3.1: Using Vectors to Simplify Descriptions and Calculations in the Cartesian Plane

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for Ohio University MATH 3110/5110 College Geometry

Topics:

- Triangle Inequalities
  - for Absolute Value
  - for Distance Functions
- The Vector Space $\mathbb{R}^2$
  - Definition
  - Properties
- Using Vector Notation and Calculations to Simplify Geometry
  - Calculations involving the Euclidean distance function
  - Describing Cartesian lines using vectors
  - Describing Euclidean rulers using vectors

Reading: Section 3.1 An Alternative Description of the Cartesian Plan,

p 42 – 46 in *Geometry: A Metric Approach with Models, Second Edition* by Millman & Parker

Homework: Section 3.1#5, 7, 8, 9b and some custom problems

Recall Important Things From Section 2.1

Definition of Abstract Geometry

An ***abstract geometry*** $\mathcal{A}$ is an ordered pair $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ where $\mathcal{P}$ denotes a set whose elements are called **points** and $\mathcal{L}$ denotes a non-empty set whose elements are called **lines**, which are **sets of points** satisfying the following two requirements, called ***axioms***:

(i) For every two distinct points $A, B \in \mathcal{P}$, there exists **at least one** line $l \in \mathcal{L}$ such that $A \in l$ and $B \in l$.

(ii) For every line $l \in L$ there exist at least two distinct points that are elements of the line.

Definition of Incidence Geometry

An ***incidence geometry*** $\mathcal{A}$ is an *abstract geometry* $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ that satisfies the following two additional requirements, called ***axioms***:

(i) For every two distinct points $A, B \in \mathcal{P}$, there exists **exactly one** line $l \in \mathcal{L}$ such that $A \in l$ and $B \in l$.

(ii) There exist (at least) three *non-collinear* points.

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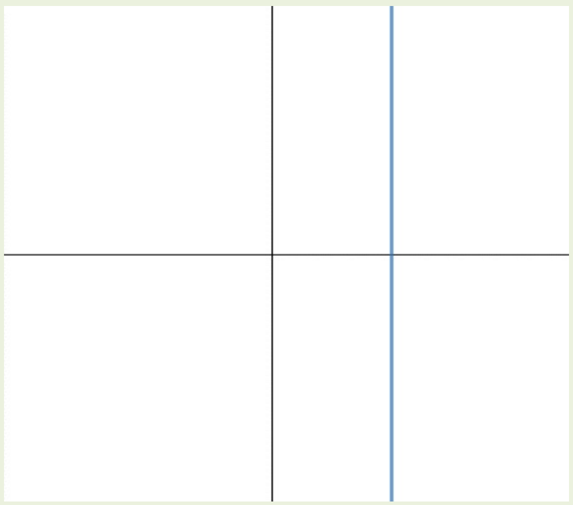
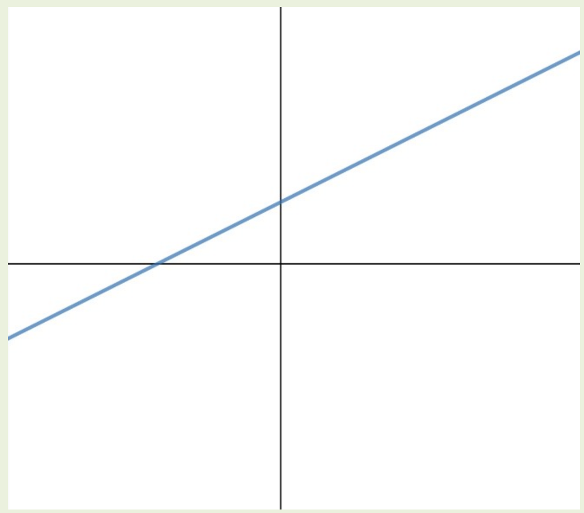
The Cartesian Plane

Definition: The ***Cartesian Plane***, $\mathcal{C}$, is the pair $\mathcal{C} = (\mathbb{R}^2, \mathcal{L}_E)$ where

- The set of points is the set $\mathbb{R}^2$ of ordered pairs of real numbers.
- The set of lines is the set $\mathcal{L}_E$ containing lines (sets of points) of two types:
  - A ***vertical line*** is a set of the form $L_a = \{(x,y) \in \mathbb{R}^2 | x = a\}$, where $a \in \mathbb{R}$
  - A ***non-vertical line*** is a set of the form $L_{m,b} = \{(x,y) \in \mathbb{R}^2 | y = mx + b\}$, where $a, b \in \mathbb{R}$





Proposition 2.1.4

The *Cartesian Plane* $\mathcal{C} = (\mathbb{R}^2, \mathcal{L}_E)$ satisfies the definition of *incidence geometry* .

That is, the *Cartesian Plane* $\mathcal{C} = (\mathbb{R}^2, \mathcal{L}_E)$ is a *model* of *incidence geometry* .

Procedure for Finding the *Cartesian* Line Passing through Two Distinct Points in $\mathbb{R}^2$

Suppose $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are any two distinct points of $\mathbb{R}^2$.

If $x_1 = x_2$ then let $a = x_1 = x_2$. In this case, $L_a \in \mathcal{L}_H$ and $P, Q \in L_a$.

If $x_1 \neq x_2$ then define constants m, b by the following formulas:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$b = y_2 - mx_2$$

Then $P, \in L_{m,b}$ and $Q \in L_{m,b}$.

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And important things from Section 2.2

Definition of Distance Function

words: d is a distance function on set S

meaning: d is a function $d: S \times S \to \mathbb{R}$ that satisfies these requirements

(i) $\forall P, Q \in S(d(P,Q) \geq 0)$

(ii) $d(P,Q) = 0$ if and only if $P = Q$

(iii) $d(P,Q) = d(Q,P)$

Some Examples of Distance Functions

Definition of the Absolute Value Distance Function on $\mathbb{R}$

symbol: $d_{\mathbb{R}}$

meaning: the function $d_{\mathbb{R}}: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $d_{\mathbb{R}}(x, y) = |x - y|$

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Definition of the Euclidean Distance Function on $\mathbb{R}^2$

symbol: d_E

meaning: the function $d_E: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$d_E\big((x_1, y_1), (x_2, y_2)\big) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Definition of the Taxicab Distance Function on $\mathbb{R}^2$

symbol: d_T

meaning: the function $d_T: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$d_T\big((x_1, y_1), (x_2, y_2)\big) = |x_1 - x_2| + |y_1 - y_2|$$

Definition of a Ruler for a Line

words: f *is a ruler for line* l

alternate words: f *is a coordinate system for line* l

alternate words: f *is a coordinate function for line* l

usage: There is an incidence geometry $(\mathcal{P}, \mathcal{L})$ in the discussion, and there is a distance function d on the set of points $\mathcal{P}$ in the discussion, and $l \in \mathcal{L}$.

meaning: f is a function $f: l \to \mathbb{R}$ that satisfies these requirements

(i) f is a bijection.

(ii) f "agrees with" the distance function d in the following way:

For each pair of points P and Q (not necessarily distinct) on line l, this equation is true:

$$|f(P) - f(Q)| = d(P, Q)$$

Additional Terminology:

The equation above is called the **Ruler Equation**.

The number $f(P)$ is called the **coordinate of P with respect to f**.

Definition of Metric Geometry

A ***metric geometry*** $\mathcal{M}$ is an ordered triple $\mathcal{M} = (\mathcal{P}, \mathcal{L}, d)$ that satisfies the following:

- $(\mathcal{P}, \mathcal{L})$ is an *incidence geometry*.
- d is a *distance function* on the set of points $\mathcal{P}$
- Every line $l \in \mathcal{L}$ has a *ruler*. This is requirement is called the **Ruler Postulate**.

Proposition 2.2.4: The triple $(\mathbb{R}^2, \mathcal{L}_E, d_E)$ is consisting of the *Cartesian Plane* $\mathcal{C} = (\mathbb{R}^2, \mathcal{L}_E)$ *incidence geometry* along with the *Euclidean distance function* d_E satisfies the *Ruler Postulate*, so the triple $(\mathbb{R}^2, \mathcal{L}_E, d_E)$ is qualified to be called a *metric geometry*.

Definition: The **Euclidean Plane** $\mathcal{E}$ is defined to be the *metric geometry* $\mathcal{E} = (\mathbb{R}^2, \mathcal{L}_E, d_E)$.

Proposition 2.2.7: The triple $(\mathbb{R}^2, \mathcal{L}_E, d_T)$ is consisting of the *Cartesian Plane* $\mathcal{C} = (\mathbb{R}^2, \mathcal{L}_E)$ *incidence geometry* along with the *taxicab distance function* d_T satisfies the *Ruler Postulate*, so the triple $(\mathbb{R}^2, \mathcal{L}_E, d_T)$ is qualified to be called a *metric geometry*.

Definition: The **Taxicab Plane** $\mathcal{T}$ is defined to be the *metric geometry* $\mathcal{T} = (\mathbb{R}^2, \mathcal{L}_E, d_T)$.

Rulers for Some of our Geometries

Incidence Geometry	Distance Function	Type of Line	Standard Ruler
$\mathcal{C} = (\mathbb{R}^2, \mathcal{L}_E)$	d_E	$L_a = \{(a, y) \in \mathbb{R}^2\}$ $L_{m,b} = \{(x, y) \in \mathbb{R}^2 \mid y = mx + b\}$	$f(a, y) = y$ $f(x, y) = x\sqrt{1 + m^2}$
$\mathcal{C} = (\mathbb{R}^2, \mathcal{L}_E)$	d_T	$L_a = \{(a, y) \in \mathbb{R}^2\}$ $L_{m,b} = \{(x, y) \in \mathbb{R}^2 \mid y = mx + b\}$	$f(a, y) = y$ $f(x, y) = x(1 + \lvert m \rvert)$

Triangle Inequalities

The Triangle Inequality for the Absolute Value

For all $a \in \mathbb{R}$, this inequality is true:

$$-|a| \leq a \leq |a|$$

Of course, the symbol b could be used as well.

$$-|b| \leq b \leq |b|$$

Those two inequalities can be used to give us a new inequality:

$$-|a| - |b| \leq a + b \leq |a| + |b|$$

This new inequality tells us that the following inequality is true

$$|a + b| \leq |a| + |b|$$

This inequality is called the **triangle inequality for the absolute value**. It is true for all $a, b \in \mathbb{R}$.

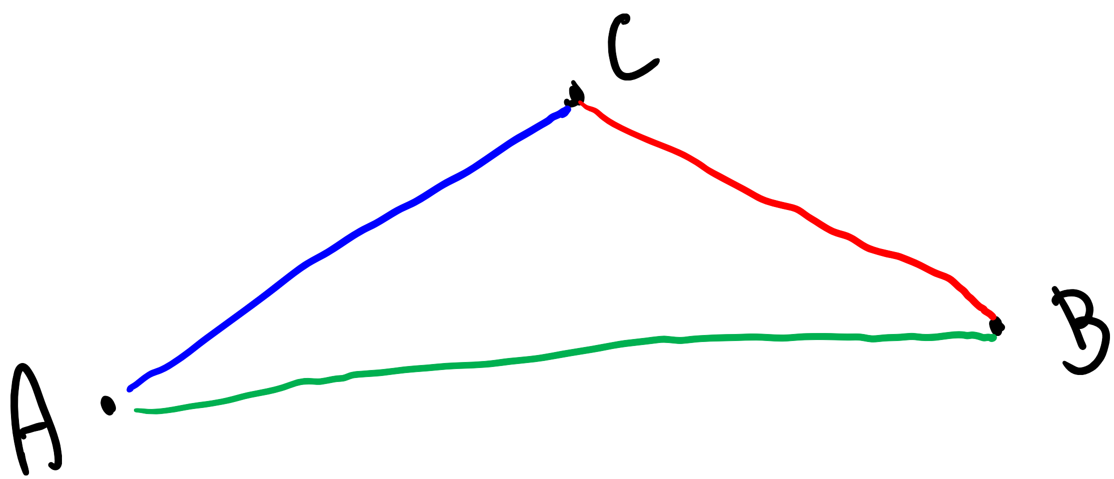
There are other things called triangle inequality, and they are not necessarily true.

Definition of the triangle inequality for distance functions

Words: Distance function d on set $\mathcal{P}$ satisfies the triangle inequality.

Meaning: For all $A, B, C \in \mathcal{P}$, the inequality $d(A,C) \leq d(A,B) + d(B,C)$ is true

In Symbols: $\forall A, B, C \in \mathcal{P}\big(d(A,C) \leq d(A,B) + d(B,C)\big)$



It is easy to show that the Absolute Value Distance Function satisfies the Triangle Inequality for Distance Functions.

> **Definition of the Absolute Value Distance Function on $\mathbb{R}$**
>
> **symbol:** $d_{\mathbb{R}}$
>
> **meaning:** the function $d_{\mathbb{R}}: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $d_{\mathbb{R}}(x, y) = |x - y|$

Proof

Suppose $x, y, z \in \mathbb{R}$

Subtract & add something

$$d_{\mathbb{R}}(x,z) = |x - z| = |x \overbrace{- y + y}^{0} - z| \leq |x - y| + |y - z| = d_{\mathbb{R}}(x,y) + d_{\mathbb{R}}(y,z)$$

by definition of $d_{\mathbb{R}}$

trick

by the absolute value triangle inequality

by definition of $d_{\mathbb{R}}$

End of Proof

What About Other Distance Functions?

In your homework problems 3.1#7,8, you are to show that the *Taxicab distance* d_T on $\mathbb{R}^2$ and the *Max distance* d_S on $\mathbb{R}^2$ both **do satisfy** the *triangle inequality for distance functions*. The proofs can be done in a similar manner

- at some point, use the trick of subtracting and adding the same quantity
- then use the *triangle inequality for the absolute value*.

But not all distance functions satisfy the *triangle inequality for distance functions*. Indeed, in your homework problem 3.1#9b, you will show that new distance function d_F on $\mathbb{R}^2$ **does not**.

> **Obvious Question:** Does the *Euclidean distance function* d_E on $\mathbb{R}^2$ satisfy the *triangle inequality for distance functions*?

If one attempts to show that the *triangle inequality* is satisfied using the formula for the Euclidean distance function, d_E, the computation gets messy very fast. It turns out that the computation can be made easier by using computations involving *vectors*. That (and some other reasons) lead us to study the *vector space* $\mathbb{R}^2$.

The Vector Space $\mathbb{R}^2$

Recall that points in the Cartesian Plane are elements of the ***set*** $\mathcal{P} = \mathbb{R}^2$. That is, they are ordered pairs (x, y) of real numbers. There are no operations involving points aside from the distance function on the set of pairs of points and the coordinate functions on the set of points on a line. One can not add points to each other, and one can not multiply points by numbers. But in the *vector space* $\mathbb{R}^2$, one can add ordered pairs of real numbers, and one can multiply an ordered pair by a real number.

Definition of the *vector space* $(\mathbb{R}^2, +, \boldsymbol{scalar\ mult})$.

For vectors $A = (x_A, y_A) \in \mathbb{R}^2$ and $B = (x_B, y_B) \in \mathbb{R}^2$ and scalar $r \in \mathbb{R}$, we define

- The ***sum*** of two vectors: $A + B = (x_A + x_B, y_A + y_B) \in \mathbb{R}^2$
- the ***scalar multiplication*** of a vector by a number: $rA = (rx_A, ry_A) \in \mathbb{R}^2$
- The ***difference*** of two vectors: $A - B = A + (-1)B = (x_A - x_B, y_A - y_B) \in \mathbb{R}^2$
- The ***inner product*** of two vectors: $\langle A, B \rangle = x_A x_B + y_A y_B \in \mathbb{R}$ the "dot product"
- The ***norm*** of a vector: $\|A\| = \sqrt{\langle A, A \rangle} = \sqrt{x_A x_A + y_A y_A} = \sqrt{x_A^2 + y_A^2} \in \mathbb{R}$

Basic properties of the the *vector space* $\mathbb{R}^2$ are articulated in the following proposition:

Proposition 3.1.1 Basic properties of the *vector space* $(\mathbb{R}^2, +, \boldsymbol{scalar\ mult})$.

For all vectors $A, B, C \in \mathbb{R}^2$ and scalars $r, s \in \mathbb{R}$,

(i) $A + B = B + A$

(ii) $(A + B) + C = A + (B + C)$

(iii) $r(A + B) = rA + rB$

(iv) $(r + s)A = rA + sA$

(v) $\langle A, B\rangle = \langle B, A\rangle$

(vi) $\langle rA, B\rangle = r\langle A, B\rangle$

(vii) $\langle A + B, C\rangle = \langle A, C\rangle + \langle B, C\rangle$

(viii) $\|rA\| = |r|\|A\|$

(ix) $\|A\| > 0$ if $A \neq (0,0)$

You should be able to prove this proposition. Its proof amounts to straightforward calculations that verify each property.

A sophisticated property of the the *vector space* $\mathbb{R}^2$ is articulated in the following proposition:

Proposition 3.1.5 Cauchy-Schwarz Inequality for the *vector space* $(\mathbb{R}^2, +, \textit{scalar mult})$.

For all vectors $A, B \in \mathbb{R}^2$, the inequality $|\langle A, B \rangle| \leq \|A\| \cdot \|B\|$ is true

The authors give an interesting proof of the Cauchy-Schwarz Inequality on page 45.

Another sophisticated property of the *vector space* $\mathbb{R}^2$ is not presented as a *proposition* in the book, but it is presenting the property here, in a nice green box.

Triangle Inequality satisfied by the *norm* for the *vector space* $(\mathbb{R}^2, +, \textit{scalar mult})$.

For all vectors $A, B \in \mathbb{R}^2$, the inequality $\|A + B\| \leq \|A\| + \|B\|$ is true

The authors give an interesting proof of this triangle inequality on page 46.

I won't present either of those proofs here, because they are very clearly presented in the book. But in a class meeting, we will consider some particular examples of vectors A and B that illustrate the inequalities, and you will do the same thing on a quiz.

I won't present either of those proofs here, because they are very clearly presented in the book. But in a class meeting, we will consider some particular examples of vectors A and B that illustrate the inequalities, and you will do the same thing on a quiz.

The point of introducing the *vector space* aspect of $\mathbb{R}^2$ is subtle: We are still interested in points as elements of the ***set*** $\mathcal{P} = \mathbb{R}^2$. But it turns out that some calculations involving points are made simpler if we use the operations of the vector space.

Proposition 3.1.3 Relationship between the *Euclidean distance function* and the *norm*

For all $A, B \in \mathbb{R}^2$, the equation $\underbrace{d_E(A,B)}_{\text{Computation from the Euclidean Plane}} = \underbrace{\|A - B\|}_{\text{Computation from the vector space } \mathbb{R}^2}$

The proof is straightforward.

Suppose $A = (x_A, y_A) \in \mathbb{R}^2$ and $B = (x_B, y_B) \in \mathbb{R}^2$

Computing the left side, we find

$$d_E(A,B) = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}$$

Computing the right side, we find

$$A - B = (x_A - x_B, y_A - y_B)$$

so

$$\|A - B\| = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}$$

This confirms that the equation is true

End of Proof

The equation $d_E(A, B) = \|A - B\|$ is certainly interesting, but you might notice that it doesn't seem to offer any real advantages. The left and right sides of the equation are simply two different ways of denoting the same (messy) calculation.

But now let's return to a question from earlier:

> **Obvious Question:** Does the *Euclidean distance function* d_E on $\mathbb{R}^2$ satisfy the *triangle inequality for distance functions*?

It turns out that we are prepared to give an answer now:

Proposition 3.1.6

The *Euclidean distance function* satisfies the *triangle inequality for distance functions*

For all $A, B, C \in \mathbb{R}^2$, the inequality $d_E(A, C) \leq d_E(A, C) + d_E(A, C)$ is true

The proof is wonderfully simple subtract + add B

$$d_E(A, C) \underset{\substack{\uparrow \\ \text{by proposition} \\ 3.1.3}}{=} \|A - C\| \underset{\substack{\uparrow \\ \text{trick}}}{=} \|A \overbrace{- B + B}^{} - C\| \underset{\substack{\uparrow \\ \text{the norm} \\ \text{satisfies the} \\ \text{triangle inequality}}}{\leq} \|A - B\| + \|B - C\| \underset{\substack{\uparrow \\ \text{By prop 3.1.3}}}{=} d_E(A, B) + d_E(B, C)$$

(The proof in the book is longer, but that is because the authors first prove the *triangle inequality* satisfied by the *norm*.)

Using Vectors to Describe Lines in the Cartesian Plane

Recall that in the *Cartesian plane*, given two distinct points A, B the description of the line through those two points depended on whether or not A, B had the same x coordinate.

But using vectors, we can give a description that works for all cases.

Using vectors to describe Cartesian lines

Given two distinct points $A, B \in \mathbb{R}^2$ line $\overleftrightarrow{AB}$ can be described using vectors as follows:

$$L_{AB} = \{X \in \mathbb{R}^2 | X = A + t(B - A) \text{ for some } t \in \mathbb{R}\}$$

Observe that the use of the letter X is not really necessary.

$$L_{AB} = \{A + t(B - A) | t \in \mathbb{R}\}$$

[Example 1]

(a) Let $A = (3,5)$ and $B = (5,9)$. Use vectors to describe the *Cartesian line* $\overleftrightarrow{AB}$. Simplify the result and illustrate it.

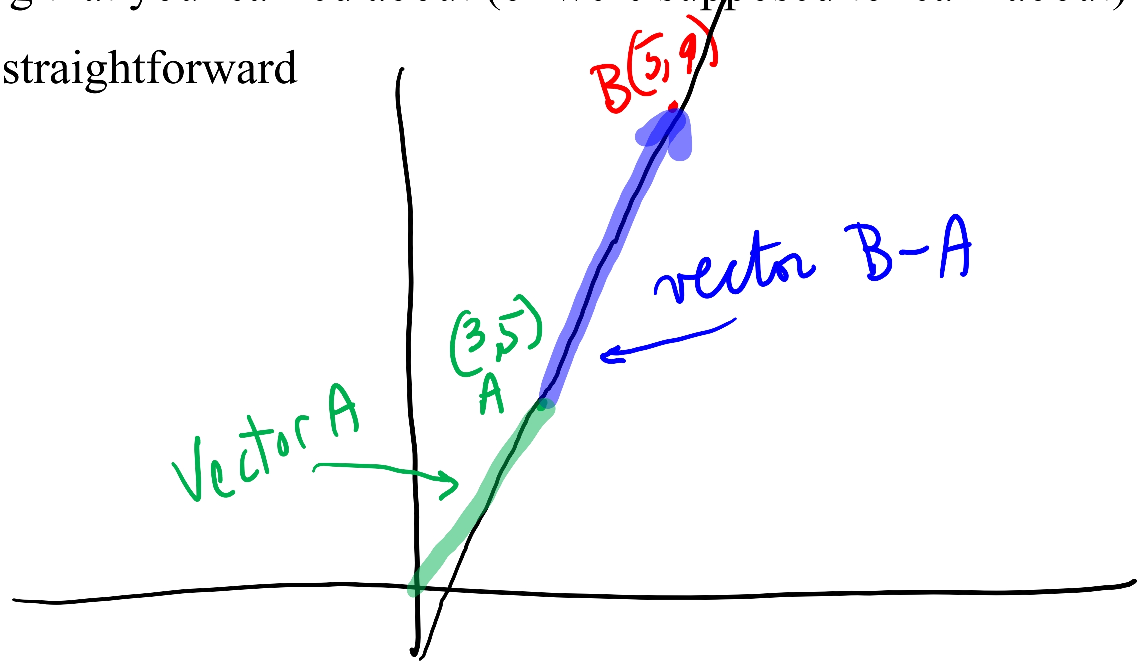
Solution:

$$L_{AB} = \{(3,5) + t\big((5,9) - (3,5)\big) | t \in \mathbb{R}\}$$

This can be simplified:

$$L_{AB} = \{(3,5) \ + t(2,4) | t \in \mathbb{R}\} = \{(3,5) \ + (2t, 4t) | t \in \mathbb{R}\} = \{(3 + 2t, 5 + 4t) | t \in \mathbb{R}\}$$

Note that the final form of the description is the familiar parametric form of the description of a line, something that you learned about (or were supposed to learn about) in Calculus. The illustration is straightforward



Remark: There is a lot of subtlety hidden in the notation of the symbol

$$L_{AB} = \{(3,5) + t\big((5,9) - (3,5)\big) | t \in \mathbb{R}\}$$

Starting with *ordered pairs* $A = (3,5)$ and $B = (5,9)$ representing *points* in $\mathcal{P} = \mathbb{R}^2$.

- Think of those ordered pairs as *vectors* in the *vector space* $(\mathbb{R}^{\mathbf{2}}, +, \boldsymbol{scalar\ mult})$
- Perform *vector addition* to obtain a new *vector* $(5,9) - (3,5)$.
- Perfom *scalar multiplication* to obtain a new *vector* $t\big((5,9) - (3,5)\big)$.
- Perform *vector addition* to obtain a new *vector* $(3,5) + t\big((5,9) - (3,5)\big)$.
- Think of the result as a *point* in $\mathcal{P} = \mathbb{R}^2$.

Realize that there is less subtlety in the symbol

Parametric form

$$L_{AB} = \{(3 + 2t, 5 + 4t) | t \in \mathbb{R}\}$$

In this symbol, the multiplication and addition are *real number* operations, *not vector* operations.

(b) Let $A = (3,5)$ and $C = (3,9)$. Use vectors to describe the *Cartesian line* $\overleftrightarrow{AB}$. Simplify the result and illustrate it.

Solution:

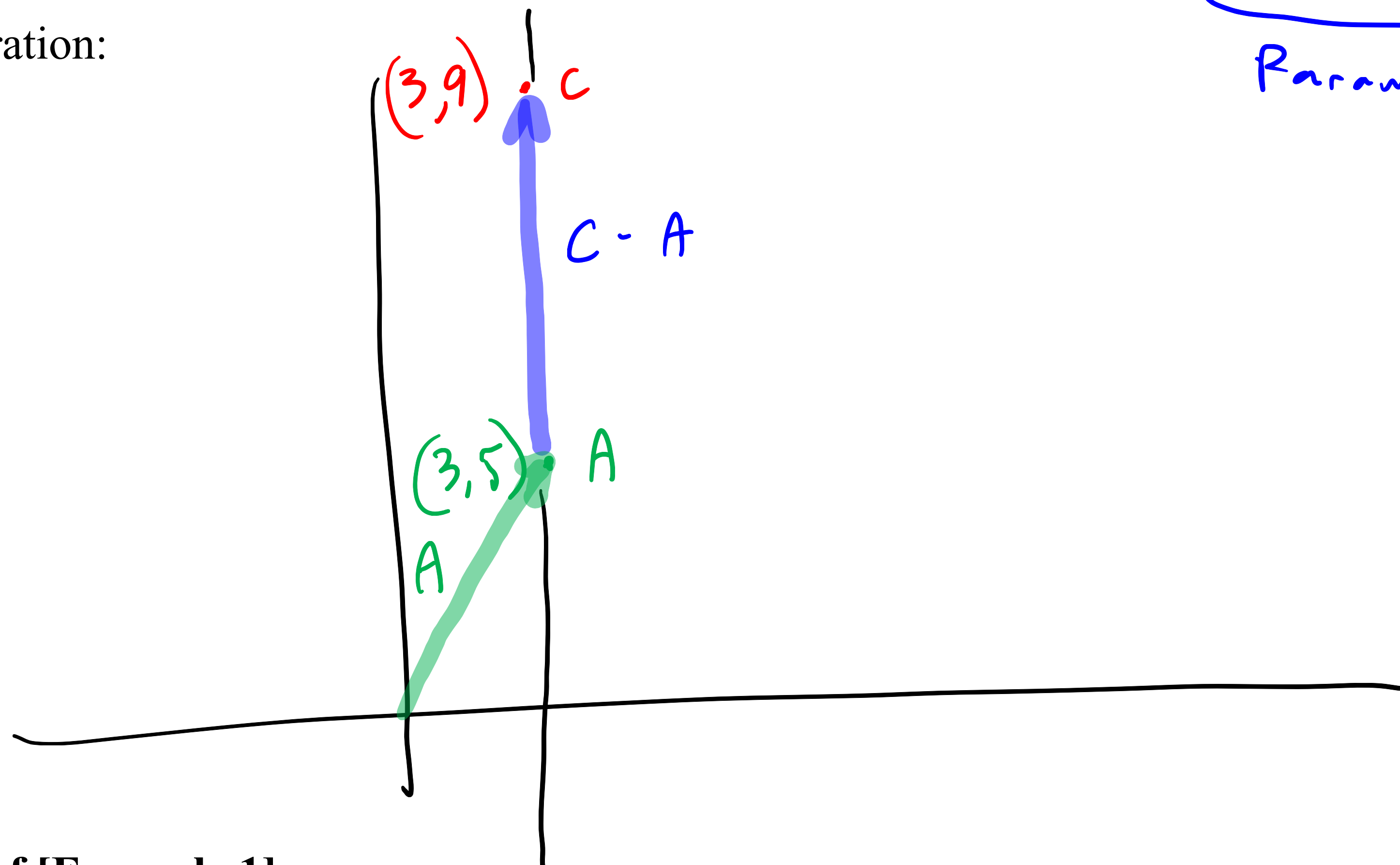
$$L_{AC} = \{(3,5) + t\big((3,9) - (3,5)\big) | t \in \mathbb{R}\}$$

This can be simplified:

$$L_{AC} = \{(3,5) + t(0,4) | t \in \mathbb{R}\} = \{(3,5) + (0,4t) | t \in \mathbb{R}\} = \{(3,5 + 4t) | t \in \mathbb{R}\}$$

Parametric form

Illustration:



End of [Example 1]

Remark: Notice that the line in part **(a)** is a non-vertical line, while the line in part **(b)** is a vertical line. However, that distinction played no role in our construction of the descriptions of the lines using **vectors**. We simply built the expression

$$L_{PQ} = \{P + t(Q - P) | t \in \mathbb{R}\}$$

and the result was a perfectly valid description of the line $\overleftrightarrow{PQ}$.

Vectors can also be used to define rulers!

Proposition 3.1.4 Using vectors to describe rulers in the Euclidean plane

If L_{AB} is a cartesian line, then $f: L_{AB} \to \mathbb{R}$ defined by

$$f\big(A + t(B - A)\big) = t\|B - A\|$$

is a ruler for the line L_{AB} in the *Euclidean Plane*.

The proof that the formula shown really does define a ruler is presented in the book, and I won't discuss it here.

However it is worth exploring the vector description a bit. We will see that although the vector description of a ruler is easy to write down (just as the vector description of a line is easy to write down), the resulting ruler is not so convenient to use to actually find the coordinates of a point!

[Example 2] Vector Description of Rulers.

(a) Let $A = (3,5)$ and $B = (5,9)$. Use vectors to describe a ruler for the *Cartesian line* $\overleftrightarrow{AB}$.

Solution:

$$f\Big((3,5) + t\big((5,9) - (3,5)\big)\Big) = t\|(5,9) - (3,5)\|$$

This can be simplified:

$$f\big((3,5) + t(2,4)\big) = t\|(2,4)\|$$

This can also be simplified:

$$\|(2,4)\| = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$$

So the ruler can be written

$$f\big((3,5) + t(2,4)\big) = t2\sqrt{5}$$

And this can be simplified even further:

$$f(3 + 2t, 5 + 4t) = t2\sqrt{5}$$

(b) Using that ruler, find the coordinate of point $P = (4,7)$ that lies on line $\overleftrightarrow{AB}$.

Solution:

We must first find the value of t

$$(4,7) = (3 + 2t, 5 + 4t)$$

We find that

$$t = \frac{1}{2}$$

Therefore, the coordinate of P is

$$f(P) = f\left(3 + 2\left(\frac{1}{2}\right), 5 + 4\left(\frac{1}{2}\right)\right) = \left(\frac{1}{2}\right)2\sqrt{5} = \sqrt{5} \approx 2.236$$

(c) Using that ruler, find the coordinate of point $Q = (6,11)$ that lies on line $\overleftrightarrow{AB}$.

Solution:

We must first find the value of t

$$(6,11) = (3 + 2t, 5 + 4t)$$

Parametric description of the line, from earlier.

We find that

$$t = \frac{3}{2}$$

Therefore, the coordinate of Q is

$$f(Q) = f\left(3 + 2\left(\frac{3}{2}\right), 5 + 4\left(\frac{3}{2}\right)\right) = \left(\frac{3}{2}\right)2\sqrt{5} = 3\sqrt{5} \approx 6.708$$

(d) Using the coordinates $f(P)$ and $f(Q)$ that you obtained in **(b)** and **(c)**, compute the value of

$$|f(P) - f(Q)|$$

Solution:

$$|f(P) - f(Q)| = \left|\sqrt{5} - 3\sqrt{5}\right| = \left|-2\sqrt{5}\right| = 2\sqrt{5} \approx 4.472$$

(e) Compare the result to the value of $d_E(P, Q)$.

Solution:

$$d_E(P, Q) = \sqrt{\left(x_P - x_Q\right)^2 + \left(y_P - y_Q\right)^2} = \sqrt{(4 - 6)^2 + (7 - 11)^2} = \cdots = 2\sqrt{5}$$

We see that the ruler equation is satisfied

$$|f(P) - f(Q)| = 2\sqrt{5} = d_E(P, Q)$$

Remark: We see that the vector form of the ruler is simple to *build*, but it is cumbersome to *use* to find the actual coordinates of an actual point, because one must first find the value of t.

End of [Example 2]

The Vector Description of a Ruler is a *Special Ruler*!

The preceeding example illustrated that although the vector description of a ruler is easy to write down (just as the vector description of a line is easy to write down), the resulting ruler is not so convenient to use to actually find the coordinates of a point! However, the vector description of rulers is handy from a theoretical standpoint. Here is the first example of a situation where the vector description is handy.

Consider two given unique points $A, B \in \mathbb{R}^2$. They determine a unique line $\overleftrightarrow{AB}$ in the *Cartesian plane*, and the function $f: L_{AB} \to \mathbb{R}$ defined by

$$f\big(A + t(B - A)\big) = t\|B - A\|$$

is a ruler for that line $\overleftrightarrow{AB}$ in the *Euclidean Plane*. Let's find the coordinates of points A and B using this ruler.

To find the coordinate of point A on line $\overleftrightarrow{AB}$, we must first find the value of t

$$A = A + t(B - A)$$

Solving this equation for t, we find that

$$t = 0$$

Therefore, the coordinate of A is

$$f(A) = f\big(A + 0(B - A)\big) = 0\|B - A\| = 0$$

To find the coordinate of point B on line $\overleftrightarrow{AB}$, we must first find the value of t

$$B = A + t(B - A)$$

Solving this equation for t, we find that

$$t = 1$$

Therefore, the coordinate of B is

$$f(B) = f\big(A + 1(B - A)\big) = 1\|B - A\| = \text{positive}$$

So we see that the ruler f will have the property that $f(A) = 0$ and $f(B)$ is *positive*.

In my videos, I have called such a ruler a *special ruler* for points A and B.

The book calls this a **ruler with *A* as origin and *B* positive**.

Recall in Section 2.3 that a lot of work went into building a special ruler. By comparison, we were able to build a special ruler using a vector description very quickly. This is one example of a situation where the vector description of rulers is handy from a theoretical standpoint. We will see other situations where the vector description of rulers is handy in coming sections.

End of Video