2.1a: Abstract Geometry

produced by Mark Barsamian, 2021.02.04

for Ohio University MATH 3110/5110 College Geometry

Topics:

- Abstract Geometry
    - Definition
    - Models
        - Finite Geometries
        - Cartesian Plane
        - Poincaré Plane
        - Riemann Sphere
- Additional Terminology Involving Points and Lines
    - Collinear and Non-Collinear Sets of Points
    - Distinct Lines
    - Intersecting lines and Parallel Lines
- Two Recurring Questions in Geometry

Reading: pages 17 – 21 of Section 2.1 Definition and Models of Incidence Geometry in the book

Millman & Parker, *Geometry: A Metric Approach with Models, Second Edition*

(Springer, 1991, ISBN 3-540-97412-1)

Homework: Section 2.1 # 1, 3, 5, 6, 8, 10, 11, 12

Abstract Geometry

> **Definition of Abstract Geometry**
>
> An ***abstract geometry*** $\mathcal{A}$ is an ordered pair $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ where $\mathcal{P}$ denotes a set whose elements are called **points** and $\mathcal{L}$ denotes a non-empty set whose elements are called **lines**, which are **sets of points** satisfying the following two requirements, called ***axioms***:
>
> **(i)** For every two distinct points $A, B \in \mathcal{P}$, there exists at least one line $l \in \mathcal{L}$ such that $A \in l$ and $B \in l$.
>
> **(ii)** For every line $l \in \overset{\mathcal{L}}{\cancel{L}}$ there exist at least two distinct points that are elements of the line.
>
> **Additional Terminology**
>
> **Words:** P *lies on* l or l *passes through* P.
>
> **Usage:** $P \in \mathcal{P}$ and $l \in \mathcal{L}$
>
> **Meaning:** $P \in l$

Requirements (i),(ii) are called the ***abstract geometry axioms***. They are simply the requriements that sets $\mathcal{P}, \mathcal{L}$ must satisfy (in addition to $\mathcal{L}$ being non-empty) in order for the pair $(\mathcal{P}, \mathcal{L})$ to be qualified to be called an *abstract geometry* .

Remarks on a Mistake in the Book's Definition of Abstract Geometry

Here is the book's definition:.

Book's Definition of Abstract Geometry $\{\mathcal{P}, \mathcal{L}\}$

An ***abstract geometry*** $\mathcal{A}$ consists of a set $\mathcal{P}$, whose elements are called **points**, together with a collection $\mathcal{L}$ of non-empty subsets of $\mathcal{P}$, called **lines**, such that:

(i) For every two points $A, B \in \mathcal{P}$, there is a line $l \in \mathcal{L}$ with $a \in l$ and $b \in l$.

(ii) Every line has at least two points.

An *abstract geometry* is written as a *set*: $\mathcal{A} = \{\mathcal{P}, \mathcal{L}\}$.

There are some subtle differences between the book's definition (in the red box) and my definition (in the green box).

Notice that my definition contains the qualifiers *distinct points* and *at least one line* for clarity.

Notice also that in my definition, an *abstract geometry* is presented as an ordered pair $\mathcal{A} = (\mathcal{P}, \mathcal{L})$, whereas in the book's definition, an *abstract geometry* is presented as a set $\mathcal{A} = \{\mathcal{P}, \mathcal{L}\}$. The use of ordered pairs (or more generally, ordered *n-tuples*) is more standard in math. The symbol representing the set of points must be on the left in the ordered pair; the symbol representing the set of line, on the right.

More importantly, I feel that the use of the phrase *non-empty* in the book's definition is a mistake, for two reasons.

Reason #1: Observe that axiom (ii) will ensure that each line l will be a non-empty subset, so stating explicitly that the lines are …*non-empty subsets*… is redundant. You will see that our book is very spare in its presentation. There is never any redundancy. That there is redundancy in this definition is a sign that it is a mistke.

Reason #2: If $\mathcal{P}$ contains a single point A and $\mathcal{L}$ is the empty set containing no lines, then $\{\mathcal{P}, \mathcal{L}\}$ satisfies the book's definition! That is, the book's definition does not require that there be any lines.

On the other hand, notice that the use of the phrase *non-empty* in *my* definition (the definition in the green box) allows us to prove a theorem:

Theorem: An *abstract geometry* must contain at least one line and at least two distinct points.

Proof

(1) Suppose that $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ is an *abstract geometry* .

(2) There exists at least one line $l \in \mathcal{L}$ (by (1) and the definition of *abstract geometry* .)

(3) There exist at least two distinct points $A, B \in l$ (by (2) and *Abstract Geometry Axiom* (ii))

End of Proof

Because of the problems with the book's definiton of *abstract geometry* (the definition in the red box above), we will not use that definition in this course. We will use my definition of *abstract geometry* (the definition in the green box above).

Models of Abstract Geometry

A **model** for *abstract geometry* is simply an example of a pair $(\mathcal{P}, \mathcal{L})$ that satisfies the definition of *abstract geometry* .

The book presents three examples of *abstract Geometries* (three models) in Section 2.1.

- The Cartesian plane
- The Poincaré plane
- The Riemann Sphere

But will start with simpler models: *finite geometries*.

Finite Geometries

Observe that there is nothing in the definition of *abstract geometry* that says that there must be an infinite set of points. Indeed, it is possible for a pair $(\mathcal{P}, \mathcal{L})$ to qualify as an *abstract geometry* with only a finite set of points.

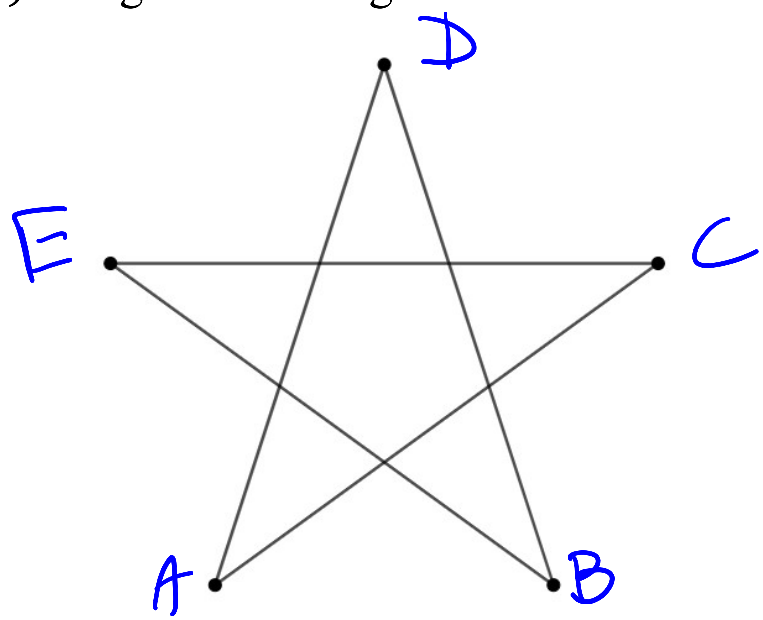
Definition: A ***finite geometry*** is an *abstract geometry* in which the set of points $\mathcal{P}$ is a *finite set*.

[Example 1] Finite sets that may or may not qualify to be called *abstract geometries*.

(a) Consider the pair $(\mathcal{P}, \mathcal{L})$ with

- points $\mathcal{P} = \{A, B, C, D, E\}$
- lines $\mathcal{L} = \{\{A, C\}, \{A, D\}, \cancel{\{A, E\}}, \{B, D\}, \{B, E\}, \{C, E\}, \}$

Here is an illustration of $(\mathcal{P}, \mathcal{L})$ using dots and segments.



Not an abstract geometry

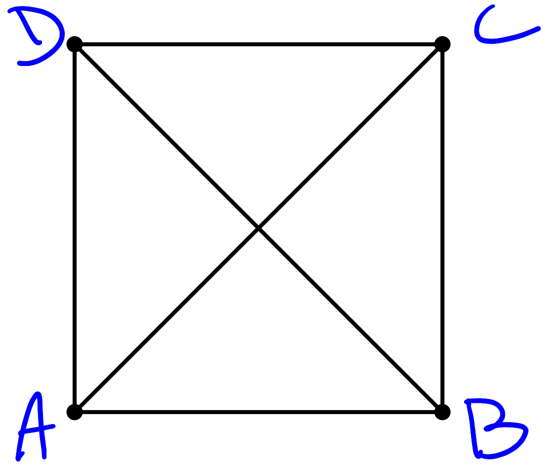
Is $(\mathcal{P}, \mathcal{L})$ qualified to be called an *abstract geometry* ? Explain why or why not.

No! Does not satisfy Axiom (i). For instance, there is no line that passes through B and C.

(It does satisfy Axiom (ii). Every line does contain at least two points.

(b) Consider the pair $(\mathcal{P}, \mathcal{L})$ with

- points $\mathcal{P} = \{A, B, C, D, \cancel{E}\}$
- lines $\mathcal{L} = \big\{\{A, B\}, \{A, C\}, \{B, C\}, \{B, D\}, \{C, D\}, \{D, A\}\big\}$



An abstract geometry

Is $(\mathcal{P}, \mathcal{L})$ qualified to be called an *abstract geometry* ? Explain why or why not.

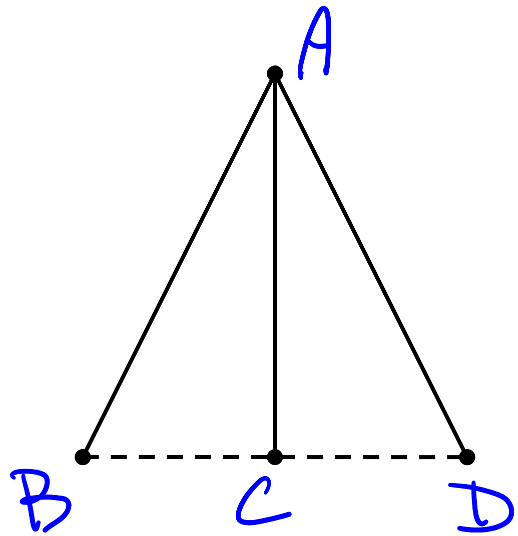
yes

Satisfies axiom (i) For each pair of points, there <u>is</u> a line that passes through both points

Satisfies axiom (ii) Each line contains at least two points

(c) Consider the pair $(\mathcal{P}, \mathcal{L})$ with

- points $\mathcal{P} = \{A, B, C, D\}$
- lines $\mathcal{L} = \{\{A, B\}, \{A, C\}, \{A, D\}, \{B, C, D\},\}$



an abstract geometry

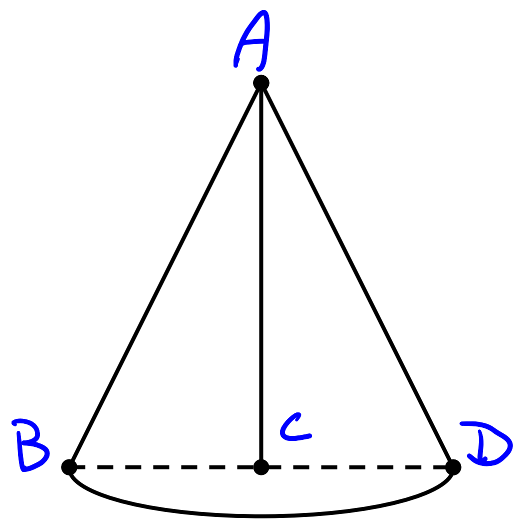
Is $(\mathcal{P}, \mathcal{L})$ qualified to be called an *abstract geometry* ? Explain why or why not.

Satisfies axiom(i) ✓

Satisfies axiom(ii) ✓

(d) Consider the pair $(\mathcal{P}, \mathcal{L})$ with

- points $\mathcal{P} = \{A, B, C, D\}$
- lines $\mathcal{L} = \big\{\{A, B\}, \{A, C\}, \{A, D\}, \{B, C, D\}, \{B, D\}\big\}$



an abstract geometry

Is $(\mathcal{P}, \mathcal{L})$ qualified to be called an *abstract geometry* ? Explain why or why not.

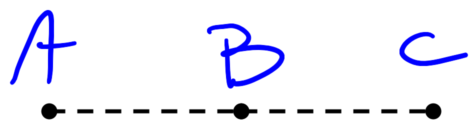
axiom (i) is satisfied ✓

axiom (ii) is satisfied ✓

(e) Consider the pair $(\mathcal{P}, \mathcal{L})$ with

- points $\mathcal{P} = \cancel{\{B, C\}}$ $\{A, B, C\}$
- lines $\mathcal{L} = \{\{A, B, C\}\}$

an abstract geometry



Is $(\mathcal{P}, \mathcal{L})$ qualified to be called an *abstract geometry* ? Explain why or why not.

axiom (i) is satisfied

axiom (ii) is satisfied

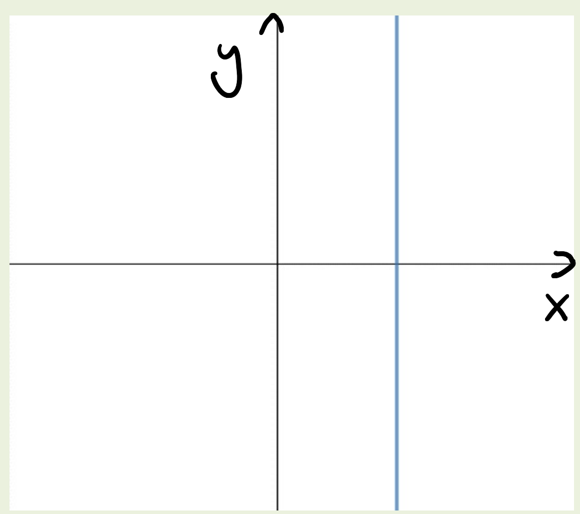
End of [Example 1]

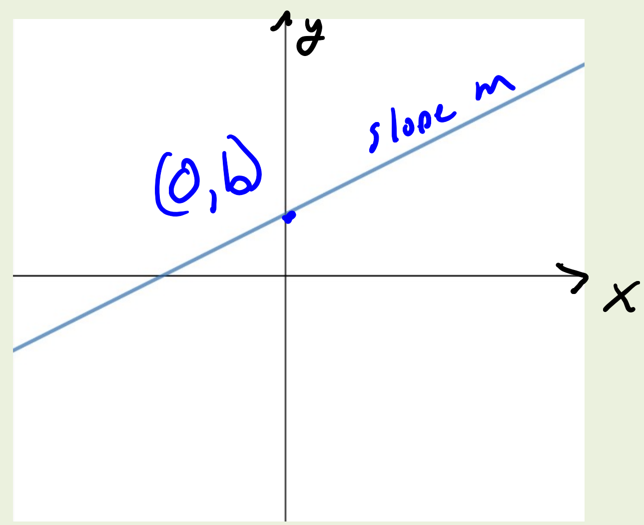
The Cartesian Plane

Definition: The ***Cartesian Plane***, $\mathcal{C}$, is the pair $\mathcal{C} = (\mathbb{R}^2, \mathcal{L}_E)$ where

- The set of points is the set $\mathbb{R}^2$ of ordered pairs of real numbers.
- The set of lines is the set $\mathcal{L}_E$ containing lines (sets of points) of two types:
  - A ***vertical line*** is a set of the form $L_a = \{(x,y) \in \mathbb{R}^2 | x = a\}$, where $a \in \mathbb{R}$
  - A ***non-vertical line*** is a set of the form $L_{m,b} = \{(x,y) \in \mathbb{R}^2 | y = mx + b\}$, where $\cancel{a}, b \in \mathbb{R}$

m, b





x = a

Proposition 2.1.1

The *Cartesian Plane* $\mathcal{C} = (\mathbb{R}^2, \mathcal{L}_E)$ satisfies the definition of *abstract geometry* .

That is, the *Cartesian Plane* $\mathcal{C} = (\mathbb{R}^2, \mathcal{L}_E)$ is a *model* of *abstract geometry* .

The authors provide a nice proof of this proposition on page 18 of the book.

Proposition 2.1.1. *Let* $\mathscr{S} = \mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$. *We define a set of "lines" as follows. A* **vertical line** *is any subset of* $\mathbb{R}^2$ *of the form*

$$L_a = \{(x, y) \in \mathbb{R}^2 \mid x = a\} \tag{1-1}$$

where a is a fixed real number. A **non-vertical line** *is any subset of* $\mathbb{R}^2$ *of the form*

$$L_{m,b} = \{(x, y) \in \mathbb{R}^2 \mid y = mx + b\} \tag{1-2}$$

where m and b are fixed real numbers. (*See Figure* 2-1.) *Let* $\mathscr{L}_E$ *be the set of all vertical and non-vertical lines. Then* $\mathscr{E} = \{\mathbb{R}^2, \mathscr{L}_E\}$ *is an abstract geometry.*

Proof. We must show that if $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are any two distinct points of $\mathbb{R}^2$ then there is an $l \in \mathscr{L}_E$ containing both. This is done by considering two cases.

Case 1. If $x_1 = x_2$ let $a = x_1 = x_2$. Then both P and Q belong to $l = L_a \in \mathscr{L}_E$.

Case 2. If $x_1 \neq x_2$ we show how to find m and b with $P, Q \in L_{m,b}$. Motivated by the idea of the "slope" of a line we define m and b by the equations:

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{and} \quad b = y_2 - mx_2.$$

It is easy to show that $y_2 = mx_2 + b$ and that $y_1 = mx_1 + b$, so that both P and Q belong to $l = L_{m,b} \in \mathscr{L}_E$.

It is easy to see that each line has at least two points so that $\mathscr{E}$ is an abstract geometry. $\square$

Definition. The model $\mathscr{E} = \{\mathbb{R}^2, \mathscr{L}_E\}$ is called the **Euclidean Plane** (The notation L_a and $L_{m,b}$ will be reserved for the lines of the Euclidean Plane, $\mathscr{E}$.)

There are some things about the book proof that may make it hard to understand. In particular, the book's proof of Proposition 2.1.1 does not have numbered statements and does not have any kind of headings that indicate the proof structure. This is typical for a book written at the level of our book. Authors assume that the reader is skilled in reading and writing proofs.

Although the book's proofs may initially be difficult for a MATH 3110/5110 student to understand, the proof skills that the student acquired in MATH 3050 or CS 3000 can can help them make sense of the proofs.

In most cases, the key to understanding the book's proofs is to consider proof structure. In general, in MATH 3110/5110, I do not intend to duplicate proofs that are in the book with proofs written at a more introductory level, but I will discuss how the student should read the book's proof of Proposition 2.1.1 and *add* structure to it, either mentally or on paper, and thereby make sense of it.

To prove that $(\mathbb{R}^2, \mathcal{L}_E)$ is an *abstract geometry* means proving that $(\mathbb{R}^2, \mathcal{L}_E)$ satisfies the axioms. There are two axioms, so the proof will need to have two parts.

Proof Part 1: Prove that $(\mathbb{R}^2, \mathcal{L}_E)$ satisfies *abstract geometry* axiom (i)

Proof Part 2: Prove that $(\mathbb{R}^2, \mathcal{L}_E)$ satisfies *abstract geometry* axiom (ii)

The authors don't provide either heading, but they do start their proof with the following words.

We must show that if $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are any two distinct points of $\mathbb{R}^2$, then there is an $l \in \mathcal{L}$ contining both.

That is the reader's clue that the book's proof is going to start with what I have called **Proof Part 1** above.

Notice that the final sentence of the book's proof is the following:

It is easy to see that that each line has at least two points so that $\mathcal{E}$ is an Abstract Geometry .

Realize that the first half of that final sentence what I have called **Proof Part 2** above.

The second half of that final sentence is a belated explanation of what the whole proof has been about: Proving that $(\mathbb{R}^2, \mathcal{L}_E)$ is a *model* of *abstract geometry* .

Now, consider how **Proof Part 1** must be structured. Observe that *abstract geometry* axiom (i) is a universal statement.

> **(i)** For every two distinct points $A, B \in \mathcal{P}$, there exists at least one line $l \in \mathcal{L}$ such that $A \in l$ and $B \in l$.

Therefore, a proof that $(\mathbb{R}^2, \mathcal{L}_E)$ satisfies axiom (i) must have the following structure.

Proof that $(\mathbb{R}^2, \mathcal{L}_E)$ satisfies *abstract geometry* axiom (i)

(1) Suppose $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are any two distinct points of $\mathbb{R}^2$.

some step here

(*) There exists at least one line $l \in \mathcal{L}_E$ such that $P \in l$ and $Q \in l$. (some justification)

End of Proof

Furthermore, it is easy to articulate an OR statement for the second statement of the proof.

Proof that $(\mathbb{R}^2, \mathcal{L}_E)$ satisfies *abstract geometry* axiom (i)

(1) Suppose $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are any two points of $\mathbb{R}^2$.

(2) $x_1 = x_2$ or $x_1 \neq x_2$ (property of real numbers)

some steps here

(*) There exists at least one line $l \in \mathcal{L}_E$ such that $P \in l$ and $Q \in l$. (some justification)

End of Proof

The presence of the OR statement enables a proof by cases. The boldface content added to the proof outline below shows that structure.

Proof that $(\mathbb{R}^2, \mathcal{L}_E)$ satisfies *abstract geometry* axiom (i)

(1) Suppose $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are any two points of $\mathbb{R}^2$.

(2) $x_1 = x_2$ or $x_1 \neq x_2$ (property of real numbers)

(3) (Case 1) Suppose that $\boldsymbol{x_1 = x_2}$

.......some steps here.......

(**) There exists at least one line $\boldsymbol{l \in \mathcal{L}_E}$ such that $\boldsymbol{P \in l}$ and $\boldsymbol{Q \in l}$ in this case. (some justification)

(***) (Case 2) Suppose that $\boldsymbol{x_1 \neq x_2}$

.......some steps here.......

(****) There exists at least one line $\boldsymbol{l \in \mathcal{L}_E}$ such that $\boldsymbol{P \in l}$ and $\boldsymbol{Q \in l}$ in this case. (some justification)

(*) **(Conclusion of Cases)** Therefore, there exists at least one line $l \in \mathcal{L}_E$ such that $P \in l$ and $Q \in l$. (**because it is true in every case**)

End of Proof

Finally, we can fill in the missing steps with details taken from the book's proof.

Proof that $(\mathbb{R}^2, \mathcal{L}_E)$ satisfies *abstract geometry* axiom (i)

(1) Suppose $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are any two points of $\mathbb{R}^2$.

(2) $x_1 = x_2$ or $x_1 \neq x_2$ (property of real numbers)

(3) (Case 1) Suppose that $x_1 = x_2$

(4) Let $\boldsymbol{a = x_1 = x_2}$.

(5) Observe that $\boldsymbol{L_a \in \mathcal{L}_E}$ and that $\boldsymbol{P \in L_a}$ and $\boldsymbol{Q \in L_a}$.

(6) There exists at least one line $l \in \mathcal{L}_E$ such that $P \in l$ and $Q \in l$ in this case. (**by (5)**)

(7) (Case 2) Suppose that $x_1 \neq x_2$

(8) Let $\boldsymbol{m = \frac{y_2 - y_1}{x_2 - x_1}}$ and $\boldsymbol{b = y_2 - m x_2}$.

(9) Observe that $\boldsymbol{L_{m,b} \in \mathcal{L}_E}$. Basic arithmetic shows that $\boldsymbol{P \in L_{m,b}}$ and $\boldsymbol{Q \in L_{m,b}}$.

(10) There exists at least one line $l \in \mathcal{L}_E$ such that $P \in l$ and $Q \in l$ in this case. (**by (9)**)

(11) (Conclusion of Cases) Therefore, there exists at least one line $l \in \mathcal{L}_E$ such that $P \in l$ and $Q \in l$. (Because by (6),(10), we see that it is true in every case)

End of Proof

What we end up with, finally, is the content of the book's proof, but organized with some added

Proof that $\left(\mathbb{R}^2, \mathcal{L}_E\right)$ satisfies the definition of *abstract geometry*

Part 1: Proof that $\left(\mathbb{R}^2, \mathcal{L}_E\right)$ satisfies *abstract geometry* axiom (i)

(1) Suppose $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are any two points of $\mathbb{R}^2$.

(2) $x_1 = x_2$ or $x_1 \neq x_2$ (property of real numbers)

(3) **(Case 1)** Suppose that $x_1 = x_2$

(4) Let $a = x_1 = x_2$.

(5) Observe that $L_a \in \mathcal{L}_E$ and that $P \in L_a$ and $Q \in L_a$.

(6) There exists at least one line $l \in \mathcal{L}_E$ such that $P \in l$ and $Q \in l$ in this case. (by (5))

(7) **(Case 2)** Suppose that $x_1 \neq x_2$

(8) Let $m = \frac{y_2 - y_1}{x_2 - x_1}$ and $b = y_2 - m x_2$.

(9) Observe that $L_{m,b} \in \mathcal{L}_E$. Basic arithmetic shows that $P \in L_{m,b}$ and $Q \in L_{m,b}$.

(10) There exists at least one line $l \in \mathcal{L}_E$ such that $P \in l$ and $Q \in l$ in this case. (by (9))

(11) **(Conclusion of Cases)** Therefore, there exists at least one line $l \in \mathcal{L}_E$ such that $P \in l$ and $Q \in l$. (Because by (6),(10), we see that it is true in every case)

End of Proof Part I

Proof Part 2: Prove that $\left(\mathbb{R}^2, \mathcal{L}_E\right)$ satisfies *abstract geometry* axiom (ii)

(It is easy to see that a line of either type has at least two points, so we omit this proof.)

End of Proof that $\left(\mathbb{R}^2, \mathcal{L}_E\right)$ satisfies the Definition of *Abstract Geometry*

The MATH 3110/5110 student should realize that "reading" the Millman & Parker book, or any advanced math book, entails a process like the one presented above. That is, when reading a proof in the book, the reader must (either mentally or on scrap paper) add outline form, statement numbers, additional statements, and clear justifications for each step.

Throughout the semester in MATH 3110/5110, I will occasionally assign Homework, Quiz, or Exam problems that ask the student to rewrite a proof from the book, adding outline form, statement numbers, additional statements, and clear justifications for each step.

Note that the steps in the proof of Proposition 2.1.1 provide us with a procedure for finding the Cartesian line through two distinct points $P, Q \in \mathbb{R}^2$. I will present the procedure here.

Procedure for Finding the *Cartesian* Line Passing through Two Distinct Points in $\mathbb{R}^2$

Suppose $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are any two distinct points of $\mathbb{R}^2$.

If $x_1 = x_2$ then let $a = x_1 = x_2$. In this case, $L_a \in \mathcal{L}_H$ and $P, Q \in L_a$.

If $x_1 \neq x_2$ then define constants m, b by the following formulas:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$b = y_2 - mx_2$$

y = mx+b

Then $P, \in L_{m,b}$ and $Q \in L_{m,b}$.

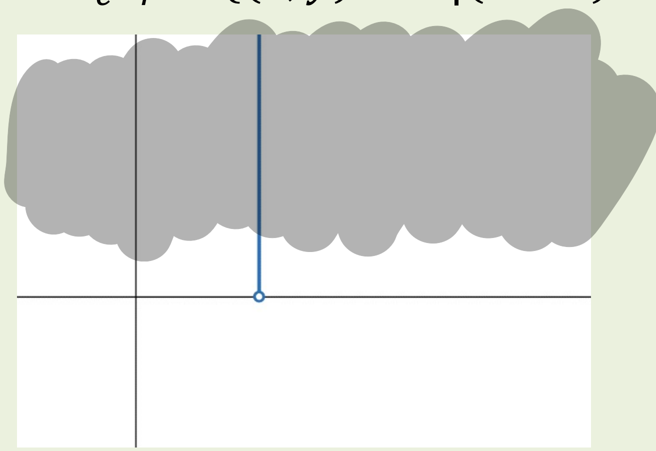
Remark: It is worth noting that the proof in Proposition 2.1.1 does not address the issue of *uniqueness* of the line. That is, the proof only shows how one can find *a Cartesian line*. The proof does not show that the resulting line is *the only such line*. But later in Section 2.1, it will be shown that, given two distinct points in $\mathbb{R}^2$, there is only *one Cartesian line* passing through both. That is why, in my presentation of the procedure, I used the phrase "…Finding *the Cartesian Line*…".

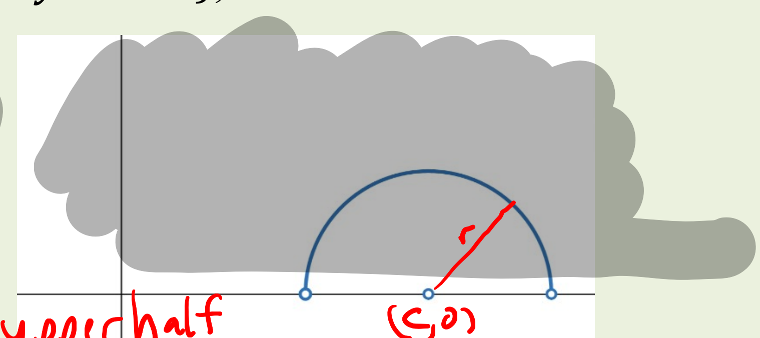
The Poincaré Plane

Definition: The ***Poincaré Plane***, $\mathcal{H}$, is the pair $\mathcal{H} = (\mathbb{H}, \mathcal{L}_H)$ Hyperbolic

- The set of points is the set $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 | y > 0\}$. (the *upper-half plane*)
- The set of lines is the set $\mathcal{L}_H$ containing lines (sets of points) of two types:
  - A ***type I line*** is a set of the form ${}_aL = \{(x, y) \in \mathbb{H} | x = a\}$, where $a \in \mathbb{R}$
  - A ***type II line*** is a set of form

$${}_cL_r = \{(x, y) \in \mathbb{H} | (x - c)^2 + y^2 = r^2\}, \text{ where } c \in \mathbb{R} \text{ and } r > 0$$





upper half of a circle centered at (c,0) with radius r

Proposition 2.1.2 The *Poincaré Plane* $\mathcal{H} = (\mathbb{H}, \mathcal{L}_H)$ is a *model* of *abstract geometry* .

The authors provide a nice proof of this proposition on page 19 of the book. The main idea of the authors' proof is the following:

Procedure for Finding the *Poincaré Line* Passing Through Two Distinct Points in $\mathbb{H}$

Suppose $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are any two distinct points of $\mathbb{H}$.

If $x_1 = x_2$ then let $a = x_1 = x_2$. In this case, ${}_aL \in \mathcal{L}_H$ and $P, Q \in {}_aL$.

If $x_1 \neq x_2$ then define constants c, r by the following formulas:

$$c = \frac{x_2^2 - x_1^2 + y_2^2 - y_1^2}{2(x_2 - x_1)}$$

$$r = \sqrt{(x_1 - c)^2 + y_1^2}$$

In this case, ${}_cL_r \in \mathcal{L}_H$ and $P, Q \in {}_cL_r$.

Remark: Later in Section 2.1, it will be shown that, given two distinct points in $\mathbb{H}$, there is *only one Poincaré line* passing through both. That is why, in my presentation of the procedure, I used the phrase "…Finding *the Poincaré Line*…".

[Example 2] Let $P = (5,3)$ and $Q = (10,2)$

(a) Find the equation for the *Cartesian line* through P and Q.

Solution:

Introduce x, y notation: $P = (5,3) = (x_1, y_1)$ and $Q = (10,2) = (x_2, y_2)$.

We follow the *Procedure for Finding the Cartesian Line Passing Through Two Distinct Points*.

Since $x_1 \neq x_2$ the *Cartesian line* that passes through P, Q will be a *non-vertical* line.

We find constants m, b by using the formulas presented in the *Procedure*:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 3}{10 - 5} = -\frac{1}{5}$$

$$b = y_2 - mx_2 = 2 - \left(\left(-\frac{1}{5}\right)10\right) = 4$$

The *Cartesian line* that passes through P, Q will be the *non-vertical* line

$$L_{-\frac{1}{5},4} = \left\{(x, y) \in \mathbb{R}^2 | y = \left(-\frac{1}{5}\right)x + 4\right\}$$

(b) Find the equation for the *Poincaré line* through P and Q.

Solution:

We follow the *Procedure for Finding the Poincaré Line Passing Through Two Distinct Points*.

Since $x_1 \neq x_2$ the *Poincaré line* that passes through P, Q will be a *type II* line.

We find constants c, r by using the formulas presented in the *Procedure*:

$$c = \frac{x_2^2 - x_1^2 + y_2^2 - y_1^2}{2(x_2 - x_1)} = \frac{2^2 - 3^2 + 10^2 - 5^2}{2(10 - 5)} = 7$$

$$r = \sqrt{(x_1 - c)^2 + y_1^2} = \sqrt{(5 - 7)^2 + 3^2} = \sqrt{13}$$

The *Poincaré line* that passes through P, Q will be the *type II* line

$$_7L_{\sqrt{13}} = \{(x, y) \in \mathbb{H} | (x - 7)^2 + y^2 = 13\}$$

(c) Illustrate your solutions to **(a)** and **(b)** with drawings.

Solution:

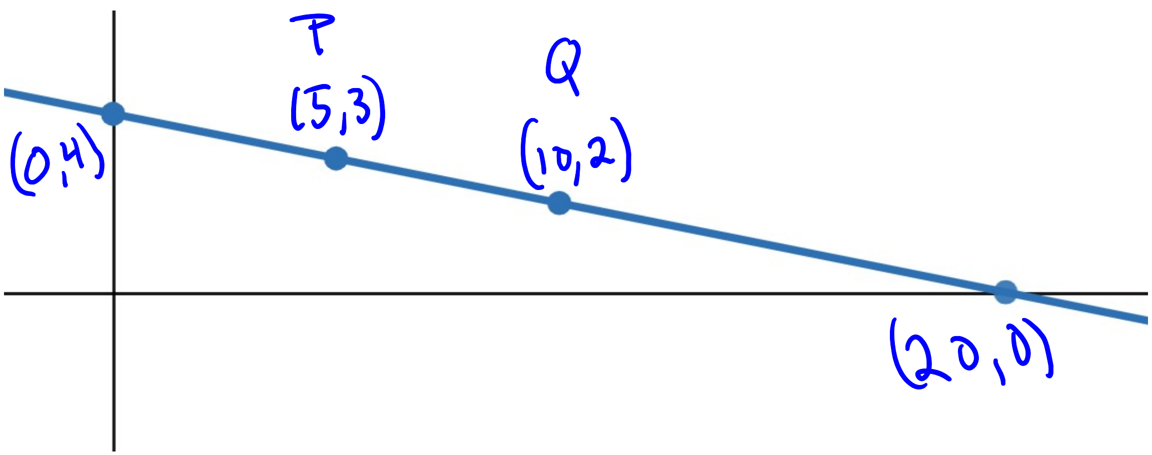
The *Cartesian line* $L_{-\frac{1}{5},4}$ will be a straight-looking line with slope $m = -\frac{1}{5}$ and y intercept $(0,4)$.

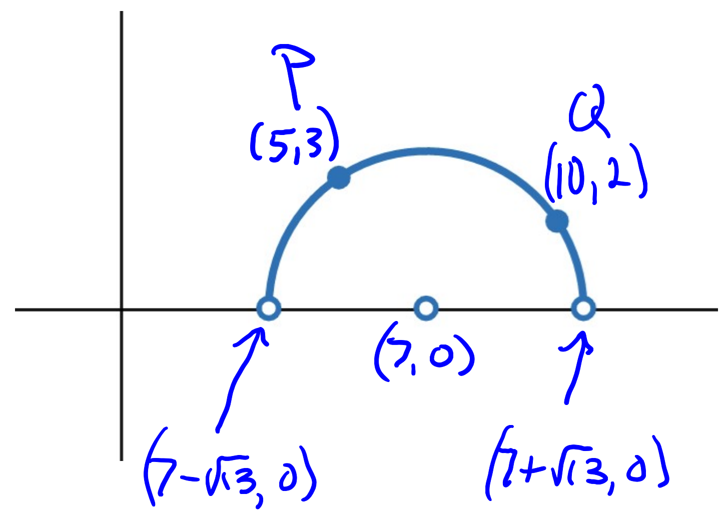
The *Poincaré line* ${}_{7}L_{\sqrt{13}}$ will be the upper half of a circle centered at $(7,0)$ with radius $\sqrt{13}$.

We plot these lines on different axes, to be clear that the lines live in different worlds.

Observe the important points that we include in the drawings, with their coordinates labeled:

The points P, Q, all axis intercepts, and the center of the circle.





End of [Example 2]

The Riemann Sphere

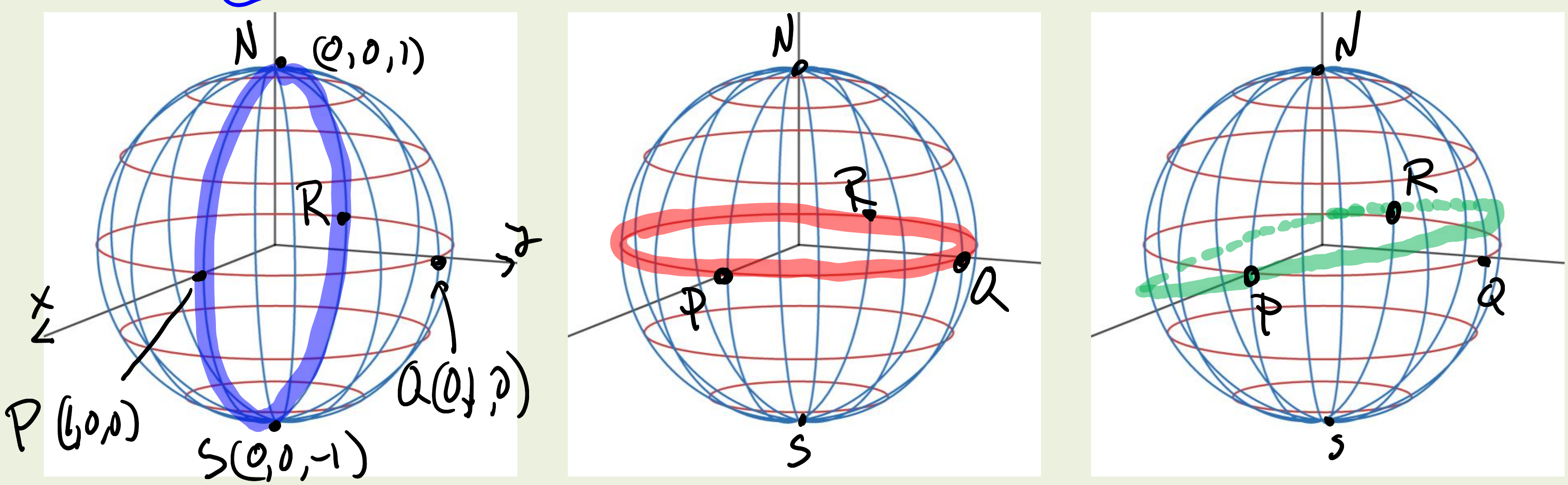
Definition: The ***Riemann Sphere***, $\mathcal{R}$, is the pair $\mathcal{R} = (S^2, \mathcal{L}_R)$

$\mathbb{R}^3$

- The set of points is the set $S^2 = \{(x, y, z) \in \cancel{\mathbb{R}^2}\ \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$. (the *unit sphere* in $\mathbb{R}^2$)
- The set of lines, $\mathcal{L}_R$, containing lines of the form

script G

$$\mathcal{G}_{a,b,c} = \{(x, y, z) \in S^2 | ax + by + cz = 0\}, \text{ where } a, b, c \in \mathbb{R}$$



all lines of longitude are "lines" in the set $\mathcal{L}_R$

most lines of latitude are not "lines" in the set $\mathcal{L}_R$

Only the equator is a "line"

Remarks about the definition of *lines* on the *Riemann Sphere*.

- The *line* $\mathcal{G}_{a,b,c}$ is the intersection of S^2 with a plane through the origin in $\mathbb{R}^3$. So the set $\mathcal{G}_{a,b,c}$ will actually be a circle.
- All circles that lie on S^2 can be described as the intersection of S^2 with a plane. But not all of the planes are *planes through the origin*. So not all circles that lie on S^2 qualify to be called *lines* in $\mathcal{R} = (S^2, \mathcal{L}_R)$.
- Consider the picture on the left. All of the *blue* lines, which would be called "lines of longitude" on a globe, do lie on planes through the origin. So all "lines of longitude" on a globe qualify to be called *lines* in $\mathcal{R} = (S^2, \mathcal{L}_R)$.
- Consider the picture in the middle. The red circles, which would be called "lines of latitude" on a globe, mostly lie in planes that are *not* through the origin. Only one "line of latitude", the *equator*, lies in a plane through the origin. So the equator is the only "line of latitude" on the globe that qualifies to be called a *line* in $\mathcal{R} = (S^2, \mathcal{L}_R)$.
- Consider the picture on the right. Besides the lines of longitude and the equator, there are other circles that lie on S^2 that do qualify to be called called *lines* in $\mathcal{R} = (S^2, \mathcal{L}_R)$.

- For the *line* $\mathcal{G}_{a,b,c}$ is the intersection of S^2 with a plane, the vector $\overrightarrow{(a,b,c)}$ is a normal vector to that plane. (Observe that there are many triples (a,b,c) that describe the same plane, so there is more than one symbol $\mathcal{G}_{a,b,c}$ that describes the same *line*.
- The *line* $\mathcal{G}_{a,b,c}$ is a circle that lies on S^2 and has radius $r = 1$, which is the same as the radius of S^2. Not all circles that lie on S^2 are this big. In the pictures above, most of the red circles have radius $r < 1$. Only the red *equator*, has $r = 1$.
- Such circles, the ones that lie on a sphere and have a radius that is the same as the radius of the sphere, are called *great circles* for the sphere. So another way of describing the *lines* in the *Riemann Sphere* $\mathcal{R} = (S^2, \mathcal{L}_R)$ is to say that the *lines* in $\mathcal{R} = (S^2, \mathcal{L}_R)$ are the *great circles* on S^2. That is why the *script capital G* character, $\mathcal{G}$, is used in the symbol $\mathcal{G}_{a,b,c}$ denoting the lines in $\mathcal{R} = (S^2, \mathcal{L}_R)$.

The book presents a short proof of the following proposition.

Proposition 2.1.3 The *Riemann Sphere* $\mathcal{R} = (S^2, \mathcal{L}_R)$ is a *model* of a*bstract geometry* .

I won't discuss the proof—it is mainly about solving equations—but it is worthwhile to make an observation about the number of *lines* that pass through *two given points*:

The proof of Proposition 2.1.3 shows that given any two distinct points $P, Q \in S^2$, there exists a great circle $\mathcal{G}$ on S^2 such that $P, Q \in \mathcal{G}$. Good. That is one of the requirements that $\mathcal{R} = (S^2, \mathcal{L}_R)$ must meet in order to qualify to be called an *abstract geometry* . But the great circle $\mathcal{G}$ is not always *unique*.

- For example, for $A = (1,0,0)$ and $B = (0,1,0)$, there is exactly one great circle $\mathcal{G}$ on S^2 such that $A, B \in \mathcal{G}$. That circle is the equator.
- But for $N = (0,0,1)$ and $S = (0,0,-1)$, there are many great circles $\mathcal{G}$ on S^2 such that $, S \in \mathcal{G}$. Every line of longitude on S^2 is a great circle $\mathcal{G}$ on S^2 such that $N, S \in \mathcal{G}$.

Observe that nothing in the definiton of *abstract geometry* forbids this behavior. Indeed, *abstract geometry* axiom (i) says

(i) For every two distinct points $A, B \in \mathcal{P}$, there exists ***at least one line*** $l \in \mathcal{L}$ such that $A, B \in l$.

If we desire that a geometry have the property that there is ***exactly one*** such line, then we will need to say that in the axioms of the geometry. We will see that done in the definition of *incidence geometry* in the next video.

Additional Terminology Involving Points and Lines

Definition of Collinear

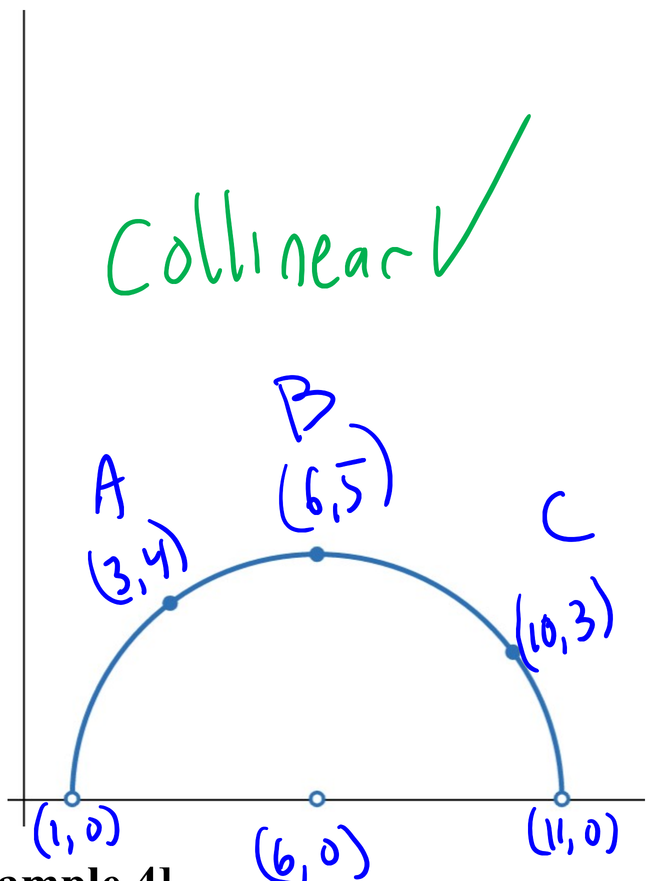
Words: The set of points $S \subset \mathcal{P}$ is ***collinear***.

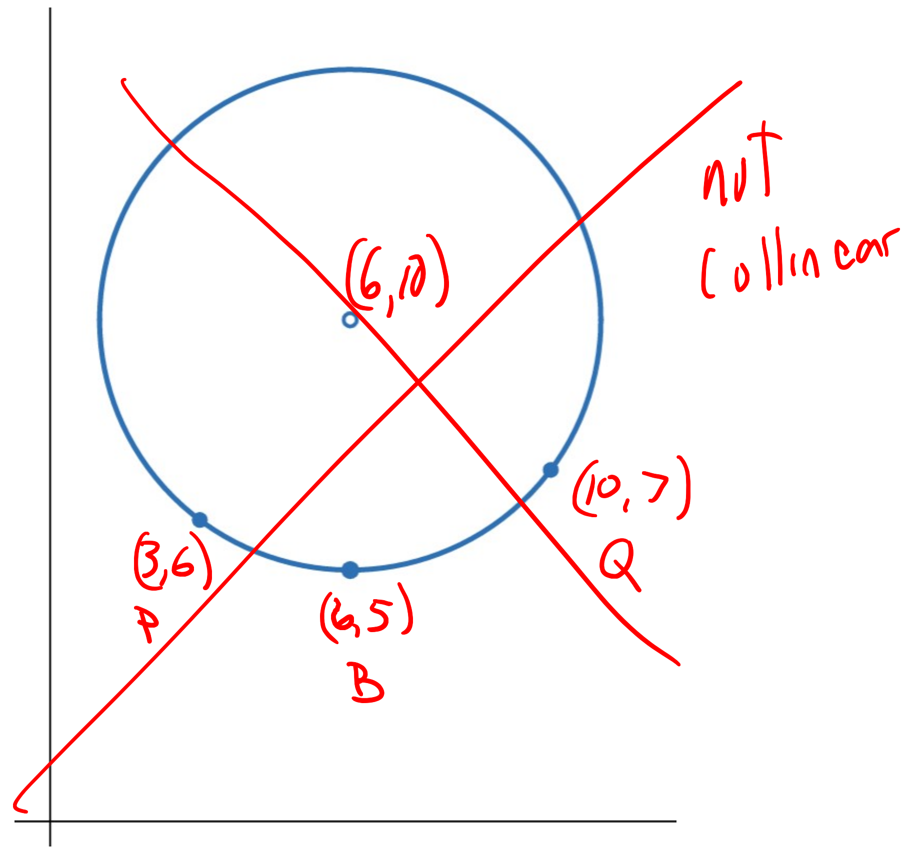
Meaning: There exists a line $l \in \mathcal{L}$ that passes through all the points in S. That is, $S \subset l$.

The reader is probably comfortable with the word *collinear* from experience. But realize that *collinear* points can have surprising configurations in abstract geometries.

[Example 3] (a) Let $A = (3,4), B = (6,5), C = (10,3)$. Observe that these three points lie on a circle of radius $r = 5$ centered at the point $(6,0)$. That is, they all lie on the *Poincaré line* ${}_6L_5$ Therefore, the set $\{A, B, C\}$ is *collinear* in the *Poincaré plane*.

(b) Let Let $P = (3,6), B = (6,5), Q = (10,7)$. Observe that these three points lie on a circle of radius $r = 5$ centered at the point $(6,10)$. This circle is not centered on the x axis, so it does not correspond to a Poincaré line. So the set $\{P, B, Q\}$ is *not* collinear in the *Poincaré plane*.





End of [Example 4]

Definition of Distinct Points and Distinct Lines

Words: *points A and B are distinct*

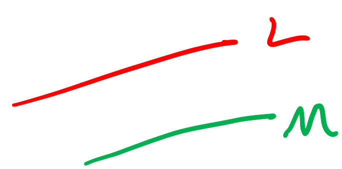
Meaning: *points A and B are not the same point.*

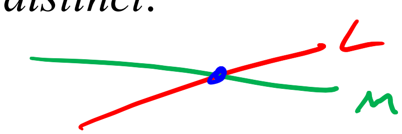
Words: *lines L and M are distinct*

Meaning: *lines L and M are not the same line.*

This may seem silly, because A, B are distinct letters. One would assume that they are not the same point. But in fact one must not assume that simply because A, B are *distinct letters*, that they must necessarily represent *distinct points*. There will be some situations where a point A is known to exist that satisfies some criteria, and a point B is known to exist that satisfies some other criteria. It might turn out that points A and B are actually the same point. But of course, it might turn out that they are distinct points.

The situation for lines is richer, though, because lines are *sets of points*. Two say that lines L, M are the same line means that they are equal as *sets of points*. That is, set L equals set M. If there are any points that are on one line and not the other line, then the lines L, M are *distinct*.

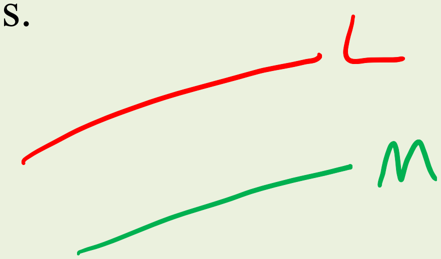




Definition of Intersecting Lines

Words: *lines L and M intersect*

Meaning: The intersection of sets L and M is not empty. That is, $L \cap M \neq \phi$. In other words, there exists at least one point P that lies on both lines.



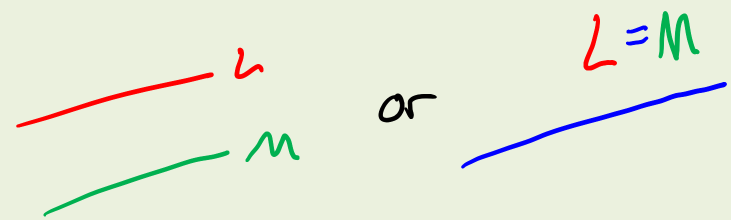
Additional Terminology

Words: *Lines L and M do not intersect*

Meaning: $L \cap M = \phi$

Definition of Parallel Lines

Words: *lines L and M are parallel*



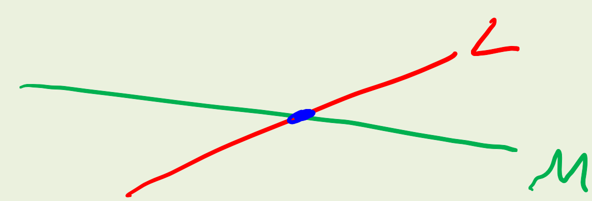
Symbol: $L \parallel M$

Meaning: Either L and M do not intersect or they are the same line.

Meaning in Symbols: $L \cap M = \phi$ or $L = M$

Additional Terminology

Words: *lines L and M are not parallel*



Symbol: $L \nparallel M$

Meaning: L and M intersect and they are not the same line.

Meaning in Symbols: $L \cap M \neq \phi$ and $L \neq M$

Two Recurring Questions in Geometry

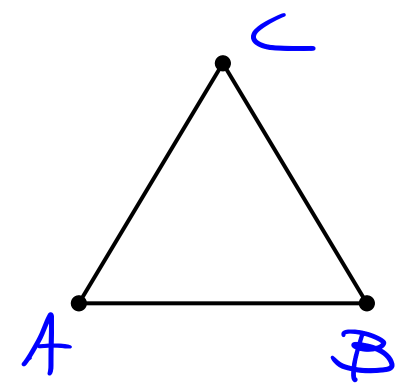
The BIG QUESTIONS

- **BIG QUESTION #1:** *Do parallel lines exist?*
- **BIG QUESTION #2:** *Given a line L and a point P not on L, how many lines exist that contain P and are parallel to L?*

[Example 5] Answers to the BIG QUESTIONS in some *finite* abstract geometries

(a) Consider the pair $(\mathcal{P}, \mathcal{L})$ with

- points $\mathcal{P} = \{A, B, C\}$
- lines $\mathcal{L} = \{\{A, B\}, \{A, C\}, \{B, C\}\}$



It is clear that $(\mathcal{P}, \mathcal{L})$ qualifies to be called an *abstract geometry* .

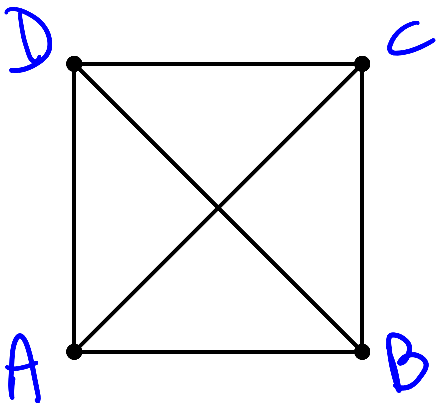
What is the answer to BIG QUESTION #1? No, there are no parallel lines.

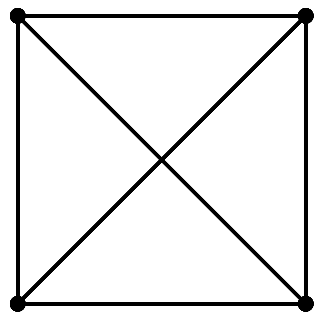
What is the answer to BIG QUESTION #2? None, because there are no parallel lines

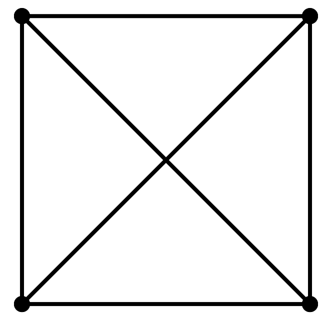
(b) Consider the pair $(\mathcal{P}, \mathcal{L})$ with

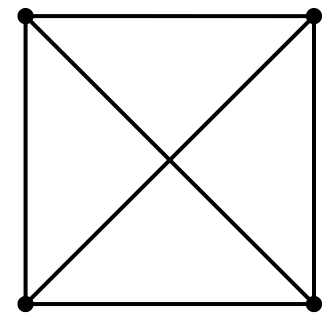
- points $\mathcal{P} = \{A, B, C, D\}$
- lines $\mathcal{L} = \{\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}\}$

It is clear that $(\mathcal{P}, \mathcal{L})$ qualifies to be called an *abstract geometry* .









What is the answer to BIG QUESTION #1?

Yes, there are parallel lines $\{A, B\} \parallel \{C, D\}$ and $\{B, D\} \parallel \{A, C\}$

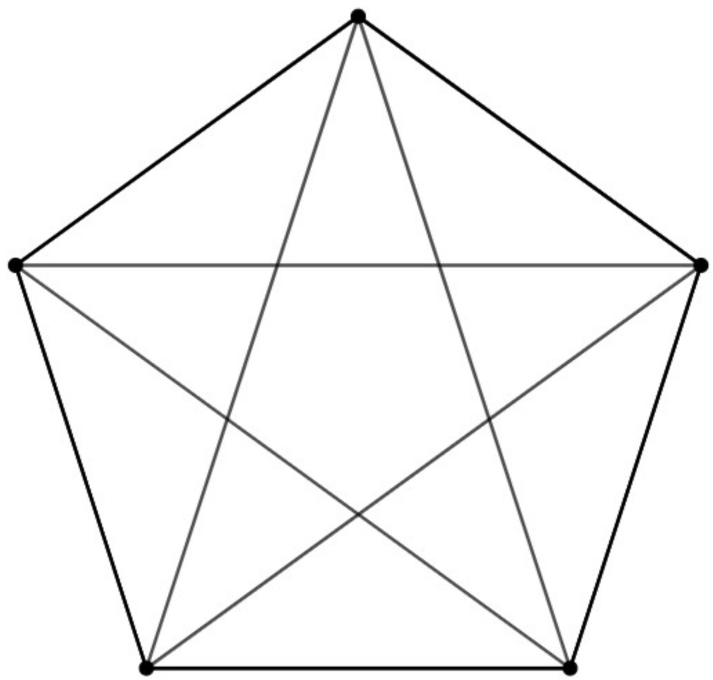
What is the answer to BIG QUESTION #2?

Always exactly one line

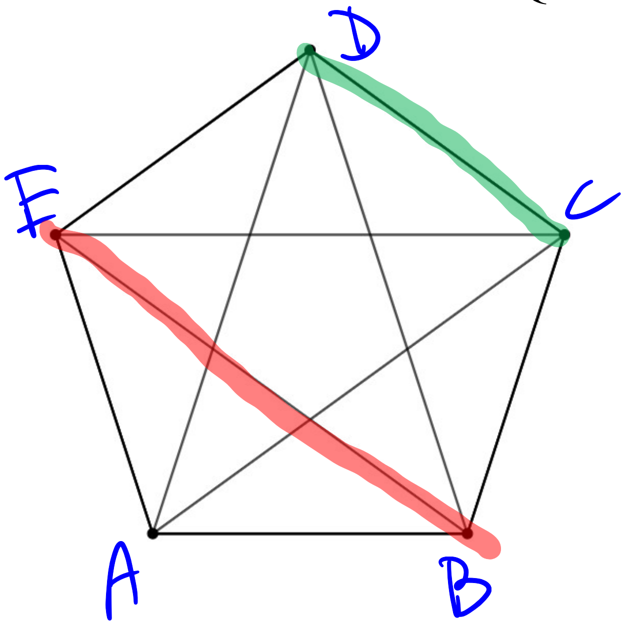
(c) Consider the pair $(\mathcal{P}, \mathcal{L})$ with

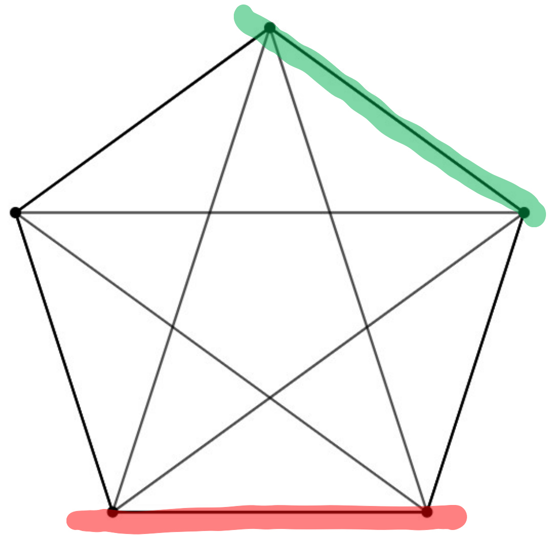
- points $\mathcal{P} = \{A, B, C, D, E\}$
- lines $\mathcal{L} = \big\{\{A, B\}, \{A, C\}, \{A, D\}, \{A, E\}, \{B, C\}, \{B, D\}, \{B, E\}, \{C, D\}, \{C, E\}, \{D, E\}\big\}$

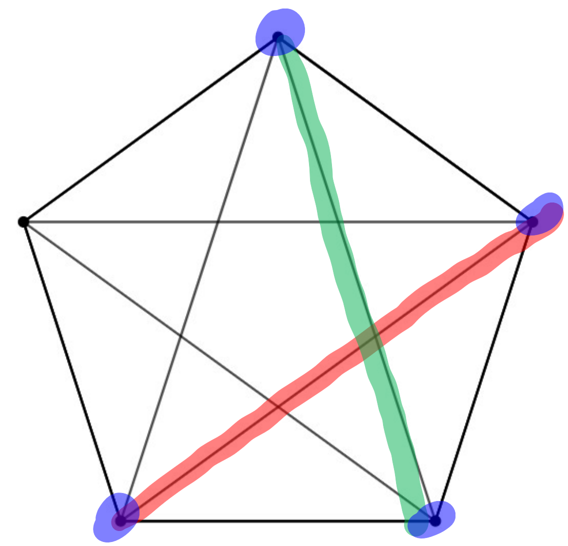
It is clear that $(\mathcal{P}, \mathcal{L})$ qualifies to be called an *abstract geometry* .



What is the answer to BIG QUESTION #1?

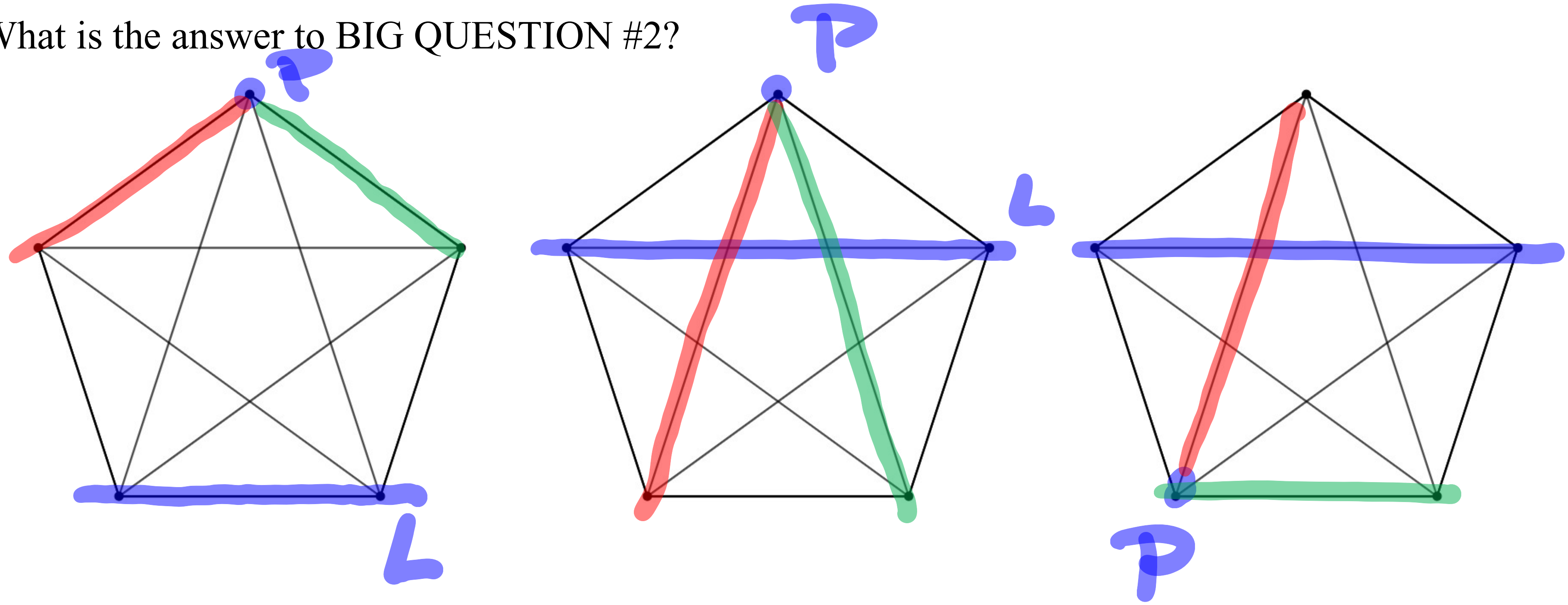






yes there are parallel lines!

What is the answer to BIG QUESTION #2?



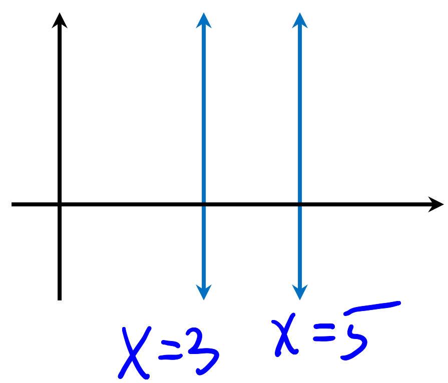
Answer: always two lines

End of [Example 5]

[Example 6] Answers to the BIG QUESTIONS in the *Cartesian Plane*

What is the answer to BIG QUESTION #1 in the *Cartesian Plane*?

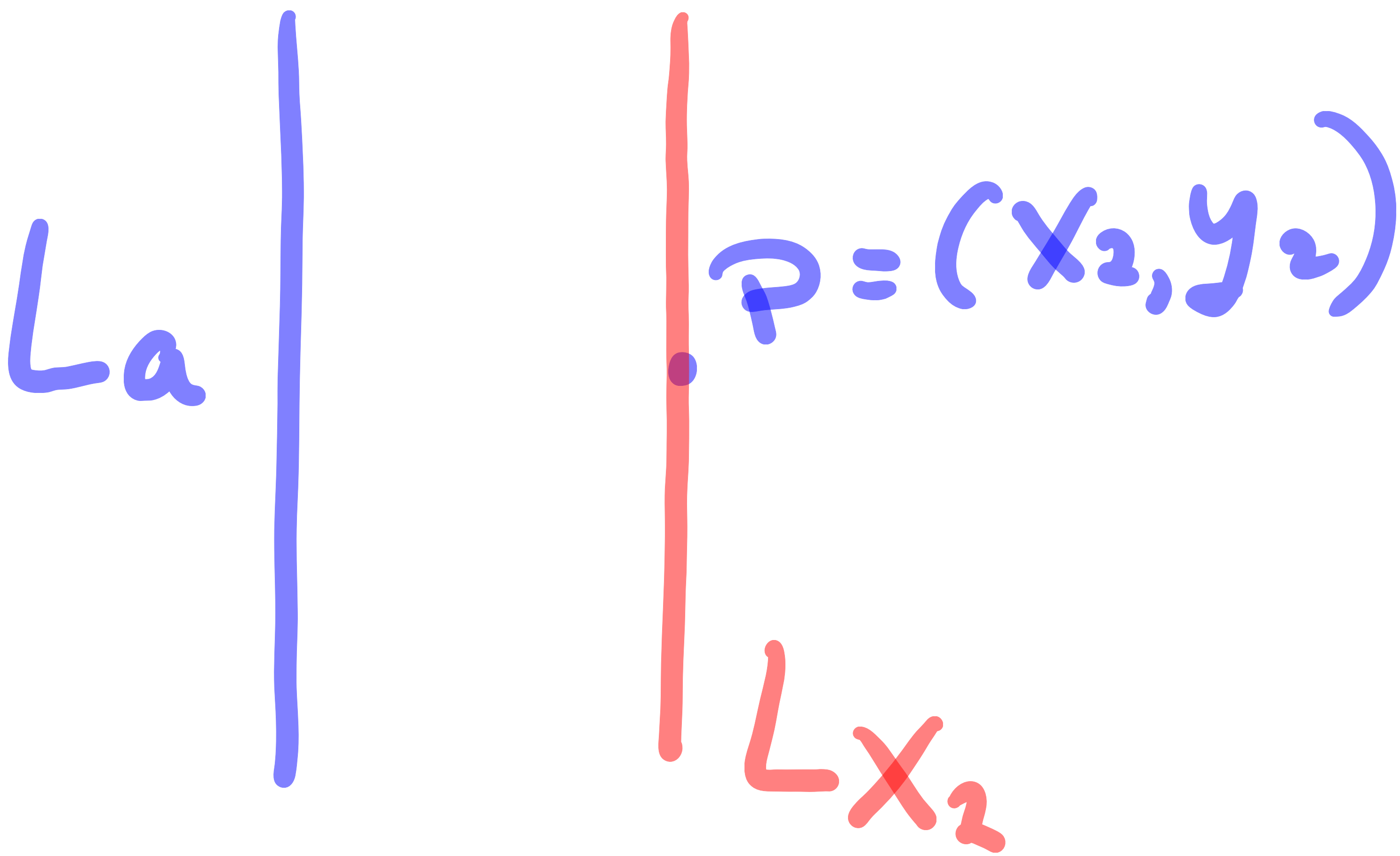
Clearly, parallel lines exist. For example, the vertical lines $L_3 \parallel L_5$



What is the answer to BIG QUESTION #2 in the *Cartesian Plane*?

You're all used to the idea that the answer to this question is 1. But it is worthwhile to present the two possible cases and explain why we know the answer is 1.

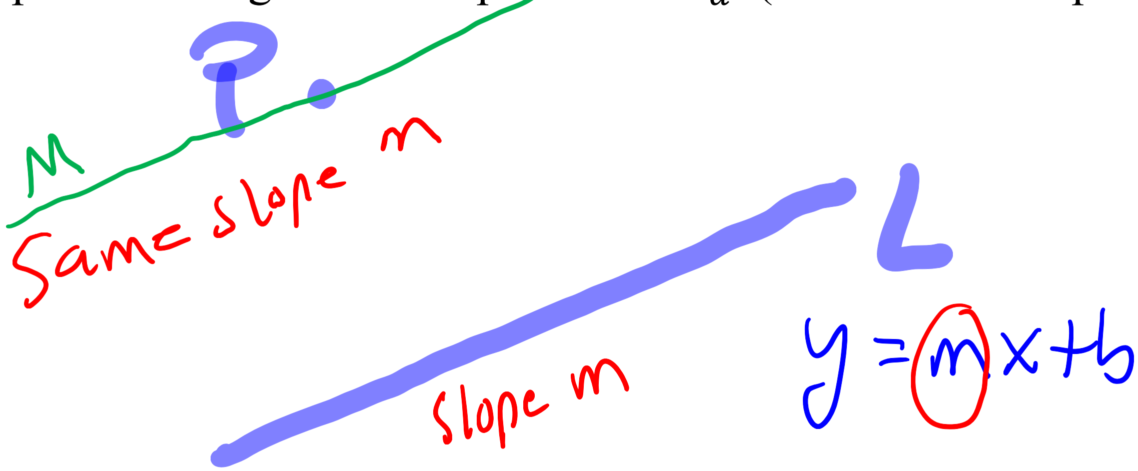
Case 1: When the given line L is a *vertical* line L_a and point $P = (x_2, y_2) \notin L_a$, then it must be that $x_2 \neq a$. Observe that the *vertical* line L_{x_2} passes through P and is parallel to L_a. Furthermore, any other lines, *vertical* or *non-vertical*, that pass through P will not be parallel to L_a. So line L_{x_2} is the only line that passes through P and is parallel to L_a. (These facts are proven by *solving equations*.)



Case 2: When the given line L is a *non-vertical* line $L_{m,b}$ and point $P = (x_2, y_2) \notin L_{m,b}$, there will be exactly one *non-vertical* line $L_{m,k}$ with the same slope m that passes through P. The y intercept k can be found by the formula

$$k = y_2 - mx_2$$

Observe that the *non-vertical* line $L_{m,k}$ passes through P and is parallel to L_a. Furthermore, any other lines, *vertical* or *non-vertical*, that pass through P will *not* be parallel to L_a. So line $L_{m,k}$ is the *only* line that passes through P and is parallel to L_a. (These facts are proven by *solving equations*.)



So we conclude that the answer to BIG QUESTION #2 in the *Cartesian Plane* is

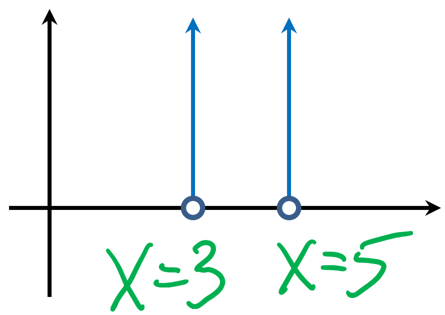
There exists exactly one line M that passes through P and is parallel to L.

End of [Example 6]

[Example 7] Answers to the BIG QUESTIONS in the *Poincaré Plane*

What is the answer to BIG QUESTION #1 in the *Poincaré Plane*?

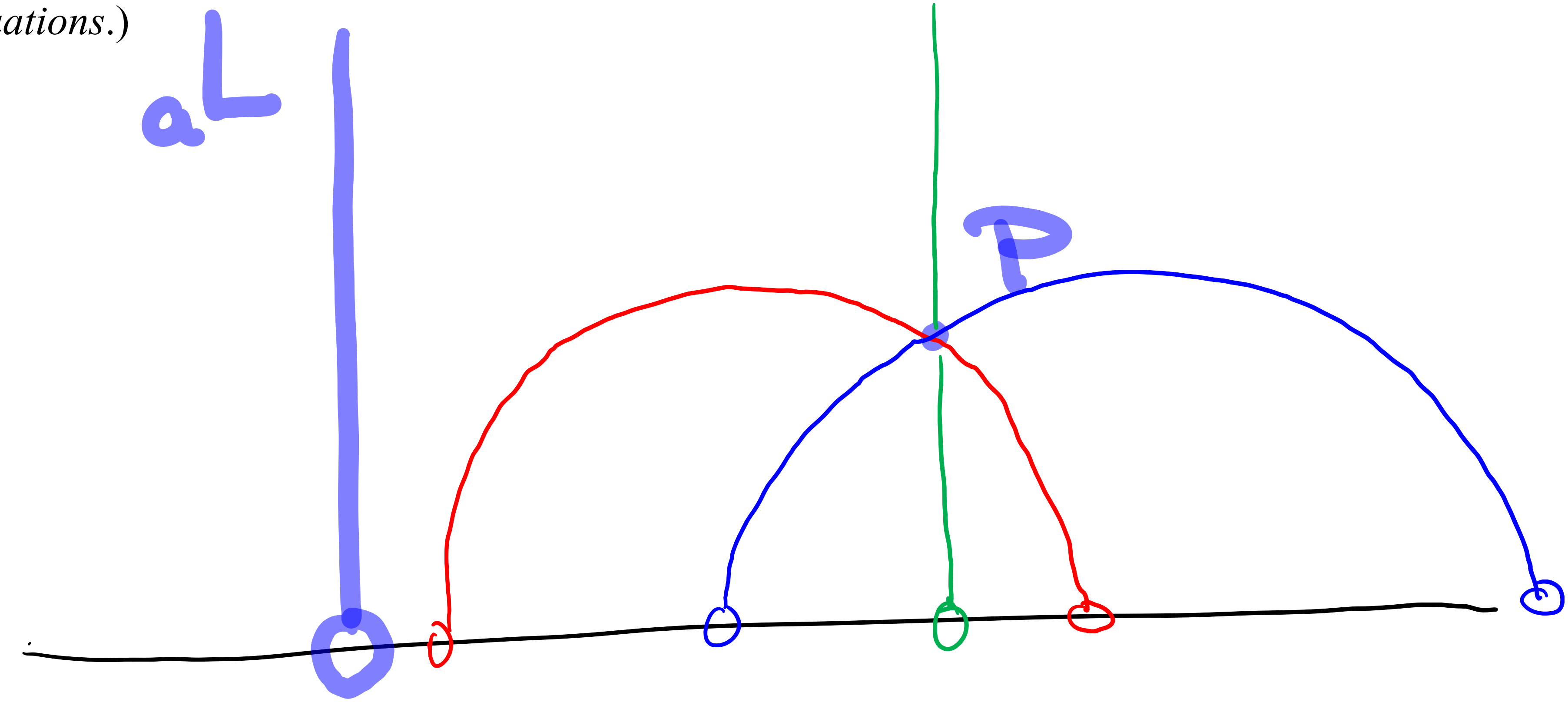
Clearly, parallel lines exist. For example, the vertical lines ${}_3L \parallel {}_5L$



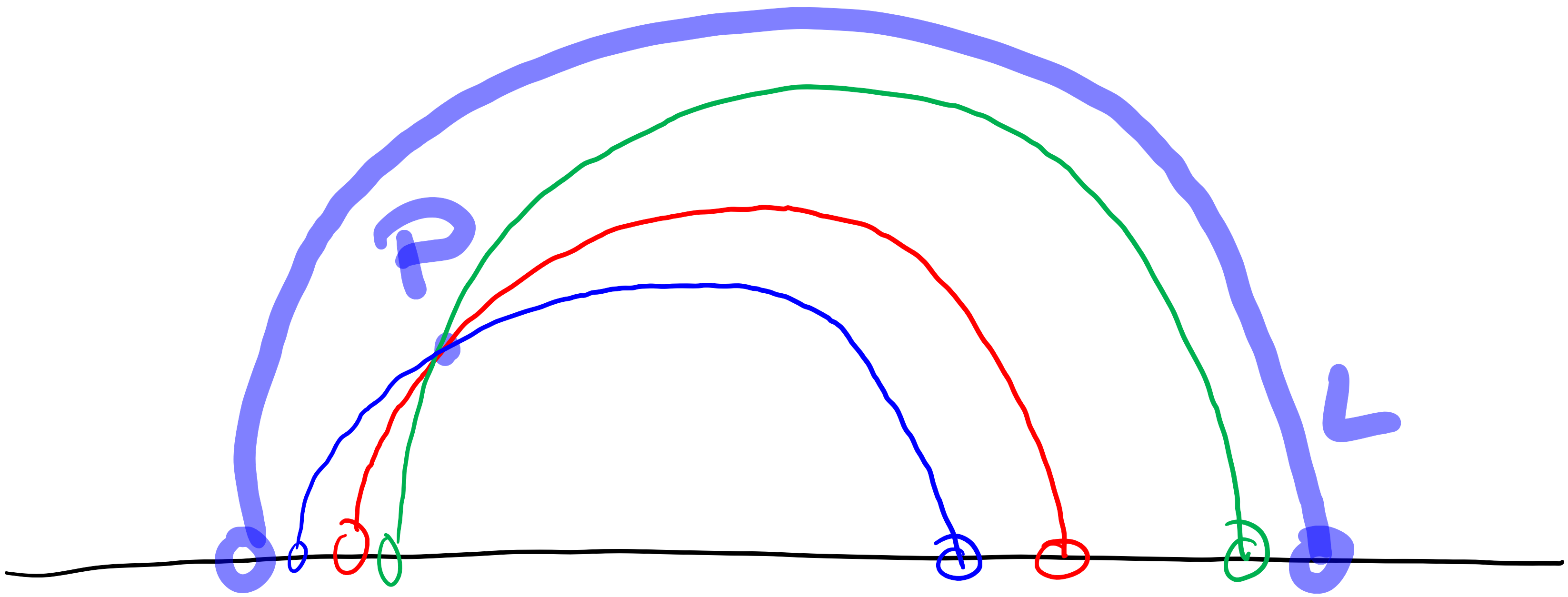
What is the answer to BIG QUESTION #2 in the *Poincaré Plane*?

As we did when we answered this question in the *Cartesian Plane*, it will be useful to consider two cases.

Case 1: When the given line L is a *type I* line ${}_aL$ and point $P = (x_2, y_2) \notin {}_aL$, then it must be that $x_2 \neq a$. Observe that the *type I* line ${}_{x_2}L$ passes through P and is parallel to ${}_aL$. But there will also be many *type II* lines that pass through P and are parallel to ${}_aL$. (These facts are proven by *solving equations*.)



Case 2: When the given line L is a *type II* line ${}_cL_r$ and point $P = (x_2, y_2) \notin {}_cL_r$, there will be many *type II* lines that pass through P and are parallel to ${}_cL_r$. There may or may not be any *type I* lines that pass through P and are parallel to ${}_cL_r$. (These facts are proven by *solving equations*.)



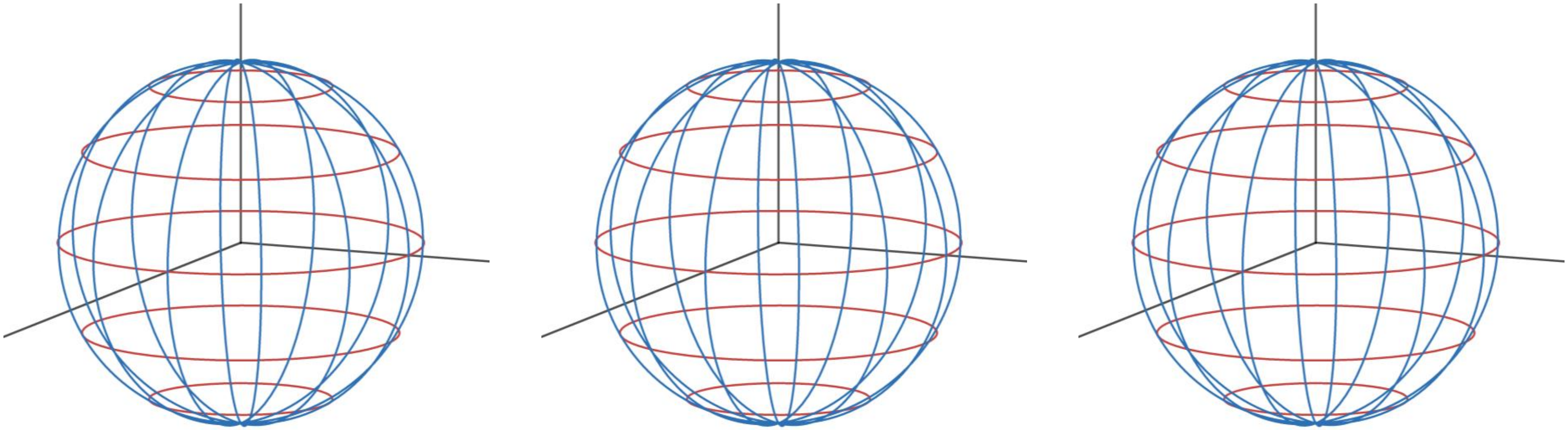
So we conclude that the answer to BIG QUESTION #2 in the *Poincaré Plane* is

There is an infinite collection of lines that pass through P and are parallel to L.

End of [Example 7]

[Example 8] Answers to the BIG QUESTIONS in the *Riemann Sphere*

What is the answer to BIG QUESTION #1 in the *Riemann Sphere?*



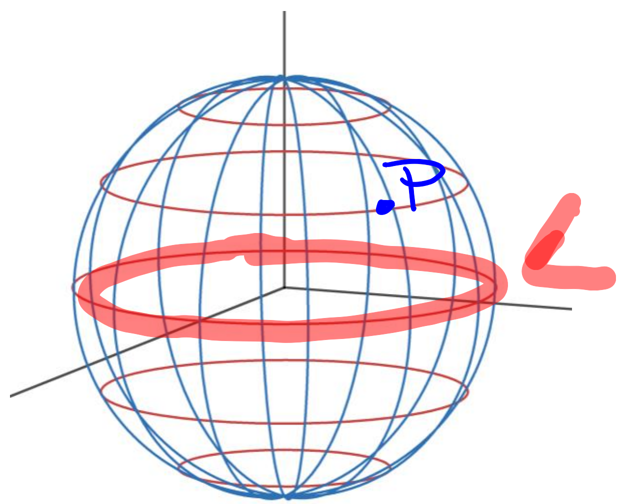
One might at first be tempted to say that parallel lines exist on the Riemann Sphere $\mathcal{R} = (S^2, \mathcal{L}_R)$, because the *lines of latitude* (the red circles) are parallel to each other. But remember that the only line of latitude that qualifies to be called a line in $\mathcal{R} = (S^2, \mathcal{L}_R)$ is the *equator*.

Notice that all of the blue lines of longitude intersect at the north and south pole, and they all intersect the equator at some point on the equator, and they all intersect the green great circles (the ones that are neither lines of longitude nor the equator) at two points. Furthermore, notice that the equator intersects every green great circle at two points. Conclude that there are no parallel lines on the *Riemann Sphere*. Therefore, the answer to BIG QUESTION #1 is that

There are no parallel lines on the Riemann Sphere.

And of course, because there are no parallel lines, the answer to BIG QUESTION #2 is,

There are no lines that pass through P and are parallel to L on the Riemann Sphere.



End of [Example 8]

[End of Video]