r = \sqrt{(x_1 - c)^2 + y_1^2}$$

- **Standard Coordinate Function:**
  - for *type I lines*: $f(x, y) = \ln(y)$
  - for *type II lines*:

$$f(x, y) = \ln\left(\frac{x - c + r}{y}\right)$$

Ruler Placement (Getting New Rulers from Old Rulers) (Section 2.3)

Challenge Theorem CT.2.3.0a Ruler Sliding Theorem

(This Theorem is presented and proven in Barsamian's Video.02.3 It is not presented as a Theorem in the book, but the proof of the fact is embedded in the book's proof of Theorem 2.3.1.)

Given:
- a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$
- a line $l \in \mathcal{L}$
- a ruler f for line l
- a number $a \in \mathbb{R}$

Claim: The function $g: l \to \mathbb{R}$ defined by

$$g(P) = f(P) - a$$

qualifies to be called a ruler for line l.

Remark: The new ruler g is analogous to the ruler that one would obtain by *sliding* a ruler f along a line l in a drawing.

Challenge Theorem CT.2.3.0b Ruler Flipping Theorem

(This Theorem is presented and proven in Barsamian's Video.02.3 It is not presented as a Theorem in the book, but the proof of the fact is embedded in the book's proof of Theorem 2.3.1.)

Given:
- a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$
- a line $l \in \mathcal{L}$
- a ruler f for line l

Claim: The function $g: l \to \mathbb{R}$ defined by

$$g(P) = -f(P)$$

qualifies to be called a ruler for line l.

Remark: The new ruler g is analogous to the ruler that one would obtain by *flipping* the direction of a ruler f on a line l in a drawing.

Theorem 2.3.1 (Ruler Sliding and Flipping Theorem)

Given:
- a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$
- a line $l \in \mathcal{L}$
- a ruler f for line l
- a number $a \in \mathbb{R}$
- a number ε that is either 1 or -1

Claim: The function $h_{a,\varepsilon}: l \to \mathbb{R}$ defined by

$$h_{a,\varepsilon}(P) = \varepsilon(f(P) - a)$$

qualifies to be called a ruler for line l.

Remark: The new ruler $h_{a,\varepsilon}$ is analogous to the ruler that one would obtain by *sliding* a ruler f along on a line l in a drawing and then (if $\varepsilon = -1$) flipping the ruler.

Theorem 2.3.2 (Ruler Placement Theorem)

Given:
- a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$
- distinct points $A, B \in \mathcal{P}$

Claim: There exists a ruler g for line $\overleftrightarrow{AB}$ such that $g(A) = 0$ and $g(B) > 0$.

Terminology: Such a ruler g is called a ***ruler with A as origin and B positive***.

Procedure 2.3.3 for Finding a *Ruler with A as Origin and B Positive* for Distinct Points A, B in a Metric Geometry

Suppose A and B are any two distinct points in a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$.

Line $\overleftrightarrow{AB}$ exists (because $(\mathcal{P}, \mathcal{L})$ is an incidence geometry and A and B are distinct points).

Let f be a ruler for line $\overleftrightarrow{AB}$. (A ruler exists because $\mathcal{M}$ is a metric geometry.)

Use f to obtain a new ruler in the following way:

Define a real number ε by $\varepsilon = sgn\big(f(B) - f(A)\big)$

(That is, ε is the *sign* of $f(B) - f(A)$. In other words

$$\varepsilon = \begin{cases} +1 & \text{if } f(B) - f(A) > 0 \\ -1 & \text{if } f(B) - f(A) < 0 \end{cases}$$

Let $h_{f(A),\varepsilon}$ be the function $h_{a,\varepsilon}: \overleftrightarrow{AB} \to \mathbb{R}$ defined by

$$h_{f(A),\varepsilon}(P) = \varepsilon\big(f(P) - f(A)\big)$$

Then $h_{f(A),\varepsilon}$ is a ***ruler with A as origin and B positive***.

Challenge Theorem CT.2.3.4 (Exercise 2.3#4) (Not Presented as a Theorem in book.)

Given:
- a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$
- distinct points $P, Q \in \mathcal{P}$

Claim: There exists a point $M \in \overleftrightarrow{PQ}$ such that $d(P, M) = d(M, Q)$

Challenge Theorem CT.2.3.5 (based on 2.3#6) (Not Presented as a Theorem in book.)

Given:
- a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$
- a point $P \in \mathcal{P}$
- a line $l \in \mathcal{L}$ through P
- a real number r

Claim: There exists a point $Q \in l$ such that $d(P, Q) = r$

The *vector space* $(\mathbb{R}^2, +, \boldsymbol{scalar\ mult})$ (Section 3.1)

Definition of the *vector space* $(\mathbb{R}^2, +, \boldsymbol{scalar\ mult})$.

For vectors $A = (x_A, y_A) \in \mathbb{R}^2$ and $B = (x_B, y_B) \in \mathbb{R}^2$ and scalar $r \in \mathbb{R}$, we define

- The ***sum*** of two vectors: $A + B = (x_A + x_B, y_A + y_B) \in \mathbb{R}^2$
- the ***scalar multiplication*** of a vector by a number: $rA = (rx_A, ry_A) \in \mathbb{R}^2$
- The ***difference*** of two vectors: $A - B = A + (-1)B = (x_A - x_B, y_A - y_B) \in \mathbb{R}^2$
- The ***inner product*** of two vectors: $\langle A, B \rangle = x_A x_B + y_A y_B \in \mathbb{R}$
- The ***norm*** of a vector: $\|A\| = \sqrt{\langle A, A \rangle} = \sqrt{x_A x_A + y_A y_A} = \sqrt{x_A^2 + y_A^2} \in \mathbb{R}$

Proposition 3.1.1 Basic properties of the *vector space* $(\mathbb{R}^2, +, \boldsymbol{scalar\ mult})$.

For all vectors $A, B, C \in \mathbb{R}^2$ and scalars $r, s \in \mathbb{R}$,

(i) $A + B = B + A$

(ii) $(A + B) + C = A + (B + C)$

(iii) $r(A + B) = rA + rB$

(iv) $(r + s)A = rA + sA$

(v) $\langle A, B \rangle = \langle B, A \rangle$

(vi) $\langle rA, B \rangle = r\langle A, B \rangle$

(vii) $\langle A + B, C \rangle = \langle A, C \rangle + \langle B, C \rangle$

(viii) $\|rA\| = |r|\|A\|$

(ix) $\|A\| > 0$ if $A \neq (0,0)$

Inequalities (Section 3.1)

(These are not all in the book, and some are in an order different from the book's order.)

Triangle Inequality for the Absolute Value (Introduced and proven in Video 3.1. Not in book.)

For for all $a, b \in \mathbb{R}$, the inequality $|a + b| \leq |a| + |b|$ is true

Definition of the triangle inequality for distance functions

(Not necessarily true for all distance functions! If it is true for a particular distance function, the proof that it is true would amount to a Proposition.)

Words: Distance function d on set $\mathcal{P}$ satisfies the triangle inequality.

Meaning: For all $A, B, C \in \mathcal{P}$, the inequality $d(A, C) \leq d(A, B) + d(B, C)$ is true

In Symbols: $\forall A, B, C \in \mathcal{P}\big(d(A, C) \leq d(A, B) + d(B, C)\big)$

The Absolute Value Distance Function satisfies the Triangle Inequality for Distance Functions. (Proven in Video 3.1)

For all $x, y, z \in \mathbb{R}$, the inequality $|x - z| \leq |x - y| + |y - z|$ is true.

That is, $d_{\mathbb{R}}(x, z) \leq d_{\mathbb{R}}(x, y) + d_{\mathbb{R}}(y, z)$.

Proposition 3.1.5 Cauchy-Schwarz Inequality for the *vector space* $(\mathbb{R}^2, +, \boldsymbol{scalar\ mult})$.

For all vectors $A, B \in \mathbb{R}^2$, the inequality $|\langle A, B\rangle| \leq \|A\| \cdot \|B\|$ is true

Proposition 3.1.3 Relationship between the *Euclidean distance function* and the *norm*

For all $A, B \in \mathbb{R}^2$, the equation $d_E(A, B) = \|A - B\|$ is true.

Triangle Inequality satisfied by the *norm* for the *vector space* $(\mathbb{R}^2, +, \boldsymbol{scalar\ mult})$

(Not stated as a Theorem in the boook, but proven on p. 46, inside their proof of Prop 3.1.6)

For all vectors $A, B \in \mathbb{R}^2$, the inequality $\|A + B\| \leq \|A\| + \|B\|$ is true

Proposition 3.1.6

The *Euclidean distance function* satisfies the *triangle inequality for distance functions*

For all $A, B, C \in \mathbb{R}^2$, the inequality $d_E(A, C) \leq d_E(A, C) + d_E(A, C)$ is true

Alternate Presentation of the Cartesian Lines and Euclidean Rulers (Section 3.1)

Proposition 3.1.2 Using Vectors to Describe Cartesian Lines

Given two distinct points $A, B \in \mathbb{R}^2$ line $\overleftrightarrow{AB}$ can be described using vectors as follows:

$$L_{AB} = \{X \in \mathbb{R}^2 | X = A + t(B - A) \text{ for some } t \in \mathbb{R}\}$$

Observe that the use of the letter X is not really necessary.

$$L_{AB} = \{A + t(B - A) | t \in \mathbb{R}\}$$

Proposition 3.1.4 Using Vectors to Describe Rulers in the Euclidean Plane

If L_{AB} is a cartesian line, then $f\colon L_{AB} \to \mathbb{R}$ defined by

$$f\big(A + t(B - A)\big) = t\|B - A\|$$

is a ruler for the line L_{AB} in the *Euclidean plane*.

Betweeness of Real Numbers (Section 3.2)

Cases of the Triangle Inequality for Distinct Real Numbers x, y, z (discussed in Video.03.2a)

The Triangle Inequality (TE)is the *inclusive inequality*

$$|x - z| \leq |x - y| + |y - z|$$

Case (i): When $x < y < z$, the TE becomes an *equality* $|x - z| = |x - y| + |y - z|$.

Case (ii): $z < y < x$, the TE becomes an *equality* $|x - z| = |x - y| + |y - z|$.

Case (iii): $y < x < z$, the TE becomes a *strict inequality* $|x - z| < |x - y| + |y - z|$.

Case (iv): $z < x < y$, the TE becomes a *strict inequality* $|x - z| < |x - y| + |y - z|$.

Case (v): $x < z < y$, the TE becomes a *strict inequality* $|x - z| < |x - y| + |y - z|$.

Case (vi): $y < z < x$, the TE becomes a *strict inequality* $|x - z| < |x - y| + |y - z|$.

Definition of Betweenness for Real Numbers

Symbol: $x * y * z$

Spoken: *y is between x and z.*

Usage: $x, y, z \in \mathbb{R}$

Meaning: $x < y < z$ or $z < y < x$

Remark: It is a property of real numbers that for given any three distinct real numbers, one is smallest, one is largest, and the other is between them.

Lemma: Betweenness for Real Numbers is Related to the Distance Between Them

(Discussed in Video.03.2a

Not presented as a Lemma in book, but concept shows up in the book's proof of Theorem3.2.3)

Given: distinct real numbers x, y, z

Claim: The following are equivalent ($TFAE$)

(a) $x * y * z$ (That is, the real number *y is between the real numbers x and z.*)

(b) $d_{\mathbb{R}}(x, z) = d_{\mathbb{R}}(x, y) + d_{\mathbb{R}}(y, z)$ (That is, $|x - z| = |x - y| + |y - z|$)

Betweeness of Points (Section 3.2)

Definition of Betweenness for Points in a Metric Geometry

Symbol: $A - B - C$

Spoken: B *is between* A *and* C.

Usage: A, B, C are points in a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$.

Meaning: The following two things are both true

- A, B, C are distinct and collinear
- $d(A, C) = d(A, B) + d(B, C)$ That is, $AC = AB + BC$

Theorem 3.2.2 (Really a Corollary of the Definition)

Given: Points A, B, C in a metric geometry

Claim: The following are equivalent (*TFAE*): (i) $A - B - C$

(ii) $C - B - A$

Theorem 3.2.3 Betweenness of Points is Related to Betweenness of Coordinates

Given: Collinear points A, B, C on line l with ruler f in a metric geometry

Claim: The following are equivalent (*TFAE*)

(i) $A - B - C$ (betweenness of *points*)

(ii) $f(A) * f(B) * f(C)$ (betweenness of *coordinates*)

Corollary 3.2.4 Fact about Three Distinct Collinear Points in a Metric Geometry

Given: Three distinct collinear points P, Q, R in a metric geometry

Claim: Exactly one of the points is between the other two.

Proposition 3.2.5 Betweenness of Points in the Euclidean Plane Expressed Using the Vector Description of a Line

Given: points A, B, C in the *Euclidean plane*, with A, C distinct points.

Claim: The following are equivalent (*TFAE*)

(i) $A - B - C$

(ii) There exists a number t with $0 < t < 1$ such that $B = A + t(C - A)$

Theorem 3.2.6 Existence of Points with Certain Betweenness Relationships

Given: Distinct points A, B in a *metric geometry*

Claim: (i) There exists a point C with $A - B - C$

(ii) There exists a point D with $A - D - B$

Definition of Betweenness for Four Points in a Metric Geometry (book page51)

Symbol: $A - B - C - D$

Meaning: $A - B - C$ and $A - B - D$ and $A - C - D$ and $B - C - D$.

Challenge Theorem CT.3.2.7 (based on 3.2#7) (Not Presented as a Theorem in book.)

Fact about Betweenness involving Four Points

If $A - B - C$ and $B - C - D$ then $A - B - D$ and $A - C - D$.

Remark: Since all four relationships are therefore true, we write $A - B - C - D$.

Segments (Section 3.3)

Definition of Segment

Symbol: $\overline{AB}$

Spoken: *segment A B*.

Usage: A, B are distinct points in a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$.

Meaning: the set

$$\overline{AB} = \{C \in \mathcal{P} | C = A \text{ or } A - C - B \text{ or } C = B\}$$

Additional Terminology

The **end points** (or **vertices**) of $\overline{AB}$ are the points A and B.

The **interior of the segment** is the set of all points of the segment that are *not* endpoints:

$$\text{int}(\overline{AB}) = \overline{AB} - \{A, B\} = \{C \in \mathcal{P} | A - C - B\}$$

Symbol: length$(\overline{AB})$

Spoken: the **length** of segment $\overline{AB}$

Meaning: the number AB. That is, the length is the number $d(A, B)$.

A **midpoint of segment $\overline{AB}$** is a point $M \in \overleftrightarrow{AB}$ such that $MA = MB$.

Challenge Theorem CT.3.3.0.a (Exercise 3.3 #11)

If M is a midpoint of segment $\overline{AB}$, then $A - M - B$.

Challenge Theorem CT.3.3.0.b (Exercise 3.3 #12)

Given a segment,

(a) The segment has a midpoint.

(b) The midpoint is unique.

Challenge Theorem CT.3.3.0.c (Exercise 3.3#13)

Order of Endpoints Does Not Matter in the Symbol for Segment

If A and B are distinct points in a metric geometry, then $\overline{AB} = \overline{BA}$

Congruence, Passing Points, Extreme Points

Definition of Segment Congruence

Symbol: $\overline{AB} \simeq \overline{CD}$

Spoken: *segment A B is congruent to segment C D* .

Meaning: The segments have the same length. That is, $AB = CD$.

Challenge Theorem CT.3.3.0.d (Exercise 3.3#4)

In any metric geometry, segment congruence is an equivalence relation.

Definition of Passing Point and Extreme Point of a Subset in a Metric Geometry

Words: B is a *passing point* of $\mathcal{A}$.

Usage: A metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$ in the discussion, and $\mathcal{A} \subset \mathcal{P}$ and $B \in \mathcal{A}$

Meaning: There exist points $X, Y \in \mathcal{A}$ such that $X - B - Y$

Words: B is an *extreme point* of $\mathcal{A}$.

Usage: Same usage

Meaning: B is *not* a *passing point* of $\mathcal{A}$.

Theorem 3.3.2 The Extreme Points of a Line Segment are the Endpoints

Given: distinct points A, B in a *metric geometry*

Claim: The only extreme points of $\overline{AB}$ are endpoints A, B; All other points are passing points. (**Corollary:** If $\overline{AB} = \overline{CD}$, then $\{A, B\} = \{C, D\}$.

Rays (Section 3.3)

Definition of Ray

Symbol: $\overrightarrow{AB}$

Spoken: *ray A B*.

Usage: A, B are distinct points in a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$.

Meaning: the set

$$\begin{aligned}\overrightarrow{AB} &= \{C \in \mathcal{P} | C = A \text{ or } A - C - B \text{ or } C = B \text{ or } A - B - C\} \\ &= \overline{AB} \cup \{C \in \mathcal{P} | A - B - C\}\end{aligned}$$

Additional Terminology

The **initial point** (or **vertex**) of $\overrightarrow{AB}$ is the point A.

The **interior of the ray** is the set of all points of the ray except the initial point:

$$\operatorname{int}(\overrightarrow{AB}) = \overrightarrow{AB} - \{A\} = \{C \in \mathcal{P} | A - C - B \text{ or } C = B \text{ or } A - B - C\}$$

Challenge Theorem 3.3.2b (Not stated as a theorem in the book)

The Only Extreme Point of a Ray is the vertex (or initial point)

Given: distinct points A, B in a *metric geometry*

Claim: The only extreme points of $\overrightarrow{AB}$ is the initial point A; All other points are passing points.

(**Corollary (This is Theorem 3.3.4(ii):** If $\overrightarrow{AB} = \overrightarrow{CD}$, then $A = C$.)

Theorem 3.3.4 Subtlety in the Notation for a Ray

(i) (Different symbols that represent the same ray.) If $C \in \overrightarrow{AB}$ and $C \neq A$, then $\overrightarrow{AC} = \overrightarrow{AB}$.

(ii) (If two rays are equal then their initial points are equal.) If $\overrightarrow{AB} = \overrightarrow{CD}$, then $A = C$.

Unions and Intersections of Rays

Challenge Theorem CT3.3.4b (Not presented as a theorem in the book.) (Exercise 3.3#14)

In a metric geometry, if $D \in \overleftrightarrow{AB} - \overrightarrow{AB}$, then $\overleftrightarrow{AB} = \overrightarrow{AD} \cup \overrightarrow{AB}$.

Challenge Theorem CT3.3.4c (Not presented as a theorem in the book.) (Exercise 3.3#14)

In a metric geometry, if A and B are distinct points, then $\overleftrightarrow{AB} = \overrightarrow{AB} \cup \overrightarrow{BA}$ and $\overline{AB} = \overrightarrow{AB} \cap \overrightarrow{BA}$.

Describing Segments and Rays using Coordinate Functions and Vectors

Proposition 3.3.3 In the ***Euclidean Plane***, segments and rays can be described as

$\overline{AB} = \{A + t(B - A) \text{ for some real number } t \text{ with } 0 \le t \le 1\}$

$\overrightarrow{AB} = \{A + t(B - A) \text{ for some real number } t \text{ with } 0 \le t\}$

Theorem 3.3.5 (Statement (i) is not part of the book's theorem, but it should be.)

If A and B are distinct points in a ***metric geometry***,

then there is a ruler f for line $\overleftrightarrow{AB}$ such that segment $\overline{AB}$ and ray $\overrightarrow{AB}$ can be described as

(i) $\overline{AB} = \{X \in \overleftrightarrow{AB} \text{ such that } 0 \le f(X) \le f(B)\} = f^{-1}([0, f(B)])$

(ii) $\overrightarrow{AB} = \{X \in \overleftrightarrow{AB} \text{ such that } 0 \le f(X)\} = f^{-1}([0, \infty))$

Theorem 3.3.6 (Segment Construction) (A Better Name: Congruent Segment Existence)

Given ray $\overrightarrow{AB}$ and segment $\overline{PQ}$ in a metric geometry,

there exists a unique point $C \in \overrightarrow{AB}$ such that $\overline{PQ} \simeq \overline{AC}$.

Theorem 3.3.8 (Segment Addition)

In a metric geometry, if $A - B - C$ and $P - Q - R$ and $\overline{AB} \simeq \overline{PQ}$ and $\overline{BC} \simeq \overline{QR}$, then $\overline{AC} \simeq \overline{PR}$.

Theorem 3.3.9 (Segment Subtraction)

In a metric geometry, if $A - B - C$ and $P - Q - R$ and $\overline{AB} \simeq \overline{PQ}$ and $\overline{AC} \simeq \overline{PR}$, then $\overline{BC} \simeq \overline{QR}$.

Angles and Triangles (Section 3.4)

Definition of Angle

Symbol: $\angle ABC$

Spoken: *angle A B C*.

Usage: A, B, C are noncollinear points in a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$.

Meaning: the set $\angle ABC = \overrightarrow{BA} \cup \overrightarrow{BC}$

Additional Terminology: The **vertex** of $\angle ABC$ is the point B.

Challenge Theorem CT3.4.0 (Not stated as a Theorem in the book.) (Exercise 3.4#1)

In a metric geometry, $\angle ABC = \angle CBA$.

Lemma 3.4.1: The Only Extreme Point of an Angle is the Vertex

Theorem 3.4.2 (Corollary) If $\angle ABC = \angle DEF$, then $B = E$.

Definition of Triangle

Symbol: ΔABC

Spoken: *triangle A B C*.

Usage: A, B, C are noncollinear points in a metric geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$.

Meaning: the set $\Delta ABC = \overline{AB} \cup \overline{BC} \cup \overline{CA}$

Additional Terminology:

The **vertices** of ΔABC are the points A, BC.

The **sides** (or **edges**) of ΔABC are the segments $\overline{AB}, \overline{BC}, \overline{CA}$.

Challenge Theorem CT3.4.2.b (Not stated as a Theorem in the book.) (Exercise 3.4#2)

If D, E, F are noncollinear points in a metric geometry, and line l contains at most one of D, E, F, then each of the lines $\overleftrightarrow{DE}, \overleftrightarrow{EF}, \overleftrightarrow{FD}$ intersects l in at most one point.

Lemma 3.4.3: The Only Extreme Points of a Triangle are the Vertices

Theorem 3.4.4 (Corollary) If $\Delta ABC = \Delta DEF$, then $\{A, B, C\} = \{D, E, F\}$.

Partitions and Convex Sets (Section 4.1)

Definition of Partition of a Set

Words: $\{A_1, A_2, A_3, \ldots\}$ *is a partition of set* A.

Meaning: The following three requirements are all satisfied.

- Each of the A_i is a non-empty subset of A.
- A is the union of all the A_i. That is,

$$A = \bigcup_i A_i$$

- The sets $A_1, A_2, A_3, \ldots$ are mutually disjoint. That is,

$$\text{If } i \neq j \text{ then } A_i \cap A_j = \phi$$

Definition of Convex

Words: S is convex

Usage: A metric geometry $(\mathcal{P}, \mathcal{L}, d)$ is given, and $S \subset \mathcal{P}$ is a set of points.

Meaning: For every two distinct points $A, B \in S$, the segment $\overline{AB}$ is a subset of S.

Quantified version: $\forall A, B \in S, A \neq B (\overline{AB} \subset S)$.

Universal Conditional Version: $\forall A, B \in \mathcal{P}, A \neq B (\text{If } A, B \in S \text{ then } \overline{AB} \subset S)$

Challenge Theorem CT4.1.0.1 (Not stated as a Theorem in the book.) (Exercise 4.1#1)

If S_1 and S_2 are convex sets in a metric geometry, then $S_1 \cap S_2$ is convex.

Challenge Theorem CT4.1.0.2 (Not stated as a Theorem in the book.) (Exercise 4.1#2)

If l is a line in a metric geometry, then l is convex.

The Plane Separation Axiom (Section 4.1)

Definition: The Plane Separation Axiom (PSA) (**Barsamian's version of the definition**)

Words: A metric Geometry $(\mathcal{P},\mathcal{L},d)$ satisfies the **plane separation axiom** (PSA)

Meaning: For every line $l \in \mathcal{L}$, there are two associated sets of points called *half planes*, denoted H_1 and H_2, with the following properties:

(i) The three sets l, H_1, H_2 form a partition of the set $\mathcal{P}$ of all points.

(ii) Each of the *half planes* is convex.

(iii) If $A \in H_1$ and $B \in H_2$, then $\overline{AB}$ *intersects* line l.

Additional Terminology:

Line l is called the ***edge*** of *half planes* H_1 and H_2.

Words: *Points A, B lie on the* ***same side*** *of line l.*

Meaning: Points A, B are elements of the same half plane associated to l.

Words: *Points A, B lie on* ***opposite sides*** *of line l.*

Meaning: Points A, B are elements of different half planes associated to l.

PSA(ii) and PSA(iii) are *conditional statements*, so their *contrapositives* are *automatically true*.

Given: A metric geometry satisfying PSA, a line l, and points A, B not on l

PSA (ii): If distinct points P, Q are in the same *half plane*, then $\overline{PQ}$ does not intersect line l.

PSA (ii) (contrapositive): If $\overline{PQ}$ does intersect line l, then P, Q are *not* in the same *half plane*.

PSA (iii) If P, Q are not in the same *half plane*, then $\overline{PQ}$ intersects line l.

PSA (iii) (contrapositive) If $\overline{PQ}$ does not intersect line l, then P, Q are distinct points in the same *half plane*.

Definitions and Theorems about the Plane Separation Axiom (Section 4.1)

Challenge Theorem CT4.1.0.3 (Not stated as a Theorem in the book.) (Exercise 4.1#4)

If L is a line in a metric geometry that satisfies the Plane Separation Axiom, then each of the half-planes determined by L contains a point.

New Challenge Theorem CT4.1.0.4 (Not stated as a Theorem in the book.) (Exercise 4.1#5)

If L is a line in a metric geometry that satisfies the Plane Separation Axiom, then each of the half-planes determined by L contains three non-collinear points.

Theorem 4.1.1 (Uniqueness of the Half Planes for a line in *PSA*)

If l is a line in a metric geometry and two sets of sets of points, denoted $\{H_1, H_2\}$ and $\{H_1', H_2'\}$, both satisfy the PSA half plane requirements,

then $\{H_1, H_2\} = \{H_1', H_2'\}$. This means that either $H_1 = H_1'$ and $H_2 = H_2'$, or $H_1 = H_2'$ and $H_2 = H_1'$,

Definition of *same side* of a line and *opposite sides* of a line. (This terminology is redundant.)

Usage: There is a metric geometry satisfying PSA in the discussion.

Words: Points P, Q are on the ***same side*** of line l.

Meaning: P, Q are in the same half-plane of l.

Words: Points P, Q are on ***opposite sides*** of line l.

Meaning: P, Q are in different half-planes of l.

Theorem 4.1.2

Given: A metric geometry satisfying PSA, a line l, and points A, B not on l

Claim:

(i) A and B are on opposite sides of l if and only if $\overline{AB} \cap l \neq \phi$.

(ii) A and B are on the same side of if and only if $\overline{AB} \cap l = \phi$ or $A = B$.

Challenge Theorem CT4.1.2.1 (Not stated as a Theorem in the book.) (Exercise 4.1#8)

Let l be a line in a metric geometry that satisfies PSA.

Define a relation $\sim$ on the set of points that are not on l by saying that $P \sim Q$ means that P and Q are on the same side of l.

Claim:

(i) $\sim$ is an equivalence relation on the set of points that are not on l.

(ii) the equivalence classes of $\sim$ are the half-planes determined by l.

Theorem 4.1.3

Given: A metric geometry satisfying PSA, and a line l

Claim:

If P and Q are on opposite sides of l, and Q and R are on opposite sides of l, then P and R are on the same side of l.

Theorem 4.1.4

Given: A metric geometry satisfying PSA, and a line l

Claim:

If P and Q are on opposite sides of l, and Q and R are on the same of l, then P and R are on opposite sides of l.

Theorem 4.1.5 (Uniqueness of the line for a half plane in *PSA*)

If H is a half-plane determined by a line l, and H is also a half-plane determined by a line l', then $l = l'$.

Definition of Edge of a Half-Plane

Words: l is the edge of half-plane H.

Meaning: H_1 is a half-plane determined by l.

The Plane Separation Axiom (PSA) for the Euclidean and Poincaré Planes (Section 4.2)

Definition of Cartesian half planes (Barsamian's Definition)

For real numbers a, m, b, define the define the *Cartesian half planes* as follows

$$H_a^+ = \{(x,y) \in \mathbb{R}^2 | x > a\}$$
$$H_a^- = \{(x,y) \in \mathbb{R}^2 | x < a\}$$
$$H_{m,b}^+ = \{(x,y) \in \mathbb{R}^2 | y > mx + b\}$$
$$H_{m,b}^- = \{(x,y) \in \mathbb{R}^2 | y < mx + b\}$$

Proposition 4.2.4 The Euclidean plane $(\mathbb{R}^2, \mathcal{L}_E, d_E)$satisfies the *PSA*

With *Cartesian half planes* H_a^+ and H_a^- (or $H_{m,b}^+$ and $H_{m,b}^-$) playing the role of half planes for the *vertical line* L_a (or for the *non-vertical line* $L_{m,b}$),
the *Euclidean plane* satisfies the *Plane Separation Axiom (PSA)*.

Challenge Theorem CT4.2.4b (Not stated as a Theorem in the book.) (exercise 4.3#5)
The Taxicab plane $(\mathbb{R}^2, \mathcal{L}_E, d_T)$ satisfies the *PSA*

With *Cartesian half planes* H_a^+ and H_a^- (or $H_{m,b}^+$ and $H_{m,b}^-$) playing the role of half planes for the *vertical line* L_a (or for the *non-vertical line* $L_{m,b}$),
the *Taxicab plane* satisfies the *Plane Separation Axiom (PSA)*.

Definition of Poincaré half planes

For real numbers a, c, r with $r > 0$, define the define the *Poincaré half planes* as follows

$$_aH^+ = \{(x,y) \in \mathbb{H} | x > a\}$$
$$_aH^- = \{(x,y) \in \mathbb{H} | x < a\}$$
$$_cH_r^+ = \{(x,y) \in \mathbb{H} | (x-c)^2 + y^2 > r^2\}$$
$$_cH_r^- = \{(x,y) \in \mathbb{H} | (x-c)^2 + y^2 < r^2\}$$

Proposition 4.2.5 The Poincaré plane satisfies the *PSA*

With *Poincaré half planes* $_aH^+$ and $_aH^-$ (or $_cH_r^+$ and $_cH_r^+$) playing the role of half planes for the *type I line* $_aL$ (or for the *type II line* $_cL_r$),
the *Poincaré plane* satisfies the *Plane Separation Axiom (PSA)*.

Pasch Geometries (Section 4.3)

Definition: Pasch's Postulate (PP)

Words: A metric Geometry $(\mathcal{P}, \mathcal{L}, d)$ satisfies **Pasch's Postulate (PP)**

Meaning: For every line and for every triangle, if the line intersects a side of the triangle at a point that is not a vertex, then the line intersects at least one of the opposite sides.

Challenge Theorem CT4.3.0a Two Equivalent Statements in a Metric Geometry (Contains all the information of the textbook's Theorems 4.3.1 and 4.3.2, and 4.3.3, and can be used in place of any of those three theorems.)

Given: Metric Geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$

Claim: The following statements are equivalent (TFAE)

(1) The metric geometry satisfies the *Plane Separation Axiom (PSA)*.

(2) The metric geometry satisfies *Pasch's Postulate (PP)*.

Definition of Pasch Geometry

A ***Pasch Geometry*** is a *metric geometry* that satisfies the *Plane Separation Axiom (PSA)*.

Remark: By the *Theorem About Two Equivalent Statements in a Metric Geometry*, we see that Pasch Geometries are also the metric geometries that satisfy *Pasch's Postulate (PP)*.

Definition: Peano's Axiom (PA) (Introduced in Exercise 4.3#1)

- **Words:** A metric Geometry $(\mathcal{P}, \mathcal{L}, d)$ satisfies ***Peano's Axiom (PA)***
- **Meaning:** Given triangle ΔABC and points D, E such that $B - C - D$ and $A - E - C$, there exists a point $F \in \overleftrightarrow{DE}$ such that $A - F - B$ and $D - E - F$.

Challenge Theorem CT.4.3.4 (Not stated as a Theorem in the book.) (Exercise 4.3#1)

If a metric geometry satisfies the *Plane Separation Axiom (PSA)*,
then the metric geometry also satisfies *Peano's Axiom (PA)*.

Challenge Theorem CT.4.3.5 (Not stated as a Theorem in the book.) (Exercise 4.3#2)

In a metric geometry that satisfies *PSA*, if ΔABC and points D, F have properties $B - C - D$ and $A - F - B$, then there exists a point E such that $D - E - F$ and $A - E - C$.

Challenge Theorem CT.4.3.6 (Not stated as a Theorem in the book.) (Exercise 4.3#3)

In a metric geometry that satisfies *PSA*, Given any ΔABC and any point P,
there exists a line through P that contains exactly two points of ΔABC.

Convex Sets and Half Planes and Facts about Them (Section 4.4)

Challenge Theorem CT4.4.0.1 (Not stated as a Theorem in the book.)

In a metric geometry, every ray is convex.

Challenge Corollary CT4.4.0.2 (Not stated as a Theorem in the book.)

In a metric geometry, the interior of every ray is convex.

Challenge Theorem CT4.4.0.3 (Not stated as a Theorem in the book.)

In a metric geometry, every segment is convex.

Challenge Corollary CT4.4.0.4 (Not stated as a Theorem in the book.)

In a metric geometry, the interior of every segment is convex.

Theorem 4.4.1 In a Pasch Geometry, if $\mathcal{A}$ is a nonempty convex set that does not intersect line l, then all points of $\mathcal{A}$ lie on the same side of l.

Theorem 4.4.2 In a Pasch Geometry, let $\mathcal{A}$ be a line, ray, segment, interior of a ray, or interior of a segment.

(i) If l is a line with $\mathcal{A} \cap l = \phi$, then all of $\mathcal{A}$ lies on one side of l.

(ii) If $A - B - C$ and $\overleftrightarrow{AC} \cap l = \{B\}$

then $\text{int}(\overrightarrow{BA})$ and $\text{int}(\overline{BA})$ both lie **(entirely)** on the same side of l,

while $\text{int}(\overrightarrow{BC})$ and $\text{int}(\overline{BC})$ both lie **(entirely)** on the other side of l.

Theorem 4.4.3 (*The Z Theorem*) In a Pasch geometry, if P and Q are on opposite sides of $\overleftrightarrow{AB}$, then $\overrightarrow{BP} \cap \overrightarrow{AQ} = \phi$. In particular, $\overline{BP} \cap \overline{AQ} = \phi$.

Angle and Triangle Interiors (Section 4.4)

Definition of More Descriptive Half Plane Notation

Symbol: $H_{\overleftrightarrow{AB},C}$

Usage: A, B, C are non-collinear points in a Pasch geometry.

Meaning: The half plane of line A, B that contains C.

Definition of Angle and Triangle Interiors

Symbol: $\text{int}(\angle ABC)$

Spoken: *the interior of angle* A, B, C

Meaning: $H_{\overleftrightarrow{BA},C} \cap H_{\overleftrightarrow{BC},A}$

Symbol: $\text{int}(\Delta ABC)$

Spoken: *the interior of triangle* A, B, C

Meaning: $H_{\overleftrightarrow{AB},C} \cap H_{\overleftrightarrow{BC},A} \cap H_{\overleftrightarrow{CA},B}$

Theorem 4.4.4 The Interior of an Angle does not depend on the particular points used.

In a Pasch geometry, if $\angle ABC = \angle A'BC'$, then $\text{int}(\angle ABC) = \text{int}(\angle A'BC')$.

Theorem 4.4.5 (Really just a Corollary of the Definitition of Angle Interior!)

Given: $\angle ABC$ and point P in a Pasch geometry.

Claim: The Following Are Equivalent (TFAE)

(i) $P \in \text{int}(\angle ABC)$

(ii) A and P are on the same side of $\overleftrightarrow{BC}$ and C and P are on the same side of $\overleftrightarrow{BA}$.

Challenge Corollary CT4.4.5.1 (Not stated as a Theorem in the book.) (Exercise 4.4#18)

In a Pasch geometry, $\text{int}(\Delta ABC) = \text{int}(\angle ABC) \cap \text{int}(\angle BCA) \cap \text{int}(\angle CAB)$

Theorem 4.4.6 Given $\angle ABC$ in a Pasch geometry, if $A - P - C$, then $P \in \text{int}(\angle ABC)$.

Immediate consequence (corollary) of Theorem 4.4.6: In a Pasch geometry, all points in the interior of one side of a triangle are in the interior of the opposite angle. That is, in any triangle ΔABC the following subset relationship is true: $\text{int}(\overline{AC}) \subset \text{int}(\angle ABC)$

The Crossbar Theorem and Related Theorems(Section 4.4)

Theorem 4.4.7 (The Crossbar Theorem)

Given: a Pasch geometry, angle $\angle ABC$, and point P.

Claim: If $P \in \text{int}(\angle ABC)$, then $\overrightarrow{BP}$ intersects $\overline{AC}$ at a point F such that $A - F - C$.

Challenge Theorem CT4.4.7.1 (Not stated as a Theorem in the book.) (Exercise 4.4#5)

Given: a Pasch geometry and $P \in \text{int}(\angle ABC)$

Claim: $\text{int}\left(\overrightarrow{BP}\right) \in \text{int}(\angle ABC)$

Challenge Theorem CT4.4.7.2 (Not stated as a Theorem in the book.) (Exercise 4.4#6)

Given: a Pasch geometry, ΔABC and D, E, F such that $B - C - D$, $A - E - C$ and $B - E - F$

Claim: $F \in \text{int}(\angle ACD)$

Challenge Theorem CT4.4.7.3 (Not stated as a Theorem in the book.) (Exercise 4.4#8)

Given: a Pasch geometry, $P \in \text{int}(\angle ABC)$ and $D = \overrightarrow{AP} \cap \overrightarrow{BC}$

Claim: $A - P - D$

Theorem 4.4.8

Given: a Pasch geometry, a line $\overleftrightarrow{AB}$, and distinct points C, P on the same side of $\overleftrightarrow{AB}$.

Claim: The Following Are Equivalent (TFAE)

(i) $P \in \text{int}(\angle ABC)$

(ii) A and C are on opposite sides of line $\overleftrightarrow{BP}$

Theorem 4.4.9

Given: a Pasch geometry, and points A, B, C, D, P such that $A - B - D$ and such that P, C are on the same side of $\overleftrightarrow{AD}$

Claim: The Following Are Equivalent (TFAE)

(i) $P \in \text{int}(\angle ABC)$

(ii) $C \in \text{int}(\angle DBP)$

Challenge Theorem CT4.4.9.1 (Not stated as a Theorem in the book.) (Exercise 4.4#11)

Given: a Pasch geometry, $\overleftrightarrow{AB}$,and distinct points C, P on the same side of $\overleftrightarrow{AB}$.

Claim: Either $\overrightarrow{BC} = \overrightarrow{BP}$ or $P \in int(\angle ABC)$ or $C \in int(\angle ABP)$

Challenge Theorem CT4.4.9.2 (Not stated as a Theorem in the book.) (Exercise 4.4#12)

The Converse of the Crossbar Theorem

Given: a Pasch geometry, angle $\angle ABC$, and point P.

Claim: If $\overrightarrow{BP}$ intersects $\overline{AC}$ at a point F such that $A - F - C$, then $P \in \text{int}(\angle ABC)$

Challenge Theorem CT4.4.9.3 (Not stated as a Theorem in the book.) (Exercise 4.4#13)

Given: a Pasch geometry, angle $\angle ABC$, $D \in \text{int}\left(\overrightarrow{BA}\right)$ and $E \in \text{int}\left(\overrightarrow{BC}\right)$ and a point P.

Claim: If $\overrightarrow{BP}$ intersects $\text{int}(\overline{AC})$ then $\overrightarrow{BP}$ also intersects $\text{int}(\overline{DE})$

Theorem 4.4.10

Given: a Pasch geometry, and ΔABC.

Claim: $\text{int}(\Delta ABC)$ is convex

Challenge Theorem CT4.4.10.1 (Not stated as a Theorem in the book.) (Exercise 4.4#17)

(Informally: A triangle cannot enclose a line.**)**

Given: a Pasch geometry

Claim: If line l intersects $\text{int}(\Delta ABC)$ then l intersects ΔABC in exactly two points.

Challenge Theorem CT4.4.10.2 (Not stated as a Theorem in the book.) (Exercise 4.4#19)

An alternative Description of the Interior of a Triangle

Given: a Pasch geometry

Claim: $\text{int}(\Delta ABC) = \{P | \text{there exists a } D \text{ such that } B - D - C \text{ and } A - P - D\}$.

Challenge Theorem CT4.4.10.3 (Not stated as a Theorem in the book.) (Exercise 4.4#23)

(Informally: An angle cannot enclose a line.**)**

Given: a Pasch geometry

Claim: If line l intersects $\text{int}(\angle ABC)$ then l intersects $\angle ABC$.

Crossbar Interiors (Section 4.4)

Definition of the Crossbar Interior of an Angle (introduced in Exercise 4.4#25)

Words: the ***crossbar interior*** of $\angle ABC$

Symbol: $\text{cint}(\angle ABC)$

Usage: A, B, C are non-collinear points in a ***metric geometry***

Definition: The set of points defined as follows:

$$\text{cint}(\angle ABC) = \left\{P \middle| D - P - E \text{ for some } D \in \text{int}\left(\overrightarrow{BA}\right) \text{ and some } E \in \text{int}\left(\overrightarrow{BC}\right)\right\}$$

Remark: Observe that the crossbar interior of an angle does not rely on the notion of half-planes, so the definition works for angles in a metric geometry.

Challenge Theorem CT4.4.10.4 (Not stated as a Theorem in the book.)

In the ***Euclidean plane***, $\text{cint}(\angle ABC) = \text{int}(\angle ABC)$

Challenge Theorem CT4.4.10.5 (Not stated as a Theorem in the book.)
(The subject of Exercise 4.4#25)

In the ***Poincaré plane***, $\text{cint}(\angle ABC) \subsetneq \text{int}(\angle ABC)$. That is, the crossbar interior of any angle is a proper subset of the interior of the angle. There are points in the interior of the angle that are *not* in the crossbar interior of the angle.

Quadrilaterals (Section 4.5)

Definition of Quadrilateral

words: quadrilateral A, B, C, D

symbol: $\square ABCD$

usage: A, B, C, D are distinct points, no three of which are collinear, and such that the segments $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$ intersect only at their endpoints.

meaning: quadrilateral A, B, C, D is the set $\square ABCD = \overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA}$

additional terminology: Points A, B, C, D are each called a ***vertex*** of the quadrilateral. Segments $\overline{AB}$ and $\overline{BC}$ and $\overline{CD}$ and $\overline{DA}$ are the ***sides*** of the quadrilateral. Segments $\overline{AC}$ and $\overline{BD}$ are the ***diagonals***.

In a *Pasch geometry*, a ***"convex-shape quadrilateral"*** is one in which all the points of any given side lie on the same side of the line determined by the opposite side. A quadrilateral that does not have this property is called a ***"non-convex-shape quadrilateral"***. (Note: the book says "convex quadrilateal" and "non-convex quadrilateral.)

Theorem 4.5.1 (Corollary of the Definition of Quadrilateral)

Given a Quadrilateral $\square ABCD$ in a metric geometry.

$$\square ABCD = \square BCDA = \square CDAB = \square DABC = \square ADCB = \square DCBA = \square CBAD = \square BADC$$

Furthermore, if both $\square ABCD$ and $\square ABDC$ exist, they are not equal.

Theorem 4.5.2 In a metric geometry, if $\square ABCD = \square PQRS$, then $\{A, B, C, D\} = \{P, Q, R, S\}$. Furthermore, if $A = P$, then $C = R$ and either $B = Q$ or $B = S$ so that the sides, angles, and diagonals of $\square ABCD$ are the same as those of $\square PQRS$.

Convex Quadrilaterals

Definition of Convex Quadrilateral

Words: A quadrilateral is ***convex***.

Usage: There is a Pasch geometry in the discussion and the quadrilateral is in that geometry.

Meaning: Each side of the quad. lies entirely in a half plane determined by its opposite side.

Remark: This is ***terrible*** terminology. Note that EVERY quadrilateral is NOT convex as a set of points. (Just as EVERY triangle is NOT convex.) It would be much better to use the term ***convex shaped quadrilateral*** for the thing that this definition is describing!

Challenge Theorem 4.5.2.1 (Not stated as a Theorem in the Book)(Exercise 4.5#5)
(Really just a corollary of the Definition of Convex)

Given a Quadrilateral in a Pasch geometry

Claim: the following are equivalent:

(i) The quadrilateral is convex.

(ii) Each side does not intersect the line determined by its opposite side.

Definition of the Interior of a Convex Quadrilateral

Symbol: int(□$ABCD$)

Spoken: *the interior of convex quadrilateral* A, B, C, D

Usage: *Quadrilateral* A, B, C, D has to be convex for this definition to work!

Meaning: $H_{\overleftrightarrow{BA},C} \cap H_{\overleftrightarrow{BC},D} \cap H_{\overleftrightarrow{CD},A} \cap H_{\overleftrightarrow{DA},B}$

Challenge Theorem 4.5.2.2 (Not stated as a Theorem in the Book)(Exercise 4.5#6)

Given a Quadrilateral in a Pasch geometry.

If the quadrilateral is convex, then its interior is a convex set of points.

Theorem 4.5.3 Given a Quadrilateral in a Pasch geometry

Claim: the following are equivalent:

(i) The quadrilateral is convex.

(ii) The vertex of each angle is contained in the interior of the opposite angle.

Challenge Theorem 4.5.3.2 (Not stated as a Theorem in the Book)(Exercise 4.5#11)

Given a Quadrilateral in a Pasch geometry.

Claim: The lines containing the diagonals of the quadrilateral intersect.

Theorem 4.5.4 Given a Quadrilateral in a Pasch geometry.

If the quadrilateral is convex, then its diagonals intersect.

Challenge Theorem 4.5.4.1 (Not stated as a Theorem in the Book)(Exercise 4.5#7)

Given a Quadrilateral in a Pasch geometry.

If the diagonals of the quadrilateral intersect, then the quadrilateral is convex.

Challenge Theorem 4.5.4.2 (Not stated as a Theorem in the Book)(Just a Corollary)

Given a Quadrilateral in a Pasch geometry.

Claim: the following are equivalent:

(i) The quadrilateral is convex.

(ii) The diagonals of the quadrilateral intersect.

Theorem 4.5.5 Given a Quadrilateral in a Pasch geometry.

If one pair of opposite sides is parallel (that is, they lie in lines that are parallel), then the quadrilateral is convex.

Angle Measure (Section 5.1)

Definition: The symbol $\mathcal{A}$ denotes the set of all angles in a Pasch Geometry

Definition of Angle Measure

Words: Angle measure (or protractor) based on r_0

Usage: There is a Pasch geometry in the discussion, and r_0 is a fixed positive real number

Meaning: a function $m: \mathcal{A} \to \mathbb{R}$ that has these three properties (the *Axioms of Angle Measure*)

(i) $0 < m(\angle ABC) < r_0$

(ii) (This statement is called the *Angle Constuction Axiom*)

Given

- a half plane H
- a ray $\overrightarrow{BC}$ on the edge of that half plane
- a number θ such that $0 < \theta < r_0$

There exists a unique ray $\overrightarrow{BA}$ with $A \in H$ such that $m(\angle ABC) = \theta$.

(iii) (This statement is called the *Angle Addition Axiom*)

If $D \in int(\angle ABC)$, then $m(\angle ABD) + m(\angle DBC) = m(\angle ABC)$

The three *Axioms of Angle Measure* are a “wish list” of properties that a function must have in order to be qualified to be called an “angle measure”.

Definition: A **Protractor Geometry** is an ordered quadruple $(\mathcal{P}, \mathcal{L}, d, m)$ such that the ordered triple $(\mathcal{P}, \mathcal{L}, d)$ is a *Pasch Geometry* and m is an *angle measure* for $(\mathcal{P}, \mathcal{L}, d)$.

Euclidean and Poincaré Angle Measure (Section 5.1)

Definition: Euclidean Angle Measure is the function $m_E \colon \mathcal{A} \to \mathbb{R}$ ($\mathcal{A}$ is the set of angles in $\mathbb{R}^2$) defined by

$$m_E(\angle ABC) = \cos^{-1}\left(\frac{\langle A-B, C-B\rangle}{\|A-B\|\|C-B\|}\right)$$

Remarks: Using *Degrees* for the $\cos^{-1}$, because we are always using $r_0 = 180$ in our course.

The expression $A - B$ is the *Euclidean Vector from B to A*.

Definition of the Euclidean Tangent to a Poincaré Ray

For distinct points $P = (x_P, y_P)$ and $Q = (x_Q, y_Q)$ in $\mathbb{H}$, the **Euclidean Tangent to Poincaré Ray $\overrightarrow{PQ}$** is denoted by the symbol T_{PQ} and is computed as follows:

If $x_P = x_Q$, then $\overleftrightarrow{PQ}$ is a *type I line* ${}_{x_P}L$ and $T_{PQ} = (0, y_Q - y_P)$

If $x_P < x_Q$, then $\overleftrightarrow{PQ}$ is a *type II line* ${}_cL_r$. Find c, r and then build $T_{PQ} = (y_P, c - x_P)$

If $x_P > x_Q$, then $\overleftrightarrow{PQ}$ is a *type II line* ${}_cL_r$. Find c, r and then build $T_{PQ} = -(y_P, c - x_P)$

Observe that in the case that $\overleftrightarrow{PQ}$ is a *type II line* ${}_cL_r$, the computation of T_{PQ} only needs the value of c for the line. The value of r is not used for the computation of T_{PQ}, but it turns out that the value of r will always equal the norm, $\|T_{PQ}\|$. The norm is needed for the angle calculation. So it is worth finding both c and r for the line.

Definition: Poincaré Angle Measure is the function $m_E \colon \mathcal{A} \to \mathbb{R}$ ($\mathcal{A}$ denotes the set of angles in $\mathbb{H}$) defined by

$$m_H(\angle ABC) = \cos^{-1}\left(\frac{\langle T_{BA}, T_{BC}\rangle}{\|T_{BA}\|\|T_{BC}\|}\right)$$

Theorems about Angle Measure (Section 5.3)

Challenge Theorem 5.3.0.1 (Corollary of Angle Measurement Axiom (iii) (Angle Addition))

If $D \in int(\angle ABC)$ then $m(\angle ABD) < m(\angle ABC)$.

Theorem 5.3.1: In a protractor geometry, given points C, D in the same half plane of $\overleftrightarrow{AB}$, if $m(\angle ABD) < m(\angle ABC)$ then $D \in int(\angle ABC)$

Challenge Theorem 5.3.1.1 (Corollary)

In a protractor geometry, given points C, D in the same half plane of $\overleftrightarrow{AB}$.

Claim: The following are equivalent (TFAE)

(i) $D \in int(\angle ABC)$

(ii) $m(\angle ABD) < m(\angle ABC)$

Definition of Linear Pair

Words: *Two angles from a linear pair.*

Meaning: The two angles can be labeled $\angle ABC$ and $\angle CBD$ with $A - B - D$.

Theorem 5.3.2 The Linear Pair Theorem

Given: a protractor geometry and points A and D on opposite sides of a line $\overleftrightarrow{BC}$

Claim: If $A - B - D$ (so the angles form a linear pair), then $m(\angle ABC) + m(\angle CBD) = 180$.

Theorem 5.3.3 (Converse of the Statement of the Angle Addition Axiom)

In a protractor geometry, if $m(\angle ABC) + m(\angle CBD) = m(\angle ABD)$ then $C \in int(\angle ABD)$

Theorem 5.3.4 (Converse of the Statement of the Linear Pair Theorem)

In a protractor geometry, if $m(\angle ABC) + m(\angle CBD) = 180$, then $A - B - D$ (so the angles form a Linear Pair)

Given: a protractor geometry and points A and D on opposite sides of a line $\overleftrightarrow{BC}$

Claim: If $m(\angle ABC) + m(\angle CBD) = 180$, then $A - B - D$ (so the angles form a linear pair).

Acute, Right, and Obtuse Angles (Section 5.3)

Definition of Acute, Right, Obtuse Angles

An **acute angle** is an angle whose measure is less than 90.

A **right angle** is an angle whose measure is 90. (Right angles are marked with a box.)

An **obtuse angle** is an angle whose measure is greater than 90.

Definition of Perpendicular Lines

Words: l and l' are perpendicular

Symbol: $l \perp l'$ (In a drawing, perpendicular lines are marked with a box at the angle.)

Usage: l, l' are lines in a protractor geometry

Meaning: $l \cup l'$ contains a right angle

Definition of Perpendicular Rays, Segments

Words: $\overrightarrow{AB}$ (or $\overline{AB}$) and $\overline{CD}$ (or $\overrightarrow{CD}$) are perpendicular

Usage: $\overrightarrow{AB}$ (or $\overline{AB}$) and $\overline{CD}$ (or $\overrightarrow{CD}$) are segments or rays in a protractor geometry

Meaning: The lines containing those objects are perpendicular. That is, $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$.

Remark: Notice that segments and rays can be perpendicular without intersecting.

Theorem 5.3.5 Existence of a Unique Perpendicular to a Line through a Point On the Line

In a protractor geometry, if B is a point on a line l,

then there exists a unique line l' such that l' contains B and $l' \perp l$.

Definition of Perpendicular Bisector of a Segment

Words: *a perpendicular bisector of* $\overline{AB}$

Usage: $\overline{AB}$ is a segment in a protractor geometry

Meaning: a line l such that $l \perp \overleftrightarrow{AB}$ and $l \cap \overleftrightarrow{AB} = \{M\}$ where M is the midpoint of $\overline{AB}$.

Corollary 5.3.7 Existence of a Unique Perpendicular Bisector

If $\overline{AB}$ is a segment in a protractor geometry, then $\overline{AB}$ has a *unique perpendicular bisector*.

Angle Bisectors and Congruent Angles (Section 5.3)

Definition of Angle Bisector

Words: *a bisector of* $\angle ABC$

Usage: $\angle ABC$ is an angle in a protractor geometry

Meaning: a ray $\overrightarrow{BD}$ such that $D \in \text{int}(\angle ABC)$ and $m(\angle ABD) = m(\angle DBC)$.

Theorem 5.3.8 Existence of a Unique Angle Bisector

If $\angle ABC$ is an angle in a protractor geometry, then $\angle ABC$ has a *unique angle bisector*.

Definition of Congruent Angles

Symbol: $\angle ABC \simeq \angle DEF$

Words: $\angle ABC$ *is congruent to* $\angle DEF$

Usage: $\angle ABC$ and $\angle DEF$ are angles in a protractor geometry

Meaning: $m(\angle ABC) = m(\angle DEF)$

Illustration: Congruent angles are marked with tic marks.

Definition of Vertical Pair (Bowtie Pair)

Words: *Two angles from a vertical pair (bowtie pair).*

Meaning: The two angles can be labeled $\angle ABC$ and $\angle DBE$ with $A - B - D$ and $C - B - E$.

Theorem 5.3.9 (Vertical Angle (Bowtie Angle) Theorem)

In a prot. geometry, if two angles form a vertical pair (a bowtie pair), then they are congruent.

Challenge Theorem 5.3.9.1 (Exercise 5.3#9)

If $\angle ABC$ and $\angle A'BC'$ form a vertical pair and $\angle ABC$ is a right angle,
then so are $\angle A'BC$ and $\angle A'BC'$ and $\angle ABC'$.

Challenge Theorem 5.3.9.2 (Exercise 5.3#15)

Given: $\angle ABC$ in a protractor geometry

Claim: The following are Equivalent

(i) $\angle ABC$ is a right angle

(ii) There exists a point D such that $D - B - C$ and $\angle ABC \simeq \angle ABD$.

Theorems about Congruent Angles (Section 5.3)

Theorem 5.3.10 (Congruent Angle Existence Theorem)

In a protractor geometry, given $\angle ABC$ and a ray $\overrightarrow{ED}$ that lies in the edge of a half plane H, there exists a unique ray $\overrightarrow{EF}$ with $F \in H$ such that $\angle DEF \simeq \angle ABC$.

Theorem 5.3.11 (Congruent Angle Addition Theorem)

In a protractor geometry,
if $D \in int(\angle ABC)$ and $S \in int(\angle PQR)$ and $\angle ABD \simeq \angle PQS$ and $\angle DBC \simeq \angle SQR$,
then $\angle ABC \simeq \angle PQR$.

Theorem 5.3.12 (Congruent Angle Subtraction Theorem)

In a protractor geometry,
if $D \in int(\angle ABC)$ and $S \in int(\angle PQR)$ and $\angle ABD \simeq \angle PQS$ and $\angle ABC \simeq \angle PQR$,
then $\angle DBC \simeq \angle SQR$.

Trigonometric Facts in the Euclidean Plane (Section 6.1)

Proposition 6.1.2 (Euclidean Law of Cosines) (proven in exercise 6.1#3)

If P, Q, R are three non-collinear points in $\mathbb{R}^2$,

then $\left(d_E(P,R)\right)^2 = \left(d_E(Q,P)\right)^2 + \left(d_E(Q,R)\right)^2 - 2d_E(Q,P)d_E(Q,R)\cos\left(m_E(\angle PQR)\right)$.

In other words, for triangle ΔPQR, if p, q, r are defined to be the lengths of the sides opposite those vertices and $\theta = m_E(\angle PQR)$, then $q^2 = p^2 + q^2 - 2pq\cos(\theta)$

CT6.1.2.1 The Pythagorean Theorem of Euclidean Geometry (proven in exercise 6.1#6, Corollary of the Law of Cosines)

In the Euclidean plane, if triangle ΔABC has a right angle at C, then

$\left(d_E(A,B)\right)^2 = \left(d_E(C,B)\right)^2 + \left(d_E(C,A)\right)^2$ In other words, for triangle ΔABC, if a, b, c are the lengths of the sides opposite those vertices and $\angle ACB$ is a right angle, then $c^2 = a^2 + b^2$.

CT6.1.2.2 T Corollary (*SOHCAHTOA* Interpretation of Sin, Cos, Tan) (proven in 6.1#7)

In the *Euclidean plane*, if triangle ΔABC has a right angle at C, and $m_E(\angle B) = \theta$, then

$$\sin(\theta) = \frac{AC}{AB} \quad \text{and} \quad \cos(\theta) = \frac{BC}{BA} \quad \text{and} \quad \tan(\theta) = \frac{AC}{BC}$$

Triangle Congruence (Section 6.1)

Definition of Corresponding Parts of Triangles

Words: Corresponding Parts of ΔABC and ΔDEF.

Usage: ΔABC and ΔDEF are triangles in a protractor geometry and there is a bijection $f: \{A, B, C\} \rightarrow \{D, E, F\}$ in the discussion.

Meaning: The following pairs of parts of ΔABC and ΔDEF

- Side $\overline{AB}$ in ΔABC corresponds to side $\overline{f(A)f(B)}$ in ΔDEF.
- Side $\overline{BC}$ in ΔABC corresponds to side $\overline{f(B)f(C)}$ in ΔDEF.
- Side $\overline{CA}$ in ΔABC corresponds to side $\overline{f(C)f(A)}$ in ΔDEF.
- Angle $\angle ABC$ in ΔABC corresponds to angle $\angle f(A)f(B)f(C)$ in ΔDEF.
- Angle $\angle BCA$ in ΔABC corresponds to angle $\angle f(B)f(C)f(A)$ in ΔDEF.
- Angle $\angle CAB$ in ΔABC corresponds to angle $\angle f(C)f(A)f(B)$ in ΔDEF.

Definition of Triangle Congruence

Words: A congruence between ΔABC and ΔDEF.

Usage: ΔABC and ΔDEF are triangles in a protractor geometry.

Meaning: A bijection $f: \{A, B, C\} \rightarrow \{D, E, F\}$ such that corresponding parts are congruent.

Additional Terminology: Triangle ΔABC is congruent to ΔDEF.

Meaning: There exists a congruence from ΔABC to ΔDEF.

Additional Symbol: $\Delta ABC \simeq \Delta DEF$

Meaning: The particular function $f: \{A, B, C\} \rightarrow \{D, E, F\}$ defined by $\big(f(A), f(B), f(C)\big) = (D, E, F)$ is a congruence.

Challenge Theorem CT6.1.0.1 (Exercise 6.1#1)

Congruence is an equivalence relation on the set of all triangles in a protractor geometry.

Neutral Geometry (Section 6.1)

Definition of the Side-Angle-Side Axiom

Words: *A protractor geometry satisfies the Side-Angle-Side (SAS) Axiom.*

Meaning: If there is a bijection between the vertices of two triangles, and two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle, then all the remaining corresponding parts are congruent as well, so the bijection is a congruence and the triangles are congruent.

Definition of Neutral Geometry

A **neutral geometry** (or **absolute geometry**) is a protractor geometry that satisfies *SAS*.

Proposition 6.1.3 *The Euclidean plane satisfies SAS (and therefore is a neutral geometry).*

Proposition 6.1.4 *The Poincaré plane satisfies SAS (and therefore is a neutral geometry).*

Theorem 6.1.5 (*Pons Asinorum*) (*Isosceles Triangle Theorem*) (*CS* ➔ *CA* Theorem)

In Neutral geometry, if two sides of a triangle are congruent, then the angles opposite those sides are also congruent. That is, in a triangle, if *CS* then *CA*.

Challenge Theorem CT6.1.5.1 (Exercise 6.1#5)

In a Neutral Geometry, if a triangle is equilateral, then it is equiangular.

Challenge Theorem CT6.1.5.2 (Exercise 6.1#8)

Given: Quadrilateral $\square ABCD$ in a Neutral Geometry

Claim: If $\overline{CB} \simeq \overline{CD}$ and $\overrightarrow{CA}$ is the bisector of $\angle BCD$, then $\overline{AB} \simeq \overline{AD}$.

Challenge Theorem CT6.1.5.3 (Exercise 6.1#9)

Given: A quadrilateral in a Neutral Geometry

Claim: If the diagonals bisect each other, then opposite sides are congruent.

Challenge Theorem CT6.1.5.4 (Exercise 6.1#10)

Given: A Neutral Geometry, ΔABC, and points D, E such that $A - D - B$ and $A - E - C$.

Claim: If $\overline{AD} \simeq \overline{AE}$ and $\overline{DB} \simeq \overline{EC}$, then $\angle EBC \simeq DCB$.

Section 6.2 Basic Triangle Congruence Theorems

Definition of the Angle-Side-Angle Axiom

Words: *A protractor geometry satisfies the Angle-Side-Angle (ASA) Axiom.*

Meaning: If there is a bijection between the vertices of two triangles, and two angles and the included side of one triangle are congruent to the corresponding parts of the other triangle, then all the remaining corresponding parts are congruent as well, so the bijection is a congruence and the triangles are congruent.

Theorem 6.2.1 A neutral geometry Satisfies the Angle-Side-Angle (ASA) Axiom
(In other words, if a protractor geometry satisfies SAS, then it also satisfies ASA.)

CT 6.2.1.1 (Corollary)(6.2#2) In a neutral geom., every equiangular triangle is also equilateral.

CT 6.2.1.2 (6.2#5) In a neutral geometry,
if ΔABC and points D, E satisfy $\overline{AB} \simeq \overline{BC}$, $A - D - E - C$, and $\angle ABD \simeq \angle CBE$, then $\overline{DB} \simeq \overline{EB}$.

CT 6.2.1.3 (6.2#8) In a neutral geometry, if ΔABC and points D, E satisfy $A - D - B$
and $A - E - C$ and $\angle ABE \simeq \angle ACD$, and $\angle BDC \simeq \angle BEC$ and $\overline{BE} \simeq \overline{CD}$, then ΔABC is isosceles.

Definition

Words: The internal bisector of an angle in a triangle.

Usage: A protractor geometry, a triangle, and an angle of that triangle are in the discussion.

Meaning: The segment that lies in the bisector of the angle and has endpoints on the triangle.

CT 6.2.3.3 (6.2#13) In a neutral geometry,
if a triangle is isosceles, then the internal bisectors of its base angles are congruent.

Theorem 6.2.2 (Converse of the Statement of Pons Asinorum) (CA → CS Theorem)
In neutral geometry, if two angles of a triangle are congruent, then the sides opposite those angles are also congruent. That is, in a triangle, if CA then CS.

CT 6.2.2.1 (6.2#6) In a neutral geometry,
if ΔABC and points D, E satisfy $A - D - E - C$, $\overline{AD} \simeq \overline{EC}$, and $\angle CAB \simeq \angle ACB$, then $\angle ABE \simeq \angle CBD$.

Definition of the Side-Side-Side Axiom

Words: *A protractor geometry satisfies the Side-Side-Side (SSS) Axiom.*

Meaning: If if there is a bijection between the vertices of two triangles, and the three sides of one triangle are congruent to the corresponding parts of the other triangle, then all the remaining corresponding parts are congruent as well, so the bijection is a congruence and the triangles are congruent.

Theorem 6.2.3: A neutral geometry Satisfies the Side-Side-Side (SSS) axiom.
(In other words, if a protractor geometry satisfies SAS, then it also satisfies SSS.)

CT 6.2.3.1 (Special Rays in Isosceles Triangles) In a neutral geometry,
given ΔABC with $\overline{BA} \simeq \overline{BC}$ and ray $\overrightarrow{BD}$ with $D \in \text{int}(\angle ABC)$, the following are equivalent

(i) $\overrightarrow{BD}$ is the bisector of $\angle ABC$

(ii) $\overrightarrow{BD}$ bisects $\overline{AC}$.

(iii) $\overrightarrow{BD} \perp \overline{AC}$.

CT 6.2.3.2 (6.2#7) In a neutral geometry,
if $\square ABCD$ has $\overline{AB} \simeq \overline{CD}$ and $\overline{AD} \simeq \overline{BC}$, then $\angle A \simeq \angle C$ and $\angle B \simeq \angle D$

CT 6.2.3.3 (6.2#12) In a neutral geometry,
if C, D are on the same side of $\overleftrightarrow{AB}$ and $\overline{AC} \simeq \overline{AD}$ and $\overline{BC} \simeq \overline{BD}$, then $C = D$

Theorem 6.2.4: If a Protractor Geometry satisfies ASA, then it also satisfies SAS (and is thus a Neutral Geometry)

Theorem 6.2.5 Existence of At Least One Perp. to a Line through a Point *Not On the Line*

In a neutral geometry, if B is a point not on a line l,
then there exists ***at least one*** line l' such that l' contains B and $l' \perp l$.

The Neutral Exterior Angle Theorem (Section 6.3)

Theorem 6.3.1: Given: distinct points A, B and distinct points C, D in a metric geometry

Claim: The following are equivalent (TFAE)

(i) $AB < CD$

(ii) There exists a point $G \in \text{int}(\overline{CD})$ such that $\overline{AB} \simeq \overline{CG}$

Theorem 6.3.2: Given: angles $\angle ABC$ and $\angle DEF$ in a metric geometry,

Claim: The following are equivalent (TFAE)

(i) $m(\angle ABC) < m(\angle DEF)$

(ii) There exists a point $G \in \text{int}(\angle DEF)$ such that $\angle ABC \simeq \angle DEG$

Definition of Exterior Angle and its Remote Interior Angles

An ***Exterior Angle*** for a triangle is an angle that can be labeled $\angle BCD$ for a triangle that can be labeled ΔABC, with $A - C - D$. With this labeling, angles $\angle A$ and $\angle B$ are the ***remote interior angles*** corresponding to the exterior angle $\angle BCD$.

Theorem 6.3.3 (Neutral Exterior Angle Theorem) In a neutral geometry, the measure of an exterior angle of a triangle is greater than the measure of either of its remote interior angles.

Corollaries of the Neutral Exterior Angle Theorem (Section 6.3)

Challenge Theorem 6.3.3.1 (Corollary)(Exercise 6.3#3)

In a neutral geometry, the base angles of an isosceles triangle are acute.

Challenge Theorem 6.3.3.2 (Corollary)(Exercise 6.3#10)

In a neutral geometry, if one angle of a triangle is obtuse, then the other two angles are acute.

Challenge Theorem 6.3.3.2 (Corollary) Given: a neutral geometry, a line l, and a point P not on l

Claim: There ***cannot be more than one*** line l' such that l' contains P and $l' \perp l$.

Challenge Theorem 6.3.3.3 (Corollary) Given: a neutral geometry, a line l, and a point P not on l

Claim: There is ***exactly one*** line l' such that l' contains P and $l' \perp l$.

Corollary 6.3.4 Given: a neutral geometry, a line l, and a point P (either on l or not on l),

Claim: There is *exactly one* line l' such that $P \in l'$ and $l' \perp l$.

The SAA Theorem and BABS (Section 6.3)

Definition of the Side-Angle-Angle Axiom (SAA)

Words: *A protractor geometry satisfies the Side-Angle-Angle (SAA) Axiom.*

Meaning: If if there is a bijection between the vertices of two triangles, and two angles and a non-included side of one triangle are congruent to the corresponding parts of the other triangle, then all the remaining corresponding parts are congruent as well, so the bijection is a congruence and the triangles are congruent.

Theorem 6.3.5: A Neutral Geometry Satisfies the Side-Angle-Angle (*SAA*) axiom.

Theorem 6.3.6 (The *BS* ➔ *BA* Theorem): In Neutral Geometry, if one side of a triangle is longer than another side, then the angle opposite the longer side is bigger than the angle opposite the shorter side. That is, in a triangle, *if BS then BA*. In symbols, *BS* ➔ *BA*.

Theorem 6.3.7 (The *BA* ➔ *BS* Theorem): In Neutral Geometry, if one angle of a triangle is bigger than another angle, then the side opposite the bigger angle is longer than the side opposite the smaller angle. That is, in a triangle, *if BA then BS*. In symbols, *BA* ➔ *BS*.

Triangle Inequalities and the Open Mouth Theorem (Section 6.3)

Challenge Theorem 6.3.7.1 (Exercise 6.3#7)

Given: A neutral geometry, triangle $\angle ABC$ and point $D \in \text{int}(\Delta ABC)$

Given: $AD + DC < AB + BC$ and $m(\angle ADC) > m(\angle ABC)$

Theorem 6.3.8 (The *(Strict) Triangle Inequality of Neutral Geometry*)

Version without vertices named: In Neutral Geometry, the length of one side of a triangle is strictly less than the sum of the lengths of the other two sides.

Version with vertices named: Given: any ΔABC in Neutral Geometry

Claim: The strict inequality $AC < AB + BC$ is true.

Challenge Theorem 6.3.8.1 (Corollary)(Exercise 6.3#6)

(The General (Inclusive) Triangle Inequality of Neutral Geometry)

Given: distinct points A, B, C in Neutral Geometry

Claim: The inclusive inequality $AC \leq AB + BC$ is true. Furthermore, equality holds iff $A - B - C$.

Remark on Misleading Name: Theorem holds even when distinct points A, B, C are collinear. In that case, they do not actually form a triangle. But the name *triangle inequality* is kept.

Theorem 6.3.9 (*Open Mouth Theorem*)

Given: a neutral geometry, ΔABC, and ΔDEF with $AB = DE$ and $BC = EF$,

Claim: If $m(\angle ABC) > m(\angle DEF)$, then $AC > DF$.

Theorem 6.3.10 Given: a neutral geometry, ΔABC with $AB < AC$, and point D with $B - D - C$,

Claim: $AD < AC$.

Right Triangles (Section 6.4)

Theorem 6.4.1 If a triangle has a right angle, then the other two angles are acute, and the longest side of the triangle will be the side opposite the right angle.

Definition: A ***Right Triangle*** is defined to be a triangle that has a right angle.
The ***hypotenuse*** of a right triangle is defined to be the side opposite the right angle.
The ***legs*** of a right triangle are defined to be the sides that are not the hypotenuse.

Challenge Theorem 6.4.1.1 (exercise 6.4#8)

In neutral geometry, if ΔABC has a right angle at C, and $B - D - C$, then $\angle BDA$ is obtuse.

Theorem 6.4.2 (Perpendicular Distance Theorem)

In neutral geometry, for a line l and a point P not on l, shortest distance PQ among all points $Q \in l$ is achieved by the unique point Q on l such that $\overleftrightarrow{PQ} \perp l$.

Definition of the Distance from a Point to a Line:

For a line l and a point P not on l, the **distance from P to l** is defined to be the distance PQ where Q is the unique point on l such that $\overleftrightarrow{PQ} \perp l$.

Definition of *Altitude Line*, *Foot of an Altitude Line*, *Altitude Segment:* An ***altitude line*** of a triangle is a line that passes through a vertex of the triangle and is perpendicular to the opposite side. (Note that the altitude line does not necessarily have to intersect the opposite side to be perpendicular to it. Also note that Corollary 6.3.4 tells us that there is exactly one altitude line for each vertex.) The ***foot of the altitude*** is defined to be the point of intersection of the altitude line and the line determined by the opposite side. An ***altitude segment*** has one endpoint at the vertex and the other endpoint at the foot of the altitude line drawn from that vertex.

Theorem 6.4.3 (The altitude to the longest side stays inside the triangle.)

In neutral geometry, if $\overline{AB}$ is a longest side of ΔABC and D is the foot of the altitude from C, then $A - D - B$.

Challenge Theorem 6.4.3.1 (when an altitude lands outside the triangle)(exercise 6.4#2)

In neutral geometry, if D is the foot of the altitude from vertex C of ΔABC and $A - B - D$, then $CA > CB$.

Definition of *Median*

A ***median*** of a triangle is a segment with one endpoint on a vertex of the triangle and the other endpoint at the midpoint of the opposite side.

Challenge Theorem 6.4.3.2 (about special rays in triangles)(exercise 6.4#3)

In neutral geometry, given ΔABC,

(a) If $\overline{AB} \simeq \overline{AC}$, then the following four objects are collinear:

- (i) the median from A
- (ii) the bisector of $\angle A$
- (iii) the altitude from A
- (iv) the perpendicular bisector of $\overline{BC}$

(b) If any two fo the above four objects are collinear, then $\overline{AB} \simeq \overline{AC}$.

Theorem 6.4.4 *Hypotenuse-Leg* (*HL*) In Neutral Geometry, if there is a bijection between the vertices of two right triangles such that the hypotenuse and a leg of one triangle are congruent to the corresponding parts of the other triangle, then the triangles are congruent.

Theorem 6.4.5 *Hypotenuse-Angle* (*HA*) In Neutral Geometry, if there is a bijection between the vertices of two right triangles such that the hypotenuse and one of the acute angles of one triangle are congruent to the corresponding parts of the other triangle, then the triangles are congruent.

Theorem 6.4.6 In a neutral geometry, given a segment $\overline{AB}$ and a point P, the following are equivalent:

- (i) P lies on the perpendicular bisector of $\overline{AB}$.
- (ii) P is equidistant from the endpoints of the segment. That is, $PA = PB$.

Challenge Theorem 6.4.7.1

(Combining Theorem 6.4.7 and part of the Result of Exercise 6.4#11) In a neutral geometry, given an angle $\angle ABC$ and a point $P \in \text{int}(\angle ABC)$, the following are equivalent:

(i) P lies on the bisector of $\angle ABC$.

(ii) P is equidistant from the rays of the angle. That is

the distance from P to line $\overleftrightarrow{BA}$ is equal to the distance from P to line $\overleftrightarrow{BC}$.

Challenge Theorem 6.4.7.2 (Concurrence of Angle Bisectors of a Triangle) (Exercise 6.4#12)

In a neutral geometry, the three angle bisectors of a triangle are concurrent.

Challenge Theorem 6.4.7.3 (if two perpendicular bisectors of a triangle are known to meet) (Exercise 6.4#13)

In a neutral geometry, if the perpendicular bisectors of two sides intersect intersect at P, then all three perpendicular bisectors are concurrent (at P).

Circles (Section 6.5)

Definition of Circle

Symbols: Book's symbol: $\mathcal{C}_r(C)$ **Mark's preferred symbol:** $\text{circle}(C, r)$

Spoken: The **circle** with **center** C and **radius** r.

Usage: C is a point in a metric geometry and r is a positive real number.

Meaning: the set of points that are a distance r from point C. That is,

$$\mathcal{C}_r(C) = \text{circle}(C, r) = \{P \in \mathcal{P} | PC = r\}$$

Additional Terminology:

A **chord** of a circle is a segment $\overline{AB}$ with A, B distinct points on the circle.

A **diameter segment** of a circle is a chord that contains the center of the circle.

A **radius segment**, or **radial segment**, of a circle is a segment $\overline{CA}$ where C is the center of the circle and A is a point on the circle.

The **interior** of the circle $\mathcal{C}_r(C)$, denoted $\text{int}\{\mathcal{C}_r(C)\}$, is the set of points that are a distance less than r from point C. That is, $\text{int}\{\mathcal{C}_r(C)\} = \{P \in \mathcal{P} | PC < r\}$

The **exterior** of the circle $\mathcal{C}_r(C)$, denoted $\text{ext}\{\mathcal{C}_r(C)\}$, is the set of points that are a distance greater than r from point C. That is, $\text{ext}\{\mathcal{C}_r(C)\} = \{P \in \mathcal{P} | PC > r\}$

A **tangent line** for a circle is a line that intersects the circle at exactly one point.

A **secant line** for a circle is a line that intersects the circle at exactly two points.

Theorem 6.5.3 In a neutral geometry if two circles have three points in common, then the circles are the same circle. That is, they have the same center and the same radius.

Corollary 6.5.4 In a neutral geometry, the perpendicular bisector of any chord of a circle contains the center of the circle.

Challenge Theorem CT6.5.4.1

Given: a neutral geometry and distinct points A, B, C

Claim: If A, B, C lie on a circle, then the perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$ intersect. Furthermore, the point of intersection is the center of the circle.

Automatically True: the Contrapositive of CT6.5.4.1

Given: a neutral geometry and distinct points A, B, C

Claim: If the perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$ do not intersect,
then A, B, C do not lie on a circle.

Theorem 6.5.5 In a neutral geometry, the interior of any circle is a convex set.

Theorem 6.5.6 (Version without naming names) In a neutral geometry, a tangent line for a circle is perpendicular to the radial segment at the point of tangency.

Theorem 6.5.6 (Version with names)

Given: a neutral geometry, circle(C, r), and a line L that intersects the circle at point Q.

Claim: The following are Equivalent:

(1) $L \perp \overline{CQ}$

(2) L is tangent to circle(C, r) at Q.

Corollary 6.5.7 (Existence and Uniqueness of Tangents)

Given: a neutral geometry, circle(C, r), and $Q \in$ circle(C, r)

Claim: There exists a point line L that contains Q and is tangent to circle(C, r).

Theorem 6.5.8 (Lemma about Existence of a Point with a Certain distance Property)

Given: a neutral geometry, a real number $r > 0$, and points A, B, C with $AC < r$ and $\overline{AB} \perp \overline{AC}$.

Given: Claim: There exists a point $D \in \overrightarrow{AB}$ such that $CD = r$.

Theorem 6.5.9 (Line-Circle Theorem)

In a neutral geometry, if a line intersects the interior of a circle, then the line is a secant line.

Theorem 6.5.10 (Exterior Tangent Theorem) In a neutral geometry, for every point in the exterior of a circle, there are exactly two lines that contain the given point and are tangent to the circle.

Challenge Theorem CT6.5.11 (exercise 6.5#3)

Given: a neutral geometry, circle(C, r), a chord $\overline{AB}$ that is not a diameter, with midpoint M.

Claim: $\overleftrightarrow{CM} \perp \overline{AB}$

Challenge Theorem CT6.5.12 (exercise 6.5#4)

In Neutral geometry, a line cannot intersect a circle more than twice.

Challenge Theorem CT6.5.13 (exercise 6.5#10)

Given: a neutral geometry and a circle with point A in its interior and point B in its exterior

Claim: $\overline{AB}$ intersects the circle.

Challenge Theorem CT6.5.14 (exercise 6.5#11)

Given: a neutral geometry and three distinct points A, B, C

Claim: If the perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$ intersect at a point P,

then there is a circle that A, B, C lie on.

Automatically True: the Contrapositive of CT6.5.14 (exercise 6.5#11)

Given: a neutral geometry and three distinct points A, B, C

Claim: If there is not a circle that A, B, C lie on,

then the perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$ do not intersect

Challenge Theorem CT6.5.15 (Corollary of CT6.5.4.1 and CT6.5.14)

Given: a neutral geometry and distinct points A, B, C

Claim: The following are equivalent

(1) A, B, C lie on a circle

(2) the perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$ intersect.

Remark: In the case that (1) and (2) are true, the point of intersection of the perpendicular bisectors is the center of the circle.

Challenge Theorem CT6.5.16 (exercise 6.5#14)

In a neutral geometry, if $\overline{AB}$ is a chord in circle(C, r), then $AB \leq 2r$. Furthermore, equality holds if and only if $\overline{AB}$ is a diameter segment.

Challenge Theorem CT6.5.17 (exercise 6.5#15)

Given: a neutral geometry, circle(C, r), and chords $\overline{AB}$ and $\overline{DE}$

Claim: The following are equivalent:

(i) $\overline{AB} \simeq \overline{DE}$

(ii) The distance from C to line $\overleftrightarrow{AB}$ equals the distance from C to line $\overleftrightarrow{DE}$.

Challenge Theorem CT6.5.18 (exercise 6.5#16)

In a neutral geometry, if a line intersects a circle but does not intersect the interior of the circle, then the line is tangent to the circle.

Challenge Theorem CT6.5.19 (exercise 6.5#21)

Given: a neutral geometry, circle(C, r), diameter segment $\overline{AB}$, and chord $\overline{DE}$.

Claim: The following are equivalent:

(i) $\overline{AB}$ bisects $\overline{DE}$

(ii) $\overline{AB} \perp \overline{DE}$.