<u>1.3b: Functions: Inverses and Compositions</u>

produced by Mark Barsamian, 2021.01.28

for Ohio University MATH 3110/5110 College Geometry

Subject: Functions: Inverses and Compositions

- Inverse Functions
    - Definition
    - Graphs of Inverse Functions
    - Injective & Surjective Properties of Inverse Functions
- Function Composition
    - Definition
    - Injective & Surjective Properties of Compositions of Functions
- Relationships Between Function Composition and Inverse Functions

Textbook: Millman & Parker, *Geometry: A Metric Approach with Models, Second Edition*

(Springer, 1991, ISBN 3-540-97412-1)

Reading: Section 1.3 Functions, pages 12 - 14

Homework: Section 1.3 # 6, 7, 8, 9, 10, 11, 13

From the previous video, recall the definition of a *Relation from One Set to Another Set*

Definition of *Relation from One Set to Another Set*

(s, t)

Words: R is a **relation from *S* to *T***

Meaning: S, T are sets, and $R \subset S \times T$. That is, R is a set containing ordered pairs from $S \times T$.

Additional Terminology: Set S is called the **domain** of the relation R; Set T is called the **range**.

And recall that Every relation has an *Inverse Relation*.

Definition of Inverse Relation

Symbol: R^{-1}

Spoken: R *inverse,* or *the inverse relation for* R.

Usage: There is a relation R from a set S to a set T in the discussion.

Meaning: R^{-1} is the relation from set T to set S (that is, R^{-1} is a subset of $T \times S$) defined by $R^{-1} = \{(t, s) | (s, t) \in R\}$. That is, the ordered pairs for the relation R^{-1} are obtained by reversing the order of the elements in the ordered pairs for R.

And recall the definitions of *Function* and *Bijective Function*.

Definition of *Function*

Symbol: $f: X \to Y$

Words: f is a function from X to Y

Meaning: X, Y are sets and f is a relation from X to Y (that is, $f \subset X \times Y$) that has this extra property: For each $x \in X$, there is exactly one $y \in Y$ such that $(x, y) \in f$. This unique element $y \in Y$ is denoted by the symbol $f(x)$. That is, $y = f(x)$, so $\left(x, f(x)\right) \in f$

Meaning Expressed Formally: $\forall x \in X \left(\exists! \, y \in Y\left(y = f(x)\right)\right)$

Definition of Bijective Function

Words: f *is a* ***bijective function***, or f *is a* ***bijection*** (or f *is a* ***one-to-one correspondence***)

Meaning: f is both *injective* and *surjective*. (f is both *one-to-one* and *onto*.)

Other Wording: For every element in the range, there exists ***exactly one*** element of the domain that can be used as input to cause that element of the range to be output.

Meaning Written Formally: $\forall y \in Y\left(\exists! \, x \in X(f(x) = y)\right)$

Inverse Functions

In the previous video, we saw that when R is a relation from S to T, then there is an associated *inverse relation* R^{-1} from T to S. We saw that the inverse relation R^{-1} might *not* be qualified to be called a *function*.

Now consider the special case where the relation is a *bijective function* $f: X \to Y$. In this special case, for each $y \in Y$, there exists exactly one $x \in X$ such that $f(x) = y$. In this special case, the *inverse relation* f^{-1} *is* qualified to be called a function, called the *inverse function* for f.

Definition of Inverse Function

Symbol: f^{-1}

Usage: There is a bijective function $f: X \to Y$ in the discussion.

Spoken: The inverse function for f

Meaning: f^{-1} is the function $f^{-1}: Y \to X$ defined by

$$f^{-1}(y) \underset{\text{def}}{=} \textit{the unique } x \in X \textit{ such that } f(x) = y$$

Equivalently, f^{-1} can be defined by saying that the symbol $f^{-1}(y) = x$ means $y = f(x)$.

[Example 1] At the end of the previous video, we observed that this function is bijective:

$$f: \mathbb{R} - \{3\} \to \mathbb{R} - \{1\} \text{ defined by } f(x) = \frac{x-2}{x-3}$$

Find the inverse function f^{-1}.

Solution:

The formula for $f(x)$ corresponds to the equation $y = \dfrac{x-2}{x-3}$

This equation expresses what the function f does. That is, starting with a value for x, the equation shows the computation that will produce the corresponding value of y.

If we solve the above equation for x in terms of y we get the new equation $x = \dfrac{3y-2}{y-1}$

Observe that if the value of y is known, the equation shows the computation that will produce the corresponding value of x. But that operation is exactly what the inverse function f^{-1} does.

In other words, the inverse function is

$$f^{-1}: \mathbb{R} - \{1\} \to \mathbb{R} - \{3\} \text{ defined by } f^{-1}(y) = \frac{3y-2}{y-1}$$

End of [Example 1]

The above example illustrates the following general procedure:

Finding the Inverse Function for a Function f Given by a Formula

Given a bijective function $f: X \to Y$ where $X, Y \subset \mathbb{R}$ described by a formula $f(x)$.

Starting with the associated equation (which is solved for y in terms of x),

$$y = f(x) = \text{an expression involving } x$$

solve the equation for x in terms of y to get a new equation of the form

$$x = \text{some new expression involving } y$$

This new equation describes the operation of the inverse function f^{-1}.

That is, the inverse function is the function $f^{-1}: Y \to X$ defined by the formula

$$x = f^{-1}(y) = \text{the new expression involving } y$$

Remark: Once the formula for the inverse function is obtained in the form

$$x = f^{-1}(y) = \text{the new expression involving } y$$

the variables can be changed at will. That is, one could change the variables to

$$s = f^{-1}(t) = \text{the new expression involving } t$$

or even change them to

$$y = f^{-1}(x) = \text{the new expression involving } x$$

Graphs of Inverse Functions and Inverse Relations

Recall from the definition of the *Inverse Relation* that the ordered pairs for the inverse relation R^{-1} are obtained by reversing the order of the elements in the ordered pairs for the original relation R. This is of course also true for the special case when the relation R is a *bijective function* f and the inverse relation f^{-1} is qualified to be called a function and called the *inverse function for* f.

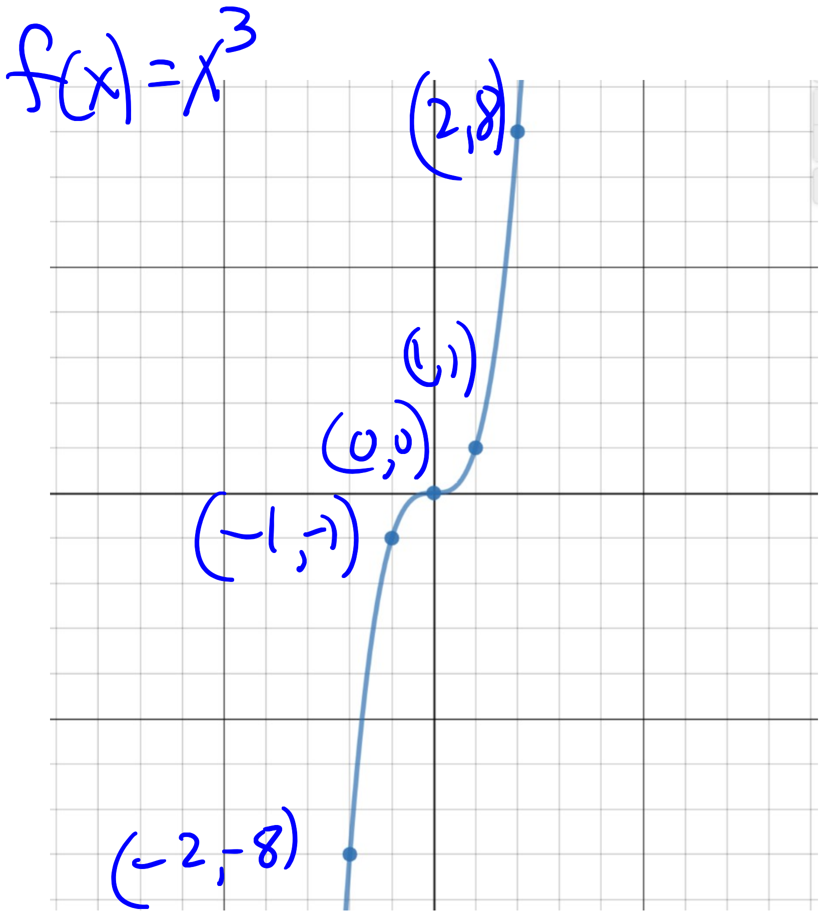
[Example 2] Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$.

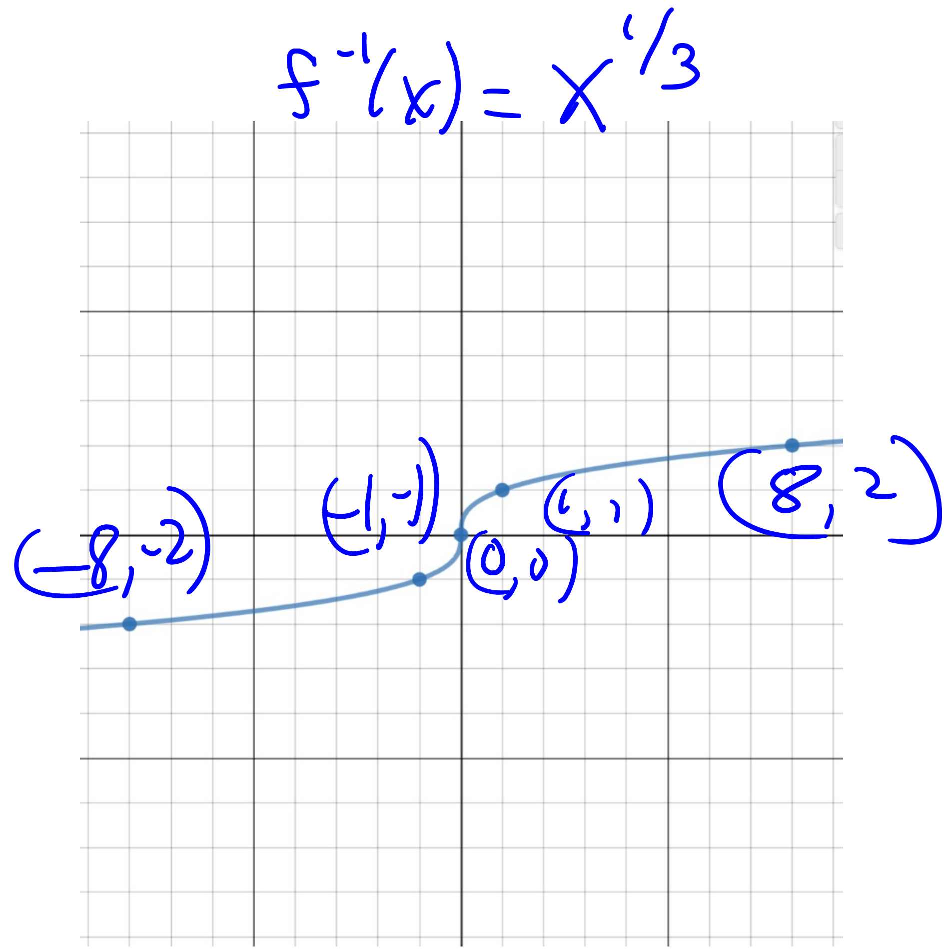
Observe that f is bijective, so the inverse relation f^{-1} is qualified to be called the *inverse function*.

The formula is for the inverse function is $f^{-1}(y) = y^{1/3}$.

In order to have the variable x in the inverse function, we can write its formula as $f^{-1}(x) = x^{1/3}$.

Compare the graphs of $f(x) = x^3$ and $f^{-1}(x) = x^{1/3}$.





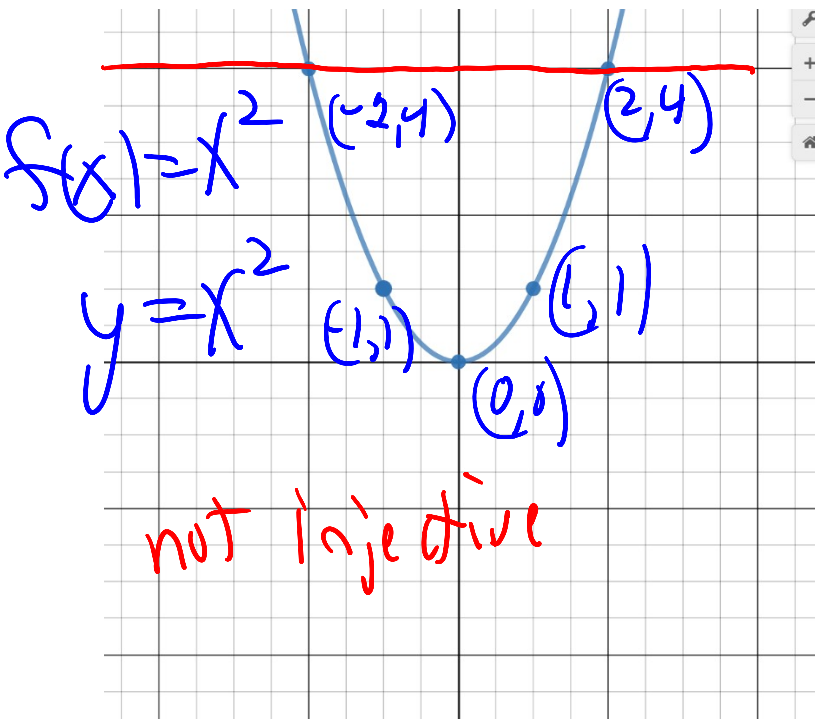
End of [Example 2]

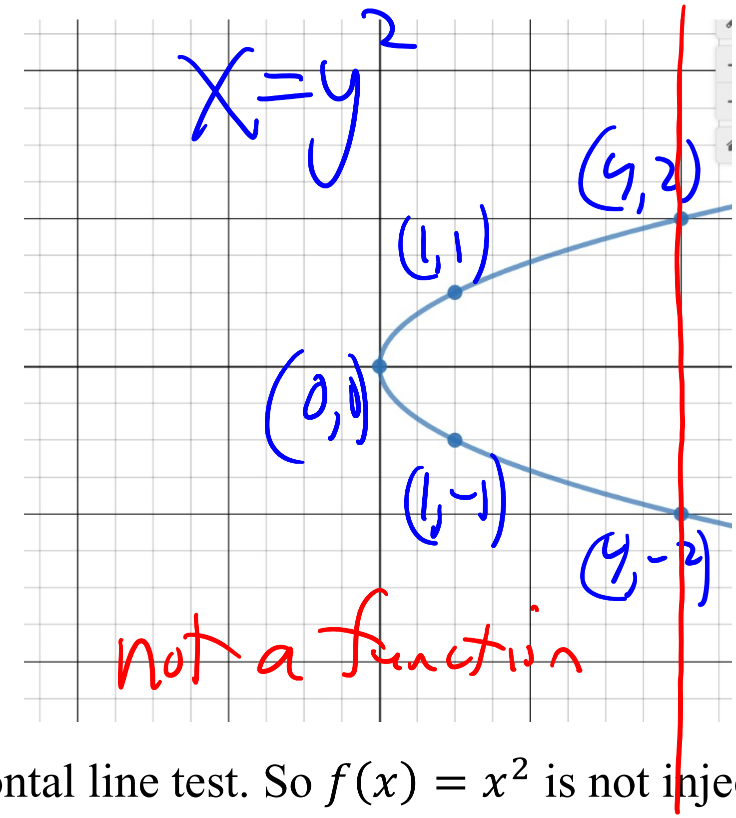
[Example 3] Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Notice that the graph fails the horizontal line test.)

Therefore, f is *not bijective* and will *not* have an *inverse function*.

But f will still have an *inverse relation*. The inverse relation is described by the equation $x = y^2$.

Compare the graphs of $y = x^2$ and $x = y^2$.





Observe that the graph of $f(x) = x^2$ fails the horizontal line test. So $f(x) = x^2$ is not injective.

Observe that the graph of the equation $x = y^2$ fails the vertical line test. So the equation $x = y^2$ does not describe y as a function of x.

End of [Example 3]

Injective & Surjective Properties of Inverse Functions

We have discussed that when $f: X \to Y$ is a bijective function, the *inverse relation for* f is qualified to be called a *function*, called the *inverse function for* f.

The inverse function for f is the function $f^{-1}: Y \to X$ defined by

$$f^{-1}(y) \underset{\text{def}}{=} \textit{the unique } x \in X \textit{ such that } f(x) = y$$

Equivalently, f^{-1} can be defined by saying that the symbol $f^{-1}(y) = x$ means $y = f(x)$.

There is an interesting fact about the inverse function that is often stated as a *theorem* in books. Here, I will present the fact as a *corollary*, because its proof involves no new ideas, but rather simply some observations about ideas that have already been presented. But realize that the reason the proof of the fact is so simple is that we have been careful to present our statements about properties of functions using precise, formal logical notation.

Corollary: If a function f is bijective, then its inverse function f^{-1} is also bijective.

Proof (Direct Proof) (A very cool proof!)

Suppose that $f: X \to Y$ is a bijective function.

The fact that f is a function from X to Y is expressed formally by the quantified statement

$$\textbf{Statement \#1: } \forall x \in X\left(\exists!\, y \in Y\big(y = f(x)\big)\right)$$

The fact that f is bijective is expressed formally by the quantified statement

$$\textbf{Statement \#2: } \forall y \in Y\big(\exists!\, x \in X(f(x) = y)\big)$$

The inverse function f^{-1} can be defined by saying that symbol $f^{-1}(y) = x$ means $y = f(x)$.

So we can replace each $f(x) = y$ in the above quantified statemens with $x = f^{-1}(y)$. The result is a new pair of quantified statements that we know are true:

$$\textbf{New Statement \#1: } \forall x \in X\left(\exists!\, y \in Y\big(x = f^{-1}(y)\big)\right)$$

$$\textbf{New Statement \#2: } \forall y \in Y\left(\exists!\, x \in X\big(x = f^{-1}(y)\big)\right)$$

Realize that New Statement #2 expresses the fact that $f^{-1}: Y \to X$ is a *function*.

And New Statement #1 tells us that $f^{-1}: Y \to X$ is *bijective*.

End of Proof

Composition of Functions

Definition of Function Composition

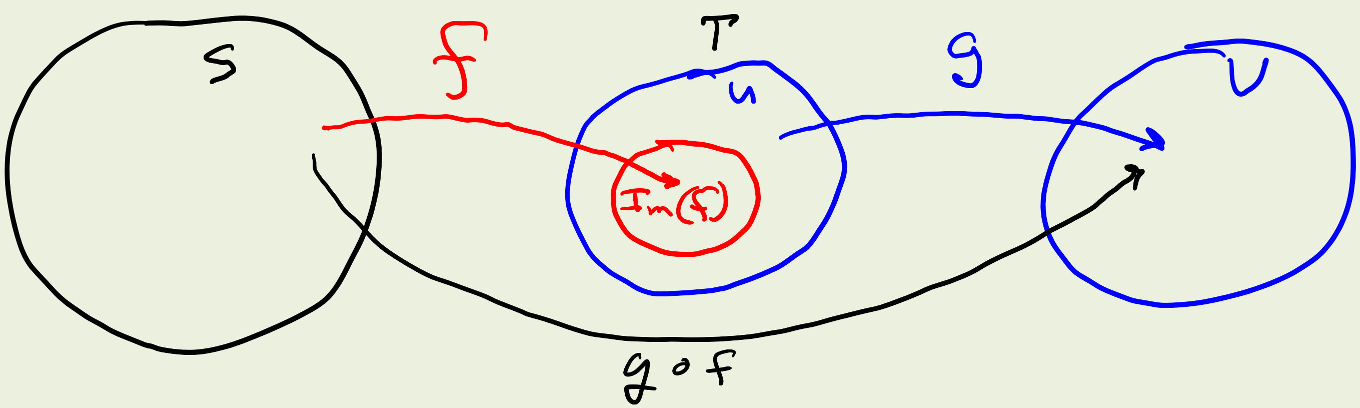
Symbol: $g \circ f$

Spoken: g composed with f

Usage: f, g are functions $f: S \to T$ and $g: U \to V$ such that $\mathrm{Im}(f) \subset U$

Meaning: the function $g \circ f: S \to V$ defined by $(g \circ f)(s) = g\big(f(s)\big)$

Diagram:



Corollary: function composition is *associative*. That is $h \circ (g \circ f) = (h \circ g) \circ f$

Proof

Suppose that f, g are functions $f: S \to T$ and $g: U \to V$ and $h: X \to Y$ such that $\mathrm{Im}(f) \subset U$ and $\mathrm{Im}(g) \subset X$.

Then symbols $h \circ (g \circ f)$ and $(h \circ g) \circ f$ both denote functions with domain S and range Y.

To show that the functions are equal, we must show that they always produce the same output when fed the same input $s \in S$. Observe

$$\begin{aligned}
\big(h \circ (g \circ f)\big)(s) &= h\big((g \circ f)(s)\big) \text{ by definition of composition} \\
&= h\Big(g\big(f(s)\big)\Big) \text{ by definition of composition} \\
&= (h \circ g)\big(f(s)\big) \text{ by definition of composition} \\
&= \big((h \circ g) \circ f\big)(s) \text{ by definition of composition}
\end{aligned}$$

End of Proof

[Example 4] Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$ and define $g: \mathbb{R} \to \mathbb{R}$ by $g(x) = x + 1$.

Then

$$(g \circ f)(x) = g\big(f(x)\big) = f(x) + 1 = x^2 + 1$$

$$(f \circ g)(x) = f\big(g(x)\big) = \big(g(x)\big)^2 = (x+1)^2 = x^2 + 2x + 1$$

It seems that $g \circ f$ is not the same function as $f \circ g$, because the formulas look different. But sometimes formulas that look different can actually represent the same function. To be sure that the functions are not the same function, we must find an example of an x value for which the functions produce different outputs.

Observe that

$$(g \circ f)(1) = (1)^2 + 1 = 2$$

$$(f \circ g)(1) = (1)^2 + 2(1) + 1 = 4$$

Since $(g \circ f)(1) \neq (f \circ g)(1)$, we conclude that $g \circ f \neq f \circ g$.

End of [Example 4]

Observation: function composition is *not commutative*. That is, in general, $g \circ f \neq f \circ g$

Injective & Surjective Properties of Compositions of Functions

Theorem A:

Given $f: S \to T$ and $g: T \to V$

If f and g are both surjective, then $g \circ f$ is surjective.

Proof

(1) Suppose $f: S \to T$ and $g: T \to V$ and that f and g are both surjective.

(2) $\forall t \in T\big(\exists s \in S(f(s) = t)\big)$) (by (1) and definition of f being surjective)

(3) $\forall v \in V\big(\exists t \in T(g(t) = v)\big)$) (by (1) and definition of g being surjective)

(4) Suppose $z \in V$

(5) Then $\exists y \in T(g(y) = z)$ (by (4) and (3). That is, because g is surjective)

(6) Then $\exists x \in S(f(s) = y)$ (by (5) and (2). That is, because f is surjective)

(7) Then $(g \circ f)(x) = g\big(f(x)\big) = g(y) = z$ (by definition of composition and (6) and (5))

(9) $\forall z \in V\left(\exists x \in S\big((g \circ f)(x) = z\big)\right)$) (by steps (4), (5), (6), (7))

(10) Thus $g \circ f$ is surjective. (by (9) and the definition of surjective)

End of Proof

Here is another very similar theorem

Theorem B:

Given $f: S \to T$ and $g: T \to V$

If f and g are both injective, then $g \circ f$ is injective.

I will leave it to you to figure out a proof of Theorem B.

Once you have proven Theorem B, it should be easy for you to prove the following theorem as a corollary.

Theorem C:

Given $f: S \to T$ and $g: T \to V$

If f and g are both bijective, then $g \circ f$ is bijective.

What about the Converse Statements?

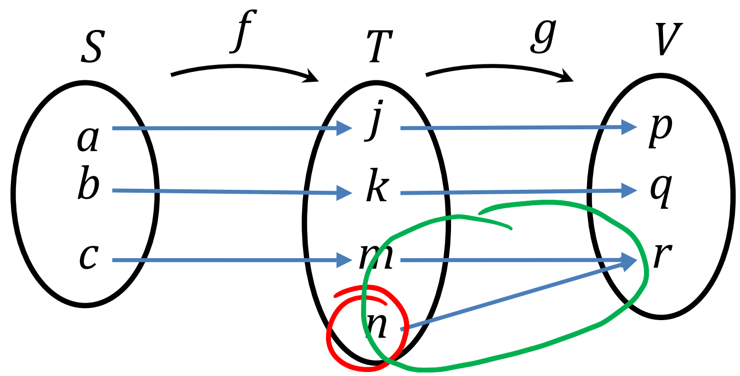
An interesting question is, are the converses of the statements of any of the theorems A, B, C valid as theorems? That is, given $f: S \to T$ and $g: T \to V$,

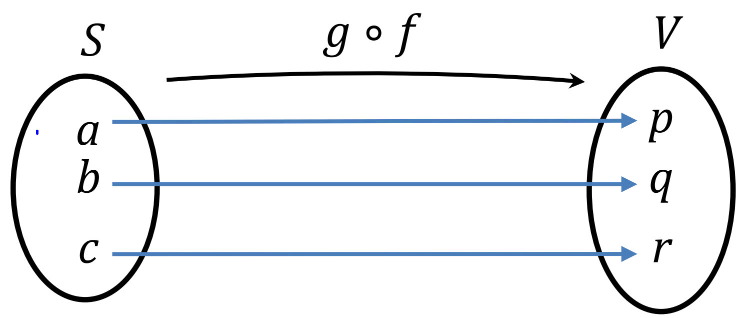
- If $g \circ f$ is surjective, does that imply that f and g are both surjective?
- If $g \circ f$ is injective, does that imply that f and g are both injective?
- If $g \circ f$ is bijective, does that imply that f and g are both bijective?

The answer is not obvious because a *conditional statement* is *not logically equivalent* to its *converse*. That is, these two statements to not mean the same thing:

- Conditional Statement: If P then Q
- Converse: If Q then P

To answer those questions, consider the following example of functions $f: S \to T$ and $g: T \to V$.





Observe that the composition $g \circ f$ is definitely a bijection.

But notice that function f is not surjective and notice that function g is not injective.

Conclude that it is possible for $g \circ f$ to be surjective without f being surjective.

And conclude that it is possible for $g \circ f$ to be injective without g being injective.

On the other hand, notice that in the diagram on the previous page, function g is surjective and function f is injective. It turns out that those functions will always have that behavior, as articulated by the following two theorems.

Theorem D:

Given $f: S \to T$ and $g: T \to V$

If $g \circ f$ is surjective, then g is surjective

Theorem E:

Given $f: S \to T$ and $g: T \to V$

If $g \circ f$ is injective, then f is injective

Both of these theorems have statements with the form of a conditional statement:

$$\text{If } P \text{ then } Q$$

Recall that the conditional statement is logically equivalent to the contrapositive.

$$\text{If } NOT(Q) \text{ then } NOT(P)$$

It turns out that for both of these theorems, the proof of the contrapositive statement is much easier than the proof of the original conditional statement. I'll do that for Theorem D on the next page.

Contrapositive of the Statement of Theorem D

If g is not surjective, then $g \circ f$ is not surjective.

Proof of Theorem D (prove the contrapositive statement, using a Direct Proof)

(1) Suppose $f: S \to T$ and $g: T \to V$ and that g is not surjective.

(2) Then $\exists v \in V\big(\forall t \in T(g(t) \neq v)\big)$ (by (1) and the definition of not surjective)

(3) Then $\forall s \in S\big(g\big(f(s)\big) \neq v\big)$ (by (2) and the definition of not surjective)

(4) Then $\forall s \in S\big((g \circ f)(s) \neq v\big)$ (by (3) and the definition of composition)

(5) Then $\exists v \in V\left(\forall s \in S\big((g \circ f)(s) \neq v\big)\right)$ (by (2),(3),(4))

(6) Thus $g \circ f$ is not surjective. (by (5) and the definition of not surjective)

End of Proof

As mentioned on the previous page, it is also to prove Theorem E by proving the contrapositive statement. I’ll leave that for you to figure out.

Relationships between Function Composition and Inverse Functions

The following Theorem is stated and proven in the book.

Theorem 1.3.10

If $f: X \to Y$, then the following two statements are equivalent. (That is, they are either both true, or they are both false.) (The book says (1) if and only if (2).)

(1) f is a bijection

(2) There exists some function $g: Y \to X$ such that $g \circ f = id_X$ and $f \circ g = id_Y$

Furthermore, the inverse function of f is the function g in this case.

I will leave it to you to read and understand the proof of this theorem.

[Example 1, revisited] Earlier in this video, we found the inverse function for this function:

$$f\colon \mathbb{R} - \{3\} \to \mathbb{R} - \{1\} \text{ defined by } \qquad f(x) = \frac{x-2}{x-3}$$

$$f^{-1}\colon \mathbb{R} - \{1\} \to \mathbb{R} - \{3\} \text{ defined by } \ f^{-1}(y) = \frac{3y-2}{y-1}$$

(a) Find $(f^{-1} \circ f)(x)$.

Solution

$$\begin{aligned}
(f^{-1} \circ f)(x) &= f^{-1}\big(f(x)\big) \\
&= \frac{3f(x) - 2}{f(x) - 1} \\
&= \frac{3\left(\frac{x-2}{x-3}\right) - 2}{\left(\frac{x-2}{x-3}\right) - 1} \\
&= \frac{3(x-2) - 2(x-3)}{(x-2) - 1(x-3)} \\
&= \frac{3x - 6 - 2x + 6}{1} \\
&= x
\end{aligned}$$

At first glance, this result seems to say simply that $(f^{-1} \circ f)(x) = x$.

But one must be careful. On the right side, notice two things:

- Consider the expression $f^{-1}(f(x))$ on the right side of line 1. Inside this expression is the expression $f(x)$. The value of $f(3)$ is undefined, because $x = 3$ is not in the domain of $f(x)$. So the value of the expression $f^{-1}(f(3))$ is undefined. Because $(f^{-1} \circ f)(x)$ is defined to mean $f^{-1}(f(x))$, we see that $(f^{-1} \circ f)(3)$ is undefined.
- The denominator $f(x) - 1$ in line 2 will be zero if $f(x) = 1$. But luckily, that will never happen, because the range of the function f is all $y \neq 1$.

So what the result actually says is this:

$$(f^{-1} \circ f)(x) = \begin{cases} x \; when \; x \neq 3 \\ undefined \; when \; x = 3 \end{cases}$$

(b) How does this result compare with the the prediction of Theorem 1.3.10?

Note that Theorem 1.3.10 predicts that $f^{-1} \circ f$ should equal id_X. To see if our result agrees with that prediction, we must first understand how the functtion id_X works.

To start, note that the symbol X denotes the domain of function f. That is, X is the set $X = \mathbb{R} - \{3\}$ The function id_X works in the following way:

$$id_X(x) = \begin{cases} x \; when \; x \in X \\ undefined \; when \; x \notin X \end{cases}$$

In other words,

$$id_X(x) = \begin{cases} x \; when \; x \neq 3 \\ undefined \; when \; x = 3 \end{cases}$$

We see that regardless of the value of the input $x \in X$, the resulting of the functions $f^{-1} \circ f$ and $id_X(x)$ will always match. That is

$$(f^{-1} \circ f)(x) = id_X(x)$$

We conclude that the two functions are equal. That is,

$$f^{-1} \circ f = id_X$$

End of [Example 1, revisited]

The following Theorem is stated in the book. Its theorem is very intersting, so I will prove it here.

Theorem 1.3.12

If $f: S \to T$ and $h: T \to V$ are bijections, then $(h \circ f)^{-1} = f^{-1} \circ h^{-1}$

Proof of Theorem 1.3.2 (Direct Proof)

(1) Suppose that $f: S \to T$ and $h: T \to V$ are bijections.

(2) h has an inverse function, $h^{-1}: V \to T$ (because g is bijective).

(3) f has an inverse function, $f^{-1}: T \to S$ (because f is bijective).

(4) $(f^{-1} \circ h^{-1}) \circ (h \circ f) = f^{-1} \circ \left(h^{-1} \circ (h \circ f)\right)$ by associativity

$= f^{-1} \circ \left((h^{-1} \circ h) \circ f\right)$ by associativity

$= f^{-1} \circ (id_T \circ f)$ by Theorem 1.3.10

$= f^{-1} \circ f$ because $id_T \circ f$ is just f

$= id_S$ by Theorem 1.3.10

(8) A similar string of equations yields the equality $(h \circ f) \circ (f^{-1} \circ h^{-1}) = id_T$

(9) The expression $(f^{-1} \circ h^{-1})$ must be the inverse function for $h \circ f$. (By (7),(8),Thm 1.3.10)

(10) But the inverse function for $h \circ f$ is denoted $(h \circ f)^{-1}$. That is, $(h \circ f)^{-1} = f^{-1} \circ h^{-1}$.

End of Proof of Theorem 1.3.12

End of Video